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Strong subband-parcelling extensions of orthodox semigroups

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The notion of subband-parcelling and strong subband-parcelling congruences on orthodox semigroups was introduced in [4] by generalizing the common properties of congruences appearing in a number of structure theorems concerning orthodox semigroups. For a list of such structure theorems the reader is referred to [4]. In particular, the concept of strong subband-parcelling congruences includes the idempotent separating congruences and the least inverse semigroups congruences on orthodox semigroups which play singificant roles in the theory of orthodox semigroups.

An orthodox semigroup T is said to be a strong subband-parcelling extension of the semigroup S if there exists a strong subband-parcelling congruence $\varkappa$ on T such that $T/\varkappa$ is isomorphic to S.

The aim of the present paper is to describe all strong subband-parcelling extensions of orthodox semigroups. All strong subband-separating extensions of orthodox semigroups are characterized in [5]. It is worth dealing with this special case separately because a much simpler construction is needed than in the general case. On the other hand, in the class of all inverse semigroups every subband-parcelling congruence is subsemilattice-separating.

In Sections 2 and 3 we introduce the construction which will be used in Section 4 to describe the strong subband-parcelling extensions of orthodox semigroups. At the end of this section we apply our results to characterize an orthodox semigroup as an extension of an inverse semigroup by the least inverse semigroup congruence. Thus we obtain a structure theorem for orthodox semigroups which describes orthodox semigroups by means of their bands of idempotents and greatest inverse semigroup homomorphic images as it was done also by YAMADA ([7]). However, our construction seems to be easier to apply in certain cases than the quasi-direct product used by him.

The notions and notations of [1] and [2] are used.

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1. Preliminary notions and results

The concept of a subband-parcelling congruence on an orthodox semigroup was introduced in [4] as follows.

Let B be a band. Suppose δ is a congruence on B with $\delta \subseteq \mathcal{D}$ and $\bar{B}$ is a subband in B with the property that $\bar{B}$ is a union of δ-classes. For brevity, if $\bar{B}$ and δ satisfy these conditions then we say that $\bar{B}, \delta$ is an *associated pair* in B. Let T be an orthodox semigroup with band of idempotents B.

Definition. The congruence relation $\varkappa$ on T is called $(\bar{B}, \delta)$*-parcelling* if the following conditions are fulfilled:

(d_1) $\delta \subseteq \varkappa | B$,

(d_2) every $\varkappa$-class containing an idempotent element contains an element of $\bar{B}$, and

(d_3) the elements of $\bar{B}$ belonging to a $\varkappa$-class form a δ-class which is the greatest one in this $\varkappa$-class. (By the order of δ-classes we mean the natural order of B/δ.)

In particular, if δ is the identical congruence then $\varkappa$ is called $\bar{B}$*-separating*. In this case (d_1) is satisfied trivially and (d_3) means that every element of $\bar{B}$ is the greatest idempotent in the $\varkappa$-class containing it.

The following proposition characterizes the subband-parcelling congruences.

Proposition 1.1. *Let T be an orthodox semigroup with band of idempotents B. The congruence relation $\varkappa$ on T is subband-parcelling if and only if there exists a greatest $\mathcal{D}$-class in the band of idempotents of each idempotent $\varkappa$-class and their union is a subband in B.*

Proof. Suppose first that $\bar{B}, \delta$ is an associated pair in B and $\varkappa$ is a $(\bar{B}, \delta)$-parcelling congruence. Let K be an idempotent $\varkappa$-class in T. Since T is regular K contains an idempotent element and hence, by (d_2) and (d_3), there exists a greatest δ-class in the band of idempotents E of K and this δ-class is just $\bar{B} \cap E$. Since $\delta \subseteq \varkappa | B$ by (d_1), $\delta | E \subseteq \mathcal{D}_E$ is implied by $\delta \subseteq \mathcal{D}_B$. Therefore $e_1 \mathcal{D}_E \leqq e_2 \mathcal{D}_E$ follows from $e_1 \delta \leqq e_2 \delta$ for every pair e_1, e_2 in E which shows that $\bar{B} \cap E$ is the greatest $\mathcal{D}$-class in E. Clearly, the union of these $\mathcal{D}$-classes for all $\varkappa$-classes is just $\bar{B}$.

Conversely, assume that $\varkappa$ is a congruence on T with the properties that the band of idempotents of each idempotent $\varkappa$-class contains a greatest $\mathcal{D}$-class and their union is a subband $\bar{B}$ in B. Consider the congruence $\delta = \varkappa | B \cap \mathcal{D}_B$ on B. Then the greatest $\mathcal{D}$-class in the band of idempotents of an idempotent $\varkappa$-class is the greatest δ-class and $\bar{B}, \delta$ is an associated pair in B. One can easily see that $\varkappa$ is a $(\bar{B}, \delta)$-parcelling congruence. The proof is complete.

The most important properties of subband-parcelling congruences proved in

[4] are drawn up in the following results. If $\bar{B}, \delta$ is an associated pair in B then $\bar{\delta}$ is used to mean $\delta|\bar{B}$.

Theorem 1.2 ([4] Theorem 2.5). *Suppose T is an orthodox semigroup with band of idempotents B and $\bar{B}, \delta$ is an associated pair in B. If there exists a $(\bar{B}, \delta)$-parcelling congruence $\varkappa$ on T then B is a band $\bar{B}/\bar{\delta}$ of the bands $F_x=\{b\in B: b\delta\leqq x$ and $b\delta\leqq y$ implies $x\leqq y$ for every y in $\bar{B}/\bar{\delta}\}$ $(x\in\bar{B}/\bar{\delta})$ with greatest δ-class x. The $\varkappa$-class containing the δ-class x is an orthodox subsemigroup in T with band of idempotents F_x.*

Remark. F_x is given in [4], Theorem 2.5, in a slightly modified form. The equivalence of these characterizations can be easily checked.

Lemma 1.3. *Suppose T is an orthodox semigroup with band of idempotents B and $\bar{B}, \delta$ is an associated pair in B. Assume that T has a $(\bar{B}, \delta)$-parcelling congruence. Let t and t^* be inverses of each other in T such that $tt^*\in F_{\bar{x}}$ and $t^*t\in F_{\bar{y}}$. Then, for every x and y in $\bar{B}/\bar{\delta}$ with $x\mathcal{R}\bar{x}$ and $y\mathcal{L}\bar{y}$, there exists an inverse t' of t such that $tt'\in F_x$ and $t't\in F_y$.*

Theorem 1.4 ([4] Theorem 2.9). *Let T be an orthodox semigroup with band of idempotents B and $\bar{B}, \delta$ an associated pair in B. Suppose T to have a $(\bar{B}, \delta)$-parcelling congruence. Then $S_B=\{t\in T: e\mathcal{R}t\mathcal{L}f$ for some e, f in $\bar{B}\}$ is an orthodox subsemigroup in T. The band of idempotents in S_B is $\bar{B}$ and the inverses of the elements in S_B belong to S_B.*

The subsemigroup S_B plays a significant role in the case of strong subband-parcelling congruences.

Definition. The $(\bar{B}, \delta)$-parcelling congruence $\varkappa$ on the orthodox semigroup T is called *strong* if every $\varkappa$-class contains an element of S_B.

Obviously, the (B, δ)-parcelling congruences are strong for $S_B=T$.

The following result is important in describing the strong subband-parcelling extensions of orthodox semigroups.

Proposition 1.5 ([4] Theorem 2.10). *Assume that T is an orthodox semigroup with band of idempotents B and $\bar{B}, \delta$ is an associated pair in B. Let $\varkappa$ be a strong $(\bar{B}, \delta)$-parcelling congruence on T. Consider two elements t and t' in T which are inverses of each other. Then there exist elements s and s' in S_B with $s\varkappa t$ and $s'\varkappa t'$ such that s and s' are inverses of each other.*

2. Semidirect product of a partial left band and a right orthodox partial semigroup

The structure of completely regular semigroups was characterized among others by WARNE in [6]. We will apply his result in the special case of bands.

Let Y be a semilattice and I_α a left zero semigroup for every α in Y. A partial groupoid on $I=\bigcup\{I_\alpha:\alpha\in Y\}$ is called a *lower associative semilattice Y of left zero semigroups I_α* $(\alpha\in Y)$ if (i) $I_\alpha\cap I_\beta=\square$ whenever $\alpha\neq\beta$, (ii) the product of elements a in I_α and b in I_β is defined if and only if $\alpha\geqq\beta$, (iii) if $\alpha\geqq\beta$ then $I_\alpha I_\beta\subseteqq I_\beta$ and (iv) if $\alpha\geqq\beta\geqq\gamma$ and $a\in I_\alpha, b\in I_\beta, c\in I_\gamma$ then $a(bc)=(ab)c$. The notion of an *upper associative semilattice of right zero semigroups* is obtained dually.

Let I be a lower associative semilattice Y of left zero semigroups $I_\alpha\,(\alpha\in Y)$ and J an upper associative semilattice Y of right zero semigroups $J_\alpha\,(\alpha\in Y)$. For every u in J, let A_u be a transformation of I and, for every a in I, let B_a be a transformation of J such that $aA_u\in I_{\alpha\beta}$ and $uB_a\in J_{\alpha\beta}$ provided $a\in I_\alpha$ and $u\in J_\beta$. Moreover, let the following conditions be fulfilled:

(W1) if $a\in I_\alpha$, $b\in I_\beta$ with $\alpha\geqq\beta$ and $u\in J$ then

(a) $uB_{ab}=uB_aB_b$,

(b) $(ab)A_u=aA_u\cdot bA_{uB_a}$;

(W2) if $u\in J_\alpha$, $v\in J_\beta$ with $\alpha\leqq\beta$ and $a\in I$ then

(a) $aA_{uv}=aA_vA_u$,

(b) $(uv)B_a=uB_{aA_v}\cdot vB_a$.

A pair A, B satisfying these conditions is termed an *(I, J)-pair*.

Define a multiplication on the set $E=\bigcup\{I_\alpha\times J_\alpha:\alpha\in Y\}$ by

$$(a,u)(b,v)=(a\cdot bA_u,\ uB_b\cdot v).$$

One can show that E is a band with respect to this multiplication. This band is called a *semidirect product of I and J* and is denoted by $\mathscr{B}(I,J;A,B)$.

Theorem 2.1 (WARNE [6]). *Every band is isomorphic to a semidirect product of some I and J where I is a lower associative semilattice Y of left zero semigroups and J is an upper associative semilattice Y of right zero semigroups for some semilattice Y.*

First we generalize the notion of an upper associative semilattice of right zero semigroups by introducing the notion of a right orthodox partial semigroup. We need the definition of the spined product of partial groupoids. The notion of spined product of semigroups is due to KIMURA [3].

Let both S and T be partial groupoids which are semilattices Y of full subgroupoids S_α and T_α $(\alpha\in Y)$, respectively. By the *spined product $S\otimes_Y T$ of S and T*

over Y we mean the subdirect product of S and T whose underlying set is

$$\cup \{S_\alpha \times T_\alpha : \alpha \in Y\}.$$

Let $\bar{Y}$ be a semilattice. Let $\bar{J}$ be an upper associative semilattice $\bar{Y}$ of right zero semigroups $\bar{J}_{\bar{\alpha}}$ $(\bar{\alpha} \in \bar{Y})$. For every $\bar{\alpha}$ in $\bar{Y}$, suppose $Y_{\bar{\alpha}}$ $(\bar{\alpha} \in \bar{Y})$ to be a semilattice with identity $\bar{\alpha}$. Assume that Y is a semilattice $\bar{Y}$ of semilattices $Y_{\bar{\alpha}}$ $(\bar{\alpha} \in \bar{Y})$ such that $\bar{Y}$ is a subsemilattice in Y. Let J be a partial groupoid with respect to the operation denoted by "$\cdot$". We say that an element σ in J is *idempotent* if $\sigma \cdot \sigma$ is defined and $\sigma \cdot \sigma = \sigma$. Suppose $J = \cup \{J_\alpha^{\bar{j}} : (\bar{j}, \alpha) \in \bar{J} \otimes_{\bar{Y}} Y\}$ where $J_\alpha^{\bar{j}} \cap J_\beta^{\bar{k}} = \square$ provided $(\bar{j}, \alpha) \neq (\bar{k}, \beta)$. Introduce the following notation: for every α in $Y_{\bar{\alpha}}$, let $J_\alpha = \cup \{J_\alpha^{\bar{j}} : \bar{j} \in \bar{J}_{\bar{\alpha}}\}$. Assume that there exists a unary operation "$'$" on J such that the following hold: for arbitrary elements ϱ in $J_\alpha^{\bar{j}}$, σ in $J_\beta^{\bar{k}}$ and τ in $J_\gamma^{\bar{l}}$ with $\varrho' \in J_{\alpha'}$, $\sigma' \in J_{\beta'}$, and $\tau' \in J_{\gamma'}$ we have

(B1) α and α' are contained in the same $Y_{\bar{\alpha}}$ and $\varrho'' \in J_\alpha$;
(B2) $\varrho \cdot \sigma$ is defined in J if and only if $\alpha' \leqq \beta$ and, in this case, $\varrho \cdot \sigma \in J_\alpha^{\bar{j} \cdot \bar{k}}$ and $(\varrho \cdot \sigma)' \in J_\xi$ for some ξ with $\xi \leqq \beta'$;
(B3) if $\alpha' \leqq \beta$ and $\beta' \leqq \gamma$ then $(\varrho \cdot \sigma) \cdot \tau = \varrho \cdot (\sigma \cdot \tau)$;
(B4) $\varrho \cdot \varrho' \cdot \varrho = \varrho$ and $\varrho' \cdot \varrho \cdot \varrho' = \varrho'$;
(B5) the idempotents in J_α form a right zero semigroup for each α in Y;
(B6) if ϱ and σ are idempotents with $\alpha \leqq \gamma$ and $\beta \leqq \gamma'$ then $(\varrho \cdot \tau)' \in J_\beta$ if and only if $(\sigma \cdot \tau')' \in J_\alpha$.

Note that both sides of the equality in (B3) are defined by (B2). Parentheses are not needed in (B4) by (B3). Moreover, J_α contains an idempotent for every α in Y as $\varrho \in J_\alpha$ implies by (B4) and (B2) that $\varrho \cdot \varrho'$ is an idempotent in J_α. In property (B2) the product $\bar{j} \cdot \bar{k}$ is defined in $\bar{J}$ since $\alpha \in Y_{\bar{\alpha}}$, $\beta \in Y_{\bar{\beta}}$ imply $\alpha' \in Y_{\bar{\alpha}}$ whence it follows by $\alpha' \leqq \beta$ that $\bar{\alpha} \leqq \bar{\beta}$. Since ϱ and σ are idempotent in (B6), we have $\alpha' \leqq \alpha$ and $\beta' \leqq \beta$. Thus the products in (B6) are defined.

A partial groupoid J fulfilling the above conditions is termed a *right orthodox partial semigroup over* $\bar{J} \otimes_{\bar{Y}} Y$. Dually, one can define a *left orthodox partial semigroup* I *over* some $\bar{I} \otimes_{\bar{Y}} Y$ where $\bar{I}$ is a lower associative semilattice $\bar{Y}$ of left zero semigroups. The duals of properties (B1)—(B6) will be referred to as (B1)*—(B6)*. If the elements of J or I are idempotent then we call them a *partial right band* and a *partial left band*, respectively.

Suppose $\varrho \in J_\alpha$ and $\varrho' \in J_{\alpha'}$. An element $\sigma \in J_\beta$ with $\sigma' \in J_{\beta'}$ is called an *inverse* of ϱ provided $\alpha' \leqq \beta$, $\beta' \leqq \alpha$ and $\varrho\sigma\varrho = \varrho$, $\sigma\varrho\sigma = \sigma$. Property (B4) means that the operation "$'$" picks out an inverse of each element. Observe that $\varrho \cdot \sigma$ and $\sigma \cdot \varrho$ are idempotent provided ϱ and σ are inverses of each other.

In what follows we draw up the basic properties of a right orthodox partial semigroup J in several lemmas. For brevity, introduce the following notations. If

$\alpha\leqq\beta$ in Y then we denote this fact also by $J_\alpha\leqq J_\beta$. If $\sigma\in J_\alpha$ then $J(\sigma)$ is used to mean J_α.

Lemma 2.2. *If* $\varrho\in J_\alpha$, $\varrho', \sigma\in J_{\alpha'}$, *and* $\sigma'\in J_{\alpha''}$ *then* $(\varrho\cdot\sigma)'\in J_{\alpha''}$.

Proof. By (B1) and (B2), the product $\varrho'\cdot(\varrho\cdot\sigma)$ is defined and $J((\varrho'\cdot(\varrho\cdot\sigma))')\leqq \leqq J((\varrho\cdot\sigma)')$. On the other hand, (B3) ensures that $(\varrho'\cdot\varrho)\cdot\sigma=\varrho'\cdot(\varrho\cdot\sigma)$. Moreover, the product $\varrho\cdot((\varrho'\cdot\varrho)\cdot\sigma)$ is also defined by (B2) and properties (B3), (B4) imply it to be equal to $\varrho\cdot\sigma$. Again by (B2), we have $J((\varrho\cdot\sigma)')\leqq J((\varrho'\cdot(\varrho\cdot\sigma))')$. Hence we obtain that $J((\varrho\cdot\sigma)')=J(((\varrho'\cdot\varrho)\cdot\sigma)')$. It follows from (B2), (B3) and (B4) that $\varrho'\cdot\varrho$ and $\sigma'\cdot\sigma$ are idempotents in $J_{\alpha'}$ and $J_{\alpha''}$, respectively. Thus, by (B6), we have $J(((\varrho'\cdot\varrho)\cdot\sigma)')=J_{\alpha''}$ as $((\sigma'\cdot\sigma)\cdot\sigma')'=\sigma''\in J_{\alpha'}$, by (B4) and (B1). Hence $J((\varrho\cdot\sigma)')=J_{\alpha''}$ which was to be proved.

Lemma 2.3. *If* $\varrho\in J_\alpha$, $\varrho'\in J_{\alpha'}$, $\sigma\in J_\beta$, $\sigma'\in J_{\beta'}$ *and* $\alpha'\leqq\beta$ *then* $(\varrho\cdot\sigma)'\in J_{\beta'}$ *implies* $\alpha'=\beta$.

Proof. By (B2), $\varrho\cdot\sigma\in J_\alpha$. Thus $(\varrho'\cdot(\varrho\cdot\sigma))'=((\varrho'\cdot\varrho)\cdot\sigma)'\in J_{\beta'}$ by Lemma 2.2. Since $\varrho'\cdot\varrho\in J_{\alpha'}$, $\sigma'\cdot\sigma\in J_{\beta'}$ and they are idempotent property (B6) ensures $((\sigma'\cdot\sigma)\cdot\sigma')'\in J_{\alpha'}$. On the other hand, (B1) and (B4) imply $((\sigma'\cdot\sigma)\cdot\sigma')'=\sigma''\in J_\beta$. Hence $\alpha'=\beta$.

Lemma 2.4. *If* ϱ *and* ϱ^* *are inverses of each other in* J *then* $J(\varrho)=J(\varrho^{*\prime})$ *and* $J(\varrho^*)=J(\varrho')$. *In particular,* $J(\varepsilon)=J(\varepsilon')$ *provided* ε *is idempotent.*

Proof. By definition, $J(\varrho')\leqq J(\varrho^*)$, $J(\varrho^{*\prime})\leqq J(\varrho)$ and $\varrho\cdot\varrho^*\cdot\varrho=\varrho$, $\varrho^*\cdot\varrho\cdot\varrho^*=\varrho^*$. Lemma 2.3 implies that $J((\varrho\cdot\varrho^*)')=J(\varrho)$. On the other hand, by (B2), we have $J((\varrho\cdot\varrho^*)')\leqq J(\varrho^{*\prime})$. Thus $J(\varrho)\leqq J(\varrho^{*\prime})$ whence we conclude the equality $J(\varrho)=J(\varrho^{*\prime})$. Similarly, starting with the equality $\varrho^*\cdot\varrho\cdot\varrho^*=\varrho^*$, the equality $J(\varrho^*)=J(\varrho')$ yields.

This lemma shows that in the case of partial right [left] bands the operation "′" can be chosen to be the identity transformation. In what follows, the operation "′" is always assumed to be identical in the case of partial right [left] bands.

Lemma 2.5. *The inverses of the idempotent elements in* J *are also idempotent.*

Proof. Suppose $\varepsilon\in J_\alpha$ is idempotent and ξ is an inverse of ε. By Lemma 2.4, $\varepsilon'\in J_\alpha$ and $\xi, \xi'\in J_\alpha$. Since $\varepsilon\cdot\varepsilon=\varepsilon$ we have $\xi=\xi\cdot\varepsilon\cdot\xi=(\xi\cdot\varepsilon)\cdot(\varepsilon\cdot\xi)$. Here $\xi\cdot\varepsilon$ and $\varepsilon\cdot\xi$ are idempotents in J_α. Thus, by (B5), their product ξ is also idempotent.

Lemma 2.6. *Let* ϱ^* *and* ϱ^{**} *be two inverses of* $\varrho\in J_\alpha$. *Then* $\varrho^*=\varrho^{**}\cdot\varepsilon$ *for some idempotent element* ε *in* J_α.

Proof. Assume that $\varrho'\in J_{\alpha'}$. Then, by Lemma 2.4, $\varrho^*, \varrho^{**}\in J_{\alpha'}$ and

$\varrho^{*\prime}, \varrho^{**\prime} \in J_\alpha$. Moreover, the elements $\varrho^* \cdot \varrho$ and $\varrho^{**} \cdot \varrho$ are idempotents in $J_{\alpha'}$. (B5) implies $(\varrho^* \cdot \varrho) \cdot (\varrho^{**} \cdot \varrho) = \varrho^{**} \cdot \varrho$. On the other hand, we obtain by (B3) that $(\varrho^* \cdot \varrho) \cdot (\varrho^{**} \cdot \varrho) = \varrho^* \cdot (\varrho \cdot \varrho^{**} \cdot \varrho) = \varrho^* \cdot \varrho$ whence we have $\varrho^* \cdot \varrho = \varrho^{**} \cdot \varrho$. Thus $\varrho^* = \varrho^* \cdot \varrho \cdot \varrho^* = \varrho^{**} \cdot (\varrho \cdot \varrho^*)$ where $\varrho \cdot \varrho^*$ is an idempotent element in J_α.

Lemma 2.7. *If $\sigma \in J_\alpha$, $\sigma' \in J_{\alpha'}$ and $\varrho = \sigma \cdot \varepsilon$ for some idempotent ε in $J_{\alpha'}$ then any inverse σ^* of σ is an inverse of ϱ.*

Proof. By Lemmas 2.2 and 2.4 we have $(\sigma \cdot \varepsilon)' \in J_{\alpha'}$. Moreover, $\sigma^* \cdot \sigma$ is an idempotent element in $J_{\alpha'}$ since $\sigma^* \in J_{\alpha'}$ by Lemma 2.4. Thus (B5) ensures $\varepsilon \cdot (\sigma^* \cdot \sigma) = \sigma^* \cdot \sigma$. By applying (B3) we obtain that

$$\varrho \cdot \sigma^* \cdot \varrho = (\sigma \cdot \varepsilon) \cdot \sigma^* \cdot (\sigma \cdot \varepsilon) = \sigma \cdot \big(\varepsilon \cdot (\sigma^* \cdot \sigma)\big) \cdot \varepsilon = \sigma \cdot (\sigma^* \cdot \sigma) \cdot \varepsilon = \sigma \cdot \varepsilon = \varrho$$

and

$$\sigma^* \cdot \varrho \cdot \sigma^* = \sigma^* \cdot (\sigma \cdot \varepsilon) \cdot \sigma^* = (\sigma^* \cdot \sigma) \cdot \varepsilon \cdot (\sigma^* \cdot \sigma) \cdot \sigma^* = (\sigma^* \cdot \sigma) \cdot (\sigma^* \cdot \sigma) \cdot \sigma^* = \sigma^*.$$

The proof is complete.

Define a relation "$\sim$" on J by $\sigma \sim \tau$ if and only if there exists a common inverse of σ and τ in J.

Lemma 2.8. (i) *The relation $\sim$ is an equivalence.*

(ii) *Let $\sigma, \tau \in J$. Then $\sigma \sim \tau$ if and only if $J(\sigma) = J(\tau)$, $J(\sigma') = J(\tau')$ and there exists an idempotent ε in $J(\sigma')$ with $\sigma = \tau \cdot \varepsilon$.*

(iii) *Let $\sigma, \tau \in J$. Then $\sigma \sim \tau$ if and only if the sets of inverses of σ and τ are equal.*

Proof. (ii) and (iii) immediately follow from Lemmas 2.6 and 2.7. Statement (i) is clear by (iii).

Lemma 2.9. *Suppose $\varrho \in J_\alpha$, $\varrho' \in J_{\alpha'}$ with $\alpha' \leqq \alpha$ and $\sigma \in J_\alpha$. Then $\varrho \cdot \sigma = \sigma$ if and only if ϱ is idempotent.*

Proof. If ϱ is idempotent then $\alpha' = \alpha$ by Lemma 2.4. Since $\sigma \cdot \sigma' \in J_\alpha$ is also idempotent we have $\varrho \cdot (\sigma \cdot \sigma') = \sigma \cdot \sigma'$ by (B5). Consequently, (B3) and (B4) imply $\varrho \cdot \sigma = \big(\varrho \cdot (\sigma \cdot \sigma')\big) \cdot \sigma = \sigma \cdot \sigma' \cdot \sigma = \sigma$. Conversely, suppose $\varrho, \sigma \in J_\alpha$, $\varrho' \in J_{\alpha'}$, $\sigma' \in J_{\beta'}$ and $\alpha' \leqq \alpha$. Let

(1) $$\varrho \cdot \sigma = \sigma.$$

Then $(\varrho \cdot \sigma)' = \sigma' \in J_{\beta'}$. Thus $\alpha' = \alpha$ follows from Lemma 2.3. Since $\varrho' \cdot \varrho$ and $\sigma \cdot \sigma'$ are idempotents in J_α property (B5) implies the equality $(\sigma \cdot \sigma') \cdot (\varrho' \cdot \varrho) = \varrho' \cdot \varrho$.

By applying (1) and (B3) we obtain that

$$\varrho = \varrho\cdot\varrho'\cdot\varrho = \varrho\cdot((\sigma\cdot\sigma')\cdot(\varrho'\cdot\varrho)) = \varrho\cdot\sigma\cdot(\sigma'\cdot(\varrho'\cdot\varrho)) = \sigma\cdot(\sigma'\cdot(\varrho'\cdot\varrho)) =$$
$$= (\sigma\cdot\sigma')\cdot(\varrho'\cdot\varrho) = \varrho'\cdot\varrho.$$

Hence ϱ is, in fact, idempotent.

Lemma 2.10. *Let* $\varrho, \sigma \in J$ *with* $J(\varrho)=J(\sigma')$. *Then* $\sigma^*\cdot\varrho=\sigma^{**}\cdot\varrho$ *holds for arbitrary inverses* σ^*, σ^{**} *of* σ.

Proof. By Lemma 2.6, $\sigma^*=\sigma^{**}\cdot\varepsilon$ for some idempotent ε in $J(\sigma')$. Since $\varepsilon\cdot\varrho=\varrho$ by Lemma 2.9 we infer that $\sigma^*\cdot\varrho=\sigma^{**}\cdot\varepsilon\cdot\varrho=\sigma^{**}\cdot\varrho$.

Lemma 2.11. *If* $\varrho\sim\sigma$ *and* τ *is a common inverse of* ϱ *and* σ *then* $\varrho\cdot\tau\cdot\sigma=\sigma$.

Proof. Clearly, $\varrho\cdot\tau\in J(\varrho)=J(\sigma)$ is idempotent. Hence Lemma 2.9 immediately implies the required equality.

In [6] WARNE has introduced the concept of a semidirect product of a lower associative semilattice Y of left zero semigroups and an upper associative semilattice Y of right groups. We gave the definition before Theorem 2.1 for the special case of right zero semigroups instead of right groups. We generalize this concept by defining a semidirect product of a partial left band over $\bar{I}\otimes_Y Y$ and a right orthodox partial semigroup over $\bar{J}\otimes_Y Y$.

Suppose we are given a semilattice $\bar{Y}$, a lower associative semilattice $\bar{Y}$ of left zero semigroups $\bar{I}_{\bar{\alpha}}$ $(\bar{\alpha}\in\bar{Y})$ denoted by $\bar{I}$ and an upper associative semilattice $\bar{Y}$ of right zero semigroups $\bar{J}_{\bar{\alpha}}$ $(\bar{\alpha}\in\bar{Y})$ denoted by $\bar{J}$. Moreover, let $Y_{\bar{\alpha}}$ be a semilattice with identity $\bar{\alpha}$ for all $\bar{\alpha}$ in $\bar{Y}$. Suppose Y is a semilattice $\bar{Y}$ of semilattices $Y_{\bar{\alpha}}$ $(\bar{\alpha}\in\bar{Y})$ such that $\bar{Y}$ is a subsemilattice in Y. Let I be a partial left band over $\bar{I}\otimes_Y Y$ and J a right orthodox partial semigroup over $\bar{J}\otimes_Y Y$. Suppose that $\bar{A}, \bar{B}$ is an $(\bar{I}, \bar{J})$-pair.

Assume that $A=\{A_\sigma : \sigma\in J\}$ is a system of transformations of I and $B=\{B_a : a\in I\}$ is a system of transformations of J such that the following are valid:

(C1) if $a\in I_\alpha^{\mathrm{i}}$, $\varrho\in J_\beta^{\mathrm{j}}$ and $\varrho'\in J_{\beta'}$ then
- (a) $aA_\varrho\in I_{\alpha_1}^{\mathrm{i}A_{\mathrm{j}}}$, $\varrho B_a\in J_{\alpha_1}^{\mathrm{j}B_{\mathrm{i}}}$ and $(\varrho B_a)'\in J_{\alpha_1'}$ where $\alpha_1\leqq\beta$ and $\alpha_1'\leqq\alpha$,
- (b) $\alpha_1=\alpha_1'=\alpha\beta$ provided ϱ is idempotent,
- (c) $\varrho B_a\sim\varrho$ whenever $\alpha=\beta'$,
- (d) if $\alpha<\beta'$ then α_1 is the element of Y for which $(\varepsilon\cdot\varrho')'\in J_{\alpha_1}$ holds provided $\varepsilon\in J_\alpha$ is idempotent;

(C2) if $a\in I_\alpha$, $b\in I_\beta$ with $\alpha\geqq\beta$ and $\varrho\in J$ then
- (a) $\varrho B_{ab}=\varrho B_a B_b$,
- (b) $(a\cdot b)A_\varrho = aA_\varrho\cdot bA_{\varrho B_a}$;

(C3) if $\varrho\in J_\alpha$, $\sigma\in J_\beta$ with $\alpha\leqq\beta$ and $a\in I$ then

(a) $aA_{\varrho\cdot\sigma}=aA_\sigma A_\varrho$,

(b) $(\varrho\cdot\sigma)B_a=\varrho B_{aA_\sigma}\cdot\sigma B_a$.

The pair A, B with these properties is called an *(I, J)-pair over $\bar{A}, \bar{B}$.*

Note that $\alpha_1\in Y_{\bar{\alpha}\bar{\beta}}$ in (C1) (a) provided $\alpha\in Y_{\bar{\alpha}}$ and $\beta\in Y_{\bar{\beta}}$ as $\bar{I}\bar{A}_{\bar{J}}\in\bar{I}_{\bar{\alpha}\bar{\beta}}$ and $\bar{J}\bar{B}_{\bar{I}}\in\bar{J}_{\bar{\alpha}\bar{\beta}}$. The idempotent ε in (C1) (d) can be chosen arbitrarily in J_α for $J((\varepsilon_1\cdot\varrho')')=J((\varepsilon_2\cdot\varrho')')$ by (B6) provided ε_1 and ε_2 are idempotents in J_α. Moreover, one can easily check by (C1) (a) that the right hand sides of the equalities in (C2) (b) and (C3) (b) are defined in I and J, respectively.

Let us define a multiplication on the set $\bigcup\{I_\alpha\times J_\alpha:\alpha\in Y\}$ by

$$(a,\varrho)(b,\sigma)=(a\cdot bA_\varrho,\ \varrho B_b\cdot\sigma). \tag{2}$$

Suppose $a\in I_\alpha$, $\varrho\in J_\alpha$ and $b\in I_\beta$, $\sigma\in J_\beta$. Then, by (C1) (a), we have $bA_\varrho\in I_{\alpha_1}$, $\varrho B_b\in J_{\alpha_1}$ and $(\varrho B_b)'\in J_{\beta_1}$ where $\alpha_1\leqq\alpha$ and $\beta_1\leqq\beta$. Therefore the products $a\cdot bA_\varrho$ and $\varrho B_b\cdot\sigma$ are defined in I and J, respectively, and we have $a\cdot bA_\varrho\in I_{\alpha_1}$, $\varrho B_b\cdot\sigma\in J_{\alpha_1}$ by (B2). Thus (2) is, in fact, a multiplication on the required set. The groupoid thus defined is called a *semidirect product of I and J* and is denoted by $\mathfrak{B}(I, J;\ A, B)$.

Before proving that $\mathfrak{B}(I, J;\ A, B)$ is an orthodox semigroup we verify six lemmas for the (I, J)-pairs.

Lemma 2.12. *If $a\in I_\alpha$ and $\varrho\in J_\beta$, $\varrho'\in J_{\beta'}$ with $\beta'\leqq\alpha$ then $\varrho B_a\sim\varrho$.*

Proof. By property (C3) (b), we have

$$\varrho B_a=(\varrho\cdot(\varrho'\cdot\varrho))B_a=\varrho B_{aA_{\varrho'\cdot\varrho}}\cdot(\varrho'\cdot\varrho)B_a. \tag{3}$$

Since $\varrho'\cdot\varrho\in J_{\beta'}$ is idempotent we have $aA_{\varrho'\cdot\varrho}\in I_{\beta'}$, $(\varrho'\cdot\varrho)B_a\in J_{\beta'}$ and $((\varrho'\cdot\varrho)B_a)'\in J_{\beta'}$ by (C1) (b). Thus $\varrho B_{aA_{\varrho'\cdot\varrho}}\sim\varrho$ follows from (C1) (c). Moreover, owing to (B2), we have $a\cdot aA_{\varrho'\cdot\varrho}\in I_{\beta'}$. Hence (C1) (c) ensures both $(\varrho'\cdot\varrho)B_{a\cdot aA_{\varrho'\cdot\varrho}}\sim\varrho'\cdot\varrho$ and $(\varrho'\cdot\varrho)B_aB_{aA_{\varrho'\cdot\varrho}}\sim(\varrho'\cdot\varrho)B_a$. Making use of Lemma 2.8 (i) we obtain by (C2) (a) that $\varrho'\cdot\varrho\sim(\varrho'\cdot\varrho)B_a$. Then Lemma 2.5 implies $(\varrho'\cdot\varrho)B_a$ to be an idempotent in $J_{\beta'}$. We have seen that $\varrho B_{aA_{\varrho'\cdot\varrho}}\sim\varrho$. Applying Lemma 2.8 (ii) one can easily infer that (3) implies $\varrho B_a\sim\varrho$.

Lemma 2.13. *Let $a\in I_\alpha$ and $\varrho\in J_\beta$, $\varrho'\in J_{\beta'}$. Suppose $\varrho B_a\in J_{\alpha_1}$. If $\beta'\not\leqq\alpha$ then $\alpha_1<\beta$.*

Proof. In the equality (3) which clearly holds by (C3) (b) now we have $aA_{\varrho'\cdot\varrho}\in I_{\alpha\beta'}$, and $(\varrho'\cdot\varrho)B_a\in J_{\alpha\beta'}$ where $\alpha\beta'<\beta'$ as $\beta'\not\leqq\alpha$. By (B2), we obtain from (3) that $J_{\alpha_1}=J(\varrho B_a)=J(\varrho B_{aA_{\varrho'\cdot\varrho}})$. Hence, owing to property (C1) (d), we have

$J_{\alpha_1}=J((\varepsilon\cdot\varrho')')$ where $\varepsilon\in J_{\alpha\beta'}$ is idempotent. (B1), (B2) and Lemma 2.3 ensure that $\alpha_1<\beta$.

Lemma 2.14. *Let α_1,α_2,β and $\beta'\in Y$ such that $\alpha_1\leqq\alpha_2\leqq\beta'$. Assume that $\varrho\in J_\beta$ and $\varrho'\in J_{\beta'}$. Then*

(i) *$I(a_1A_\varrho)\leqq I(a_2A_\varrho)$ provided $a_1\in I_{\alpha_1}$ and $a_2\in I_{\alpha_2}$.*

(ii) *$J((\varepsilon_1\cdot\varrho')')\leqq J((\varepsilon_2\cdot\varrho')')$ provided ε_1 and ε_2 are idempotents belonging to J_{α_1} and J_{α_2}, respectively.*

Proof. The product $a_2\cdot a_1$ is defined in I by (B2)* and $a_2\cdot a_1\in I_{\alpha_1}$. Therefore properties (C1) (a), (c) and (d) imply $I((a_2\cdot a_1)A_\varrho)=I(a_1A_\varrho)$. However, we have $(a_2\cdot a_1)A_\varrho=a_2A_\varrho\cdot a_1A_{\varrho B_{a_2}}$ by (C2) (b) whence it follows by (B2)* and (C1) (a) that $I((a_2\cdot a_1)A_\varrho)=I(a_1A_{\varrho B_{a_2}})\leqq I(a_2A_\varrho)$. Thus (i) is verified. Taking into consideration (C1) (c) and (d) statement (ii) is an immediate consequence of (i).

Lemma 2.15. *Let $a\in I_\alpha$, $\varrho\in J_\beta$ and $\varrho'\in J_{\beta'}$ such that $\alpha\leqq\beta'$. If $\varrho B_a\in J_{\alpha_1}$ and ε is an idempotent in J_{α_1} then $\varepsilon\cdot\varrho\sim\varrho B_a$.*

Proof. If $\varrho B_a\in J_{\alpha_1}$ then $\alpha_1\leqq\beta$ and hence $\varepsilon\cdot\varrho$ is defined. Moreover, $aA_\varrho\in I_{\alpha_1}$. Thus $\varepsilon B_{aA_\varrho}\sim\varepsilon$ by (C1) (c). Hence we infer by Lemma 2.5 that $\varepsilon B_{aA_\varrho}$ is idempotent. Then Lemma 2.9 ensures that $\varepsilon B_{aA_\varrho}\cdot\varrho B_a=\varrho B_a$. Therefore, by (C3) (b), we have $(\varepsilon\cdot\varrho)B_a=\varrho B_a$. Properties (C1) (c) and (d) imply that $(\iota\cdot\varrho')'\in J_{\alpha_1}$ provided ι is an idempotent in J_α. By (B6), this implies $(\varepsilon\cdot\varrho)'\in J_\alpha$. Thus, in consequence of (C1) (c), we have $(\varepsilon\cdot\varrho)B_a\sim\varepsilon\cdot\varrho$ which ensures that $\varrho B_a\sim\varepsilon\cdot\varrho$.

Lemma 2.16. *Let ε and η be idempotents in J_α and J_β, respectively, where $\alpha\leqq\beta$. Moreover, let $a\in I_\alpha$. Then ηB_a and $\varepsilon\cdot\eta$ are also idempotents in J_α.*

Proof. If $\alpha=\beta$ then the statement immediately follows from (C1) (c), Lemma 2.5 and (B5). Assume that $\alpha<\beta$ and $(\varepsilon\cdot\eta)'\in J_\gamma$. Then, by (B2) and Lemma 2.3, we have $\gamma<\beta$ whence we can see by utilizing (B6) and (C1) (d) that $\eta B_a\in J_\alpha$ provided $a\in I_\gamma$. Thus Lemma 2.15 implies that $\varepsilon\cdot\eta\sim\eta B_a$. By (C3) (b), we obtain the equality $\eta B_a=(\eta\cdot\eta)B_a=\eta B_{aA_\eta}\cdot\eta B_a$ whence it follows on the one hand, that $\eta B_{aA_\eta}\in J_\alpha$ and therefore $\varepsilon\cdot\eta\sim\eta B_{aA_\eta}$ by Lemma 2.15. On the other hand, Lemma 2.9 ensures ηB_{aA_η} to be idempotent. Thus, since $\eta B_{aA_\eta}\sim\varepsilon\cdot\eta\sim\eta B_a$ we conclude by Lemmas 2.5 and 2.8 (i) that both $\varepsilon\cdot\eta$ and ηB_a are idempotent. Hence $\alpha=\gamma$ as $(\varepsilon\cdot\eta)'\in J_\alpha=J_\gamma$. The proof of the lemma is complete.

Lemma 2.17. *If ϱ_1,ϱ_2 and $\sigma_1,\sigma_2\in J$ with the property that $\varrho_1\sim\varrho_2$ and $\sigma_1\sim\sigma_2$ and, moreover, $\varrho_1\cdot\sigma_1$ is defined then $\varrho_2\cdot\sigma_2$ is also defined and $\varrho_1\cdot\sigma_1\sim\varrho_2\cdot\sigma_2$.*

Proof. One can see immediately by (B2) and Lemma 2.8 (ii) that $\varrho_1\cdot\sigma_1$ is defined if and only if $\varrho_2\cdot\sigma_2$ is defined. Suppose $\varrho_1',\varrho_2'\in J_{\alpha'}$ and $\sigma_1',\sigma_2'\in J_{\beta'}$. Then, again by

Lemma 2.8 (ii), there exist idempotents ε and η in $J_{\alpha'}$ and $J_{\beta'}$, respectively, such that $\varrho_2=\varrho_1\cdot\varepsilon$ and $\sigma_2=\sigma_1\cdot\eta$. Hence $\varrho_2\cdot\sigma_2=(\varrho_1\cdot\varepsilon)\cdot(\sigma_1\cdot\eta)=\varrho_1\cdot(\varepsilon\cdot\sigma_1)\cdot\eta$ by (B3). Assume that $(\varepsilon\cdot\sigma_1)'\in J_\gamma$. Then, by (B2), $\gamma\leqq\beta'$ and we can see by (B6) and (C1) (c) or (d) that $\sigma_1B_a\in J_{\alpha'}$ provided $a\in I_\gamma$. Thus Lemma 2.15 implies $\sigma_1B_a\sim\varepsilon\cdot\sigma_1\sim\sim(\varrho_1'\cdot\varrho_1)\cdot\sigma_1$. Hence we obtain by Lemma 2.8 (ii) that $\varepsilon\cdot\sigma_1=(\varrho_1'\cdot\varrho_1)\cdot\sigma_1\cdot\bar{\eta}$ for some idempotent element $\bar{\eta}$ in $J((\varepsilon\cdot\sigma_1)')$. Therefore $\varrho_2\cdot\sigma_2=\varrho_1\cdot(\varrho_1'\cdot\varrho_1)\cdot\sigma_1\cdot\bar{\eta}\cdot\eta==\varrho_1\cdot\sigma_1\cdot(\bar{\eta}\cdot\eta)$. Here $J((\varrho_1\cdot\sigma_1)')=J(((\varrho_1'\cdot\varrho_1)\cdot\sigma_1)')$. is implied by Lemma 2.2 whence it follows that $J((\varrho_1\cdot\sigma_1)')=J((\varepsilon\cdot\sigma_1)')=J(\bar{\eta})$. Since $\bar{\eta}\cdot\eta$ is an idempotent in $J(\bar{\eta})$ by Lemma 2.16 we conclude by Lemma 2.8 (ii) that $\varrho_2\cdot\sigma_2\sim\varrho_1\cdot\sigma_1$ which was to be proved.

Now we can turn to verifying that $\mathfrak{B}(I, J;\ A, B)$ is an orthodox semigroup which is a band $\mathscr{B}(\bar{I}, \bar{J};\ \bar{A}, \bar{B})$ of orthodox semigroups.

Lemma 2.18. *$\mathfrak{B}(I, J;\ A, B)$ is a semigroup.*

Proof. A straightforward calculation shows that the operation defined in (2) is associative. We have to apply properties (C3) (a)—(b), (C1) (a), (B3)*, (B3) and (C2) (a)—(b).

Lemma 2.19. *In the semigroup $\mathfrak{B}(I, J;\ A, B)$ the elements (a, ϱ) and (b, σ) are inverses of each other if and only if ϱ and σ are inverses of each other in J.*

Proof. By definition, $(a,\varrho)(b,\sigma)(a,\varrho)=(a,\varrho)$ and $(b,\sigma)(a,\varrho)(b,\sigma)=(b,\sigma)$ hold if and only if the following four equalities are satisfied in I and J.

(4) $$a\cdot bA_\varrho\cdot aA_\sigma A_{\varrho B_b}=a,$$

(4)′ $$b\cdot aA_\sigma\cdot bA_\varrho A_{\sigma B_a}=b,$$

(5) $$\varrho B_{b\cdot aA_\sigma}\cdot\sigma B_a\cdot\varrho=\varrho,$$

(5)′ $$\sigma B_{a\cdot bA_\varrho}\cdot\varrho B_b\cdot\sigma=\sigma.$$

Suppose first that (a,ϱ) and (b,σ) satisfy the equalities (4), (4)′, (5) and (5)′. By (C1) (a), we have $I(a)\geqq I(bA_\varrho)\geqq I(aA_\sigma A_{\varrho B_b})$. The equality (4) implies by (B2)* that $I(aA_\sigma A_{\varrho B_b})=I(a)$. Hence $I(a)=I(bA_\varrho)=I(aA_\sigma A_{\varrho B_b})$. Similarly, by (4)′, we have $I(b)=I(aA_\sigma)=I(bA_\varrho A_{\sigma B_a})$. Since $I(a)$ and $I(b)$ are left zero semigroups the equalities $a\cdot bA_\varrho=a$ and $b\cdot aA_\sigma=b$ are valid. In the equality (5) we have $J(\varrho)==J(\varrho B_b)=J(\xi)$ where $\xi=\varrho B_b\cdot\sigma B_a$. On the one hand, this implies by Lemma 2.13 that $J(\varrho')\leqq J(\sigma)$ and hence, by Lemma 2.12, we conclude that $\varrho B_b\sim\varrho$. On the other hand, applying Lemma 2.9 we obtain that ξ is an idempotent element in $J(\varrho)$ and therefore $\varrho B_b=\xi\cdot\varrho B_b=\varrho B_b\cdot\sigma B_a\cdot\varrho B_b$ since $J(\varrho B_b)=J(\varrho)$ by Lemma 2.8 (ii). In the same way, we can deduce from (5)′ that $\sigma B_a\sim\sigma$ and $\sigma B_a=\sigma B_a\cdot\varrho B_b\cdot\sigma B_a$.

Thus the elements ϱB_b and σB_a are inverses of each other in J. Consequently, Lemma 2.8 (iii) ensures ϱ and σ to be inverses of each other in J which was to be proved.

Conversely, assume that $a \in I_\alpha$, $\varrho \in J_\alpha$, $b \in I_{\alpha'}$, $\sigma \in J_{\alpha'}$ and ϱ and σ are inverses of each other in J. Then we have $\varrho B_b \sim \varrho$ and $\sigma B_a \sim \sigma$ by (C1) (c). This implies by (C1) (a) that $bA_\varrho \in I_\alpha$ and $aA_\sigma \in I_{\alpha'}$ whence, in the same way, we obtain that $aA_\sigma A_{\varrho B_b} \in I_\alpha$ and $bA_\varrho A_{\sigma B_a} \in I_{\alpha'}$. Since both I_α and $I_{\alpha'}$ are left zero semigroups the equalities (4) and (4)′ follow. Moreover, we have $b \cdot aA_\sigma = b$ and $a \cdot bA_\varrho = a$ in (5) and (5)′, respectively. Then the equalities (5) and (5)′ are implied by the relations $\varrho B_b \sim \varrho$ and $\sigma B_a \sim \sigma$ by making use of Lemma 2.11. Thus (a, ϱ) and (b, σ) are inverses of each other.

Lemma 2.20. *$\mathfrak{B}(I, J;\ A, B)$ is an orthodox semigroup with band of idempotents*

(6) $\mathbf{B} = \{(a, \varepsilon)\colon\ a \in I_\alpha$ *and* ε *is an idempotent in* J_α *for some* $\alpha \in Y\}$.

Proof. By (B4), every element in J has an inverse which implies by Lemma 2.19 that $\mathfrak{B}(I, J;\ A, B)$ is regular. We show first that the set $\mathbf{B}$ defined in (6) is the set of all idempotents in $\mathfrak{B}(I, J;\ A, B)$. Suppose that (a, ε) is an idempotent element. Then, by definition, we have

(7) $$a \cdot aA_\varepsilon = a$$

and

(8) $$\varepsilon B_a \cdot \varepsilon = \varepsilon.$$

The equality (7) ensures by (B2)* that $I(aA_\varepsilon) = I(a)$. Then $J(\varepsilon B_a) = J(\varepsilon)$. On the other hand, we have $J((\varepsilon B_a)') \leqq J(\varepsilon)$ by (C1) (a). Thus Lemma 2.9 implies by (8) that εB_a is an idempotent in $J(\varepsilon)$. From $J(\varepsilon B_a) = J(\varepsilon)$ we infer by Lemma 2.13 that $J(\varepsilon') \leqq J(\varepsilon)$ which implies by Lemma 2.12 that $\varepsilon B_a \sim \varepsilon$. Hence we obtain by making use of Lemma 2.5 that ε is idempotent. Conversely, if $a \in I_\alpha$ and ε is an idempotent in J_α then we have $aA_\varepsilon \in I_\alpha$ by (C1) (b) and $\varepsilon B_a \sim \varepsilon$ by (C1) (c). The former relation implies (7) as I_α is a left zero semigroup while the latter one, taking into consideration Lemma 2.5, ensures that εB_a is an idempotent in J_α. Hence (8) follows by (B5). Thus we have verified that (a, ε) is an idempotent element in $\mathfrak{B}(I, J;\ A, B)$. Owing to Lemma 2.5, the inverses of the idempotents in J are idempotent. Therefore we obtain by Lemma 2.19 that the inverses of the elements in $\mathbf{B}$ are contained in $\mathbf{B}$. This completes the proof of the fact that $\mathfrak{B}(I, J;\ A, B)$ is orthodox.

Lemma 2.21. *The band of idempotents* $\mathbf{B}$ *of* $\mathfrak{B}(I, J;\ A, B)$ *is a semilattice* Y *of rectangular bands*

$$\mathbf{D}_\alpha = \{(a, \varepsilon)\colon\ a \in I_\alpha, \varepsilon \text{ is an idempotent in } J_\alpha\} \quad (\alpha \in Y).$$

Proof. By applying Lemma 2.19 we see that the set of all inverses of an element (a, ε) in $\mathbf{B}$ with $a \in I_\alpha$ and $\varepsilon \in J_\alpha$ is just $\mathbf{D}_\alpha$, that is, the $\mathscr{D}$-classes in $\mathbf{B}$ are the sets $\mathbf{D}_\alpha(\alpha\in Y)$. If $(a,\varepsilon)\in\mathbf{D}_\alpha$ and $(b,\zeta)\in\mathbf{D}_\beta$ then $(a,\varepsilon)(b,\zeta)=(a\cdot bA_\varepsilon, \varepsilon B_b\cdot\zeta)\in\mathbf{D}_{\alpha\beta}$. For we have $bA_\varepsilon\in I_{\alpha\beta}$ and $\varepsilon B_b\in J_{\alpha\beta}$ by (C1) (b). This implies $a\cdot bA_\varepsilon\in I_{\alpha\beta}$ and $\varepsilon B_b\cdot\zeta\in J_{\alpha\beta}$ by (B2)* and (B2), respectively. Thus we have $\mathbf{D}_\alpha\mathbf{D}_\beta\subseteq\mathbf{D}_{\alpha\beta}$ which was to be proved.

Consider the following subset in the band $\mathbf{B}$:

(9) $$\bar{\mathbf{B}}=\{(a,\varepsilon)\colon a\in I_{\bar\alpha} \text{ and } \varepsilon \text{ is an idempotent in } J_{\bar\alpha} \text{ for some } \bar\alpha\in\bar Y\}.$$

Since $\bar Y$ is a subsemilattice in Y Lemma 2.21 implies $\bar{\mathbf{B}}$ to be a subband in $\mathbf{B}$ with the property that $\bar{\mathbf{B}}$ is a union of some $\mathcal{D}$-classes of $\mathbf{B}$. For every element $(\bar\imath,\bar\jmath)$ in the band $\mathscr{B}(\bar I,\bar J;\bar A,\bar B)$, let us define a subset in $\mathfrak{B}(I,J;A,B)$ as follows: if $\bar\imath\in\bar I_{\bar\alpha}$, $\bar\jmath\in\bar J_{\bar\alpha}$ with $\bar\alpha$ in $\bar Y$ then put

(10) $$\mathbf{F}_{(\bar\imath,\bar\jmath)}=\{(a,\varrho)\colon a\in I_\alpha^{\bar\imath},\ \varrho\in J_\alpha^{\bar\jmath} \text{ for some } \alpha \text{ in } Y_{\bar\alpha}\}.$$

Lemma 2.22. *The semigroup $\mathfrak{B}(I,J;A,B)$ is a band $\mathscr{B}(\bar I,\bar J;\bar A,\bar B)$ of the orthodox semigroups $\mathbf{F}_{(\bar\imath,\bar\jmath)}((\bar\imath,\bar\jmath)\in\mathscr{B}(\bar I,\bar J;\bar A,\bar B))$. For every $(\bar\imath,\bar\jmath)$ in $\mathscr{B}(\bar I,\bar J;\bar A,\bar B)$, the greatest $\mathcal{D}$-class of idempotents in $\mathbf{F}_{(\bar\imath,\bar\jmath)}$ is $\mathbf{F}_{(\bar\imath,\bar\jmath)}\cap\bar{\mathbf{B}}$.*

Proof. Let $(a,\varrho)\in\mathbf{F}_{(\bar\imath,\bar\jmath)}$ and $(b,\sigma)\in\mathbf{F}_{(\bar k,\bar l)}$. By definition, we have $(a,\varrho)(b,\sigma)=(a\cdot bA_\varrho, \varrho B_b\cdot\sigma)$ where $bA_\varrho\in I_{\alpha_1}^{\bar k\bar A_{\bar\jmath}}$ and $\varrho B_b\in J_{\alpha_1}^{\bar\jmath\bar B_{\bar k}}$ by (C1) (a). Thus (B2)* and (B2) imply $a\cdot bA_\varrho\in I_{\alpha_1}^{\bar\imath\cdot\bar k\bar A_{\bar\jmath}}$ and $\varrho B_b\cdot\sigma\in J_{\alpha_1}^{\bar\jmath\bar B_{\bar k}\cdot\bar l}$, respectively, whence we infer $(a,\varrho)(b,\sigma)\in\mathbf{F}_{(\bar\imath,\bar\jmath)(\bar k,\bar l)}$. This shows that the equivalence relation on $\mathfrak{B}(I,J;A,B)$ defined by $(a,\varrho)\varkappa(b,\sigma)$ if and only if (a,ϱ), $(b,\sigma)\in\mathbf{F}_{(\bar\imath,\bar\jmath)}$ for some $(\bar\imath,\bar\jmath)$ in $\mathscr{B}(\bar I,\bar J;\bar A,\bar B)$ is compatible. The second assertion of the lemma immediately follows from Lemma 2.21. Thus the congruence relation $\varkappa$ is subband-parcelling by Proposition 1.1 whence we conclude by Theorem 1.2 that $\mathbf{F}_{(\bar\imath,\bar\jmath)}$ is an orthodox semigroup for every $(\bar\imath,\bar\jmath)$ in $\mathscr{B}(\bar I,\bar J;\bar A,\bar B)$. The proof is complete.

The following theorem sums up the most important properties of a semidirect product of a partial left band and a right orthodox partial semigroup.

Theorem 2.23. *Let $\bar Y$ be a semilattice, $\bar I$ a lower associative semilattice $\bar Y$ of left zero semigroups $\bar I_{\bar\alpha}$ $(\bar\alpha\in\bar Y)$ and $\bar J$ an upper associative semilattice $\bar Y$ of right zero semigroups $\bar J_{\bar\alpha}$ $(\bar\alpha\in\bar Y)$. For every $\bar\alpha$ in $\bar Y$, consider a semilattice $Y_{\bar\alpha}$ with identity $\bar\alpha$. Let Y be a semilattice $\bar Y$ of semilattices $Y_{\bar\alpha}$ $(\bar\alpha\in\bar Y)$ such that $\bar Y$ is a subsemilattice in Y. Let I be a partial left band over $\bar I\otimes_{\bar Y}Y$ and J a right orthodox partial semigroup over $\bar J\otimes_{\bar Y}Y$. Suppose $\bar A,\bar B$ is an $(\bar I,\bar J)$-pair and A,B is an (I,J)-pair over $\bar A,\bar B$. Then the semidirect product $\mathfrak{B}(I,J;A,B)$ of I and J is an orthodox semigroup with band of idempotents $\mathbf{B}$ defined in* (6). *The subset $\bar{\mathbf{B}}$ in $\mathbf{B}$ defined in* (9) *is a subband in $\mathbf{B}$. Moreover, $\mathfrak{B}(I,J;A,B)$ is a band $\mathscr{B}(\bar I,\bar J;\bar A,\bar B)$ of the orthodox semigroups $\mathbf{F}_{(\bar\imath,\bar\jmath)}$ $((\bar\imath,\bar\jmath)\in\mathscr{B}(\bar I,\bar J;\bar A,\bar B))$ defined in* (10) *where the greatest $\mathcal{D}$-class of idempotents in $\mathbf{F}_{(\bar\imath,\bar\jmath)}$ is $\mathbf{F}_{(\bar\imath,\bar\jmath)}\cap\bar{\mathbf{B}}$.*

3. The construction

In this section we introduce the construction which will be applied in the next section to describe the strong subband-parcelling extensions of orthodox semigroups.

Let $\bar{Y}, \bar{I}, \bar{J}, \bar{A}, \bar{B}, Y, I, J, A$ and B have the properties required in Theorem 2.23. Suppose that

(C4) for every α in Y, idempotents i_α and j_α in I_α and J_α, respectively, are distinguished such that $aA_{j_\beta}=i_\beta\cdot a$ and $\sigma B_{i_\beta}=\sigma\cdot j_\beta$ provided $\alpha, \beta\in Y$ with $\alpha\leqq\beta$ and $a\in I_\alpha, \sigma'\in J_\alpha$.

If $\bar{\alpha}\in\bar{Y}$ then denote by $\bar{i}_{\bar{\alpha}}$ the element $\bar{i}$ in $\bar{I}$ for which $i_{\bar{\alpha}}\in I_{\bar{\alpha}}^{\bar{i}}$ holds. Similarly, by $\bar{j}_{\bar{\alpha}}$ we mean the element $\bar{j}$ in $\bar{J}$ with the property that $j_{\bar{\alpha}}\in J_{\bar{\alpha}}^{\bar{j}}$. By (C1) (a), it follows from (C4) that $\bar{i}\bar{A}_{\bar{j}_{\bar{\beta}}}=\bar{i}_{\bar{\beta}}\cdot\bar{i}$ and $\bar{j}\bar{B}_{\bar{i}_{\bar{\beta}}}=\bar{j}\cdot\bar{j}_{\bar{\beta}}$ provided $\bar{\alpha}, \bar{\beta}\in\bar{Y}$ with $\bar{\alpha}\leqq\bar{\beta}$ and $\bar{i}\in\bar{I}_{\bar{\alpha}}, \bar{j}\in\bar{J}_{\bar{\alpha}}$.

Let S be an orthodox semigroup with band of idempotents $E=\mathscr{B}(\bar{I}, \bar{J}; \bar{A}, \bar{B})$. The band E is a semilattice $\bar{Y}$ of the rectangular bands $E_{\bar{\alpha}}=\bar{I}_{\bar{\alpha}}\times\bar{J}_{\bar{\alpha}}$ $(\bar{\alpha}\in\bar{Y})$. For every s in S we denote by $r(s)$ $[l(s)]$ the element $\bar{\alpha}$ in $\bar{Y}$ which has the property that $E_{\bar{\alpha}}\ni ss^*$ $[E_{\bar{\alpha}}\ni s^*s]$ for some inverse s^* of s. If $s\in S$ then there exists a unique inverse s' of S such that $(\bar{i}_{r(s)}, \bar{j}_{r(s)})\mathscr{L}s'\mathscr{R}(\bar{i}_{l(s)}, \bar{j}_{l(s)})$.

For every element s in S, let τ_s be an isomorphism of $r(s)Y$ onto $l(s)Y$. Suppose that $\tau_{s^*}=\tau_s^{-1}$ provided s^* is an inverse of s and $s^*E_{\bar{\alpha}}s\subseteq E_{\bar{\alpha}\tau_s}$ whenever s^* is an inverse of s and $\bar{\alpha}\in\bar{Y}$ with $\bar{\alpha}\leqq r(s)$. Since Y is a semilattice $\bar{Y}$ of the semilattices $Y_{\bar{\alpha}}$ $(\bar{\alpha}\in\bar{Y})$ with identity $\bar{\alpha}$ such that $\bar{Y}$ is a subsemilattice in Y it is not difficult to verify that $Y_{\bar{\alpha}}\tau_s\subseteq Y_{\bar{\alpha}\tau_s}$ for every $\bar{\alpha}\in\bar{Y}$ with $\bar{\alpha}\leqq r(s)$.

Let us be given mappings $h_s\colon \bigcup\{I_\alpha\colon \alpha\leqq l(s)\}\to\bigcup\{I_\alpha\colon \alpha\leqq r(s)\}$ and $\chi_s\colon \bigcup\{J_\alpha\colon \alpha\leqq r(s)\}\to\bigcup\{J_\alpha\colon \alpha\leqq l(s)\}$ for each s in S and constants $\gamma_{s,\bar{s}}$ in $J_{l(s\bar{s})}$ for each pair of elements $s, \bar{s}$ in S such that the following conditions are satisfied:

(D1) (a) if $a\in I_\alpha^{\bar{i}}$ with $\bar{i}\in\bar{I}_{\bar{\alpha}}$ and $\alpha\leqq l(s)$ then $ah_s\in I_{\alpha\tau_s^{-1}}^{\bar{k}}$ where $s(\bar{i}, \bar{j}_{\bar{\alpha}})(s(\bar{i}, \bar{j}_{\bar{\alpha}}))'= =(\bar{k}, \bar{j}_{\bar{\alpha}\tau_s^{-1}})$,

(b) if $\sigma\in J_\alpha^{\bar{j}}$ with $\bar{j}\in\bar{J}_{\bar{\alpha}}$ and $\alpha\leqq r(s)$, $\sigma'\in J_{\alpha'}$ then $\sigma\chi_s\in J_{\alpha\tau_s}^{\bar{l}}$ with $((\bar{i}_{\bar{\alpha}}, \bar{j})s)'(\bar{i}_{\bar{\alpha}}, \bar{j})s=(\bar{i}_{\bar{\alpha}\tau_s}, \bar{l})$ and $(\sigma\chi_s)'\in J_{\alpha'\tau_s}$;

(D2) (a) if $a\in I_\alpha$, $b\in I_\beta$ with $l(s)\geqq\alpha\geqq\beta$ then $ah_s\cdot bh_s=(a\cdot bA_{j_{\alpha\tau_s^{-1}}\chi_s})h_s$,

(b) if $\varrho\in J_\alpha$, $\varrho'\in J_{\alpha'}$ and $\sigma\in J_\beta$ with $\alpha'\leqq\beta\leqq r(s)$ then $\varrho\chi_s\cdot\sigma\chi_s=(\varrho B_{i_{\beta\tau_s}h_s}\cdot\sigma)\chi_s$;

(D3) if $a\in I_\alpha$ with $\alpha\leqq l(s)$ and $\sigma\in J_\beta$ with $\beta\leqq r(s)$ then

(a) $aA_{\sigma\chi_s}h_s=i_{\beta\tau_s}h_s\cdot ah_sA_\sigma$,

(b) $\sigma B_{ah_s}\chi_s=\sigma\chi_sB_a\cdot j_{\alpha\tau_s^{-1}}\chi_s$;

(D4) (a) if $a\in I_\alpha$ with $\alpha\leqq l(s\bar{s})$ then $ah_{\bar{s}}h_s=c\cdot aA_{\gamma_{s,\bar{s}}}h_{s\bar{s}}$ for some c in $I_{r(s\bar{s})}$,

(b) if $\varrho\in J_\alpha$ with $\alpha\leqq r(s\bar{s})$ then $(j_{\alpha\tau_{s\bar{s}}}\cdot\gamma_{s,\bar{s}})\cdot\varrho\chi_s\chi_{\bar{s}}=\varrho B_c\chi_{s\bar{s}}\cdot\gamma_{s,\bar{s}}$ for some c in $I_{r(s\bar{s})}$;

(D5) if $\alpha \leqq l(s\bar{s})$ then $(j_\alpha \cdot \gamma_{s,\bar{s}})' \in J_{\alpha\tau_{s\bar{s}}^{-1}\tau_s\tau_{\bar{s}}}$;

(D6) if $s, \bar{s}, \bar{\bar{s}} \in S$ then $\gamma_{s,\bar{s}\bar{\bar{s}}} \cdot \gamma_{\bar{s},\bar{\bar{s}}} \sim \gamma_{s\bar{s},\bar{\bar{s}}} \cdot (j_{r(s\bar{s}\bar{\bar{s}})\tau_{s\bar{s}}} \cdot \gamma_{s,\bar{s}})\chi_{\bar{\bar{s}}}$;

(D7) (a) if $e \in E$, $\alpha \leqq r(e) = l(e)$ and $a \in I_\alpha$ then $ah_e = c \cdot aA_{\gamma_{e,e}}$ for some c in $I_{r(e)}$,
(b) if $e \in E$, $\alpha \leqq r(e) = l(e)$ and $\varrho \in J_\alpha$ then $\varrho\chi_e = \varrho B_c \cdot \gamma_{e,e}$ for some c in $I_{r(e)}$;

(D8) γ_{s,s^*} is idempotent whenever $s \in S$ is idempotent or s is not an inverse of itself, and s^* is an inverse of s in S.

If h, χ and γ fulfil these conditions then we call them an *(S, I, J)-triple*.

Note that $\bar{\alpha}\tau_s^{-1}$ and $\bar{\alpha}\tau_s$ are defined in (D1) (a) and (b), respectively, as $\alpha \in Y_{\bar{\alpha}}$, $\alpha \leqq l(s)$ imply $\bar{\alpha} \leqq l(s)$ and, similarly, $\alpha \in Y_{\bar{\alpha}}$, $\alpha \leqq r(s)$ imply $\bar{\alpha} \leqq r(s)$. It is not difficult to check that it follows from conditions (C1) (a) and (D1) that both sides of the equalities in (D2), (D3), (D4) and (D7) are defined. Similarly, (D5) ensures that both sides of (D6) are also defined.

Before introducing the construction by means of which we describe the strong subband-parcelling extensions of orthodox semigroups we prove some lemmas concerning (S, I, J)-triples which make the computations easier.

Lemma 3.1. *If ε is an idempotent in J_α and $s \in S$ with $\alpha \leqq r(s)$ then $\varepsilon\chi_s$ is idempotent.*

Proof. By definition, $i_{\alpha\tau_s}h_s \in I_\alpha$ and hence, by (C1) (c), we have $\varepsilon B_{i_{\alpha\tau_s}h_s} \sim \varepsilon$. Therefore Lemma 2.5 ensures $\varepsilon B_{i_{\alpha\tau_s}h_s}$ to be also an idempotent in J_α, that is, $\varepsilon B_{i_{\alpha\tau_s}h_s} \cdot \varepsilon = \varepsilon$ by (B5). Consequently, (D2) (b) implies that $\varepsilon\chi_s \cdot \varepsilon\chi_s = \varepsilon\chi_s$ which was to be proved.

Lemma 3.2. *If $\varrho \in J_\alpha$ and ϱ^* is an inverse of ϱ contained in $J_{\alpha'}$ and, moreover, $s \in S$ with $\alpha \leqq r(s)$ then $\alpha' \leqq r(s)$ and $\varrho\chi_s$ and $\varrho^*\chi_s$ are inverses of each other in J.*

Proof. If $\alpha \in Y_{\bar{\alpha}}$ then $\alpha' \in Y_{\bar{\alpha}}$ follows from Lemma 2.4 and (B1). Since $r(s) \in \bar{Y}$ the relation $\alpha \leqq r(s)$ implies $\bar{\alpha} \leqq r(s)$. Thus $\alpha' \leqq \bar{\alpha} \leqq r(s)$ whence we obtain that both $\varrho\chi_s$ and $\varrho^*\chi_s$ are defined. By definition, $i_{\alpha\tau_s}h_s \in I_\alpha$ and $i_{\alpha'\tau_s}h_s \in I_{\alpha'}$. Moreover, we have $i_{\alpha\tau_s}h_s A_{\varrho^*} \in I_{\alpha'}$ by (C1) (a), (c) and Lemma 2.8 (ii). Since $I_{\alpha'}$ is a left zero semigroup the equality $i_{\alpha'\tau_s}h_s \cdot i_{\alpha\tau_s}h_s A_{\varrho^*} = i_{\alpha'\tau_s}h_s$ holds. On the other hand, we have $\varrho B_{i_{\alpha'\tau_s}h_s} \sim \varrho$ and $\varrho^* B_{i_{\alpha\tau_s}h_s} \sim \varrho^*$ by (C1) (c). Therefore we can see by applying the equality (D2) (b) twice and making use of properties (C3) (b), (C2) (a) and Lemma 2.11 that

$$\varrho\chi_s \cdot \varrho^*\chi_s \cdot \varrho\chi_s = (\varrho B_{i_{\alpha'\tau_s}h_s} \cdot \varrho^*)\chi_s \cdot \varrho\chi_s = ((\varrho B_{i_{\alpha'\tau_s}h_s} \cdot \varrho^*) B_{i_{\alpha\tau_s}h_s} \cdot \varrho)\chi_s =$$

$$= (\varrho B_{i_{\alpha'\tau_s}h_s \cdot i_{\alpha\tau_s}h_s A_{\varrho^*}} \cdot \varrho^* B_{i_{\alpha\tau_s}h_s} \cdot \varrho)\chi_s = (\varrho B_{i_{\alpha'\tau_s}h_s} \cdot \varrho^* B_{i_{\alpha\tau_s}h_s} \cdot \varrho)\chi_s = \varrho\chi_s.$$

Dually, one obtains $\varrho^*\chi_s \cdot \varrho\chi_s \cdot \varrho^*\chi_s = \varrho^*\chi_s$ which completes the proof.

Lemma 3.3. *If* $s, \bar{s}\in S$ *then* $\gamma'_{s,\bar{s}}\in J_{l(s\bar{s})}$.

Proof. By definition, $\gamma_{s,\bar{s}}\in J_{l(s\bar{s})}$ and therefore $j_{l(s\bar{s})}\cdot\gamma_{s,\bar{s}}=\gamma_{s,\bar{s}}$ by Lemma 2.9. Thus we have $\gamma'_{s,\bar{s}}\in J_{l(s,\bar{s})}$ by (D5) as $l(s\bar{s})\tau_{s\bar{s}}^{-1}\tau_s\tau_{\bar{s}}=l(s\bar{s})$.

Lemma 3.4. *Let* $a\in I_\alpha$, $\sigma\in J_\alpha$ *and* $s\in S$ *with* $\alpha\leqq l(s)$. *Then we have* $ah_s\cdot bA_\sigma h_s= =(a\cdot bA_\sigma)h_s$ *for every* b *in* I.

Proof. By (C1) (a), we have $bA_\sigma\in I_{\alpha_1}$ where $\alpha_1\leqq\alpha$. Thus both sides are defined. Taking into consideration (D2) (a), it suffices to verify that $bA_\sigma=bA_\sigma A_{j_{\alpha\tau_s^{-1}}\chi_s}$. In consequence of Lemma 3.1, $j_{\alpha\tau_s^{-1}}\chi_s$ is an idempotent in J_α. Hence $j_{\alpha\tau_s^{-1}}\chi_s\cdot\sigma=\sigma$ by Lemma 2.9 and therefore (C3) (a) implies the equality required.

The following lemma is dual to Lemma 3.4.

Lemma 3.5. *Let* $a\in I_\alpha$, $\sigma\in J_\alpha$ *and* $s\in S$ *with* $\alpha\leqq r(s)$. *Then we have* $\varrho B_a\chi_s\cdot\sigma\chi_s= =(\varrho B_a\cdot\sigma)\chi_s$ *for every* ϱ *in* J.

Proof. Suppose that $\alpha\in Y_{\bar{\alpha}}$ and $\varrho\in J_\beta$ with $\beta\in Y_{\bar{\beta}}$. By (C1) (a), we have $\varrho B_a\in J_{\alpha_1}$ and $(\varrho B_a)'\in J_{\alpha_1'}$ where $\alpha_1\leqq\beta$ and $\alpha_1'\leqq\alpha$. The remark after the definition of an (I, J)-pair ensures that $\alpha_1, \alpha_1'\in Y_{\bar{\alpha}\bar{\beta}}$. Since $\bar{\alpha}\leqq r(s)$ follows from $\alpha\leqq r(s)$ we have $\alpha_1, \alpha_1'\leqq r(s)$. Thus both sides of the equality are defined. It suffices to prove by (D2) (b) that $\varrho B_a B_{i_{\alpha\tau_s}h_s}=\varrho B_a$. Here $i_{\alpha\tau_s}h_s\in I_\alpha$. Since I_α is a left zero semigroup we have $a\cdot i_{\alpha\tau_s}h_s=a$ whence the equality to be proved follows immediately by making use of (C2) (a).

Lemma 3.6. *If* s *and* s^* *are inverses of each other in* S *then* $\gamma_{ss^*,s}$ *and* γ_{s,s^*s} *are idempotent elements in* $J_{l(s)}$.

Proof. By definition and Lemma 3.3, we obtain that both $\gamma_{ss^*,s}$ and $\gamma'_{ss^*,s}$ belong to $J_{l(s)}$. Moreover, by (D6), we have

$$\gamma_{ss^*,s}\cdot(j_{r(s)}\cdot\gamma_{ss^*,ss^*})\chi_s\sim\gamma_{ss^*,s}\cdot\gamma_{ss^*,s}.$$

Here $j_{r(s)}\cdot\gamma_{ss^*,ss^*}=\gamma_{ss^*,ss^*}$ is an idempotent in $J_{r(s)}$ by (D8) and hence, by Lemma 3.1, $(j_{r(s)}\cdot\gamma_{ss^*,ss^*})\chi_s\in J_{l(s)}$ is also idempotent. Thus Lemma 2.8 (ii) implies that $\gamma_{ss^*,s}\sim \sim\gamma_{ss^*,s}\cdot\gamma_{ss^*,s}$, that is, $\gamma_{ss^*,s}\cdot\gamma_{ss^*,s}=\gamma_{ss^*,s}\cdot\varepsilon$ for some idempotent ε in $J_{l(s)}$. Multiplying this equality on the right by the idempotent element $\gamma'_{ss^*,s}\cdot\gamma_{ss^*,s}$ in $J_{l(s)}$ and applying (B5) we obtain that $\gamma_{ss^*,s}\cdot\gamma_{ss^*,s}=\gamma_{ss^*,s}$, that is, $\gamma_{ss^*,s}$ is, indeed, idempotent. A similar argument shows that γ_{s,s^*s} is also idemdontent.

Let us define a groupoid $\mathbf{S}=\mathfrak{S}(S, I, J; h, \chi, \gamma)$ in the following way. The underlying set of $\mathbf{S}$ is

$$\mathbf{S}=\{(a, s, \sigma)\colon s\in S,\quad a\in I_\alpha^1 \text{ and } \sigma\in J_{\alpha\tau_s}^1 \text{ where } \alpha\in Y_{r(s)},\quad ss'=(\mathbf{i}, \mathbf{j}_{r(s)}) \text{ and } s's=(\mathbf{i}_{l(s)}, \mathbf{j})\}$$

and the operation is defined by

$$(11)\qquad (a, s, \sigma)(\bar{a}, \bar{s}, \bar{\sigma}) = \left(a \cdot \bar{a}A_\sigma h_s, s\bar{s}, (j_{\beta\tau_{s\bar{s}}} \cdot \gamma_{s,\bar{s}}) \cdot \sigma B_{\bar{a}}\chi_{\bar{s}} \cdot \bar{\sigma}\right)$$

where β is the element of Y with the property that $\bar{a}A_\sigma h_s \in I_\beta$.

We have to show that the products occurring in (11) are defined in I and J and, moreover, that the set S is closed under the multiplication defined in (11). Suppose that $a\in I^i_\alpha$, $\bar{a}\in I^{\bar{i}}_{\bar{\alpha}}$, $\sigma\in J^j_{\alpha\tau_s}$ and $\bar{\sigma}\in J^{\bar{j}}_{\bar{\alpha}\tau_{\bar{s}}}$ where $ss'=(i, \bar{j}_{r(s)})$, $\bar{s}\bar{s}'=(\bar{i}, \bar{j}_{r(\bar{s})})$, $s's=(\bar{i}_{l(s)}, j)$ and $\bar{s}'\bar{s}=(\bar{i}_{l(\bar{s})}, \bar{j})$. By (C1) (a) we have $\bar{a}A_\sigma \in I^{\bar{i}A_j}_{\bar{\alpha}_1}$ and $\sigma B_{\bar{a}} \in J^{jB_{\bar{i}}}_{\bar{\alpha}_1}$. As we have seen, $\bar{\alpha}_1 \in Y_{l(s)r(\bar{s})}$ since $\bar{\alpha}\in Y_{r(\bar{s})}$ and $\alpha\tau_s \in Y_{l(s)}$. Thus $\bar{\alpha}_1 \leqq l(s)$ and $\bar{\alpha}_1 \leqq r(\bar{s})$. Therefore $\bar{a}A_\sigma h_s$ and $\sigma B_{\bar{a}}\chi_{\bar{s}}$ are defined and $\bar{a}A_\sigma h_s \in I^{\bar{k}}_{\bar{\alpha}_1\tau_s^{-1}}$, $\sigma B_{\bar{a}}\chi_{\bar{s}} \in J^{\bar{l}}_{\bar{\alpha}_1\tau_{\bar{s}}}$ where, by (D1) (a), we have $s(\bar{i}A_j, \bar{j}_{l(s)r(\bar{s})})\left(s(\bar{i}A_j, \bar{j}_{l(s)r(\bar{s})})\right)' = (\bar{k}, \bar{j}_{(l(s)r(\bar{s}))\tau_s^{-1}})$ and $((\bar{i}_{l(s)r(\bar{s})}, j\bar{B}_{\bar{i}})\bar{s})'(\bar{i}_{l(s)r(\bar{s})}, j\bar{B}_{\bar{i}})\bar{s} = (\bar{i}_{(l(s)r(\bar{s}))\tau_{\bar{s}}}, \bar{l})$. Here $(l(s)r(\bar{s}))\tau_s^{-1} = r(s\bar{s})$ and $(l(s)r(\bar{s}))\tau_{\bar{s}} = l(s\bar{s})$. Hence $\beta = \bar{\alpha}_1\tau_s^{-1} \in Y_{r(s\bar{s})}$ and $\beta\tau_{s\bar{s}} \in Y_{l(s\bar{s})}$. Property (D5) implies that $(j_{\beta\tau_{s\bar{s}}} \cdot \gamma_{s,\bar{s}})' \in J_{\beta\tau_s\tau_{\bar{s}}}$ where $\beta\tau_s\tau_{\bar{s}} = \bar{\alpha}_1\tau_{\bar{s}}$. Lemma 3.2 implies that $(\sigma B_{\bar{a}})'\chi_{\bar{s}}$ is also defined and it is an inverse of $(\sigma B_{\bar{a}})\chi_{\bar{s}}$. Thus we obtain by (D1) (b) and Lemma 2.4 that $(\sigma B_{\bar{a}}\chi_{\bar{s}})' \in J_{\bar{\alpha}_1'\tau_{\bar{s}}}$ provided $(\sigma B_{\bar{a}})' \in J_{\bar{\alpha}_1'}$. Since $\bar{\alpha}_1 \leqq \alpha\tau_s$ and $\bar{\alpha}_1' \leqq \bar{\alpha}$ we have $\beta = \bar{\alpha}_1\tau_s^{-1} \leqq \alpha$, $\bar{\alpha}_1\tau_{\bar{s}} = \beta\tau_s\tau_{\bar{s}}$ and $\bar{\alpha}_1'\tau_{\bar{s}} \leqq \bar{\alpha}\tau_{\bar{s}}$. Thus we see that the products $a \cdot \bar{a}A_\sigma h_s$ and $(j_{\beta\tau_{s\bar{s}}} \cdot \gamma_{s,\bar{s}}) \cdot \sigma B_{\bar{a}}\chi_{\bar{s}} \cdot \bar{\sigma}$ in (11) are defined and $a \cdot \bar{a}A_\sigma h_s \in I^{i\cdot\bar{k}}_\beta$, $(j_{\beta\tau_{s\bar{s}}} \cdot \gamma_{s,\bar{s}}) \cdot \sigma B_{\bar{a}}\chi_{\bar{s}} \cdot \bar{\sigma} \in J^{\bar{x}\cdot\bar{l}\cdot\bar{j}}_{\beta\tau_{s\bar{s}}}$ where $\beta \in Y_{r(s\bar{s})}$ and $j_{\beta\tau_{s\bar{s}}} \cdot \gamma_{s,\bar{s}} \in J^{\bar{x}}_{\beta\tau_{s\bar{s}}}$. Here $\bar{x}\in \bar{J}_{l(s\bar{s})}$ and $\bar{l} \in \bar{J}_{l(s\bar{s})}$ whence we infer that $\bar{x}\cdot\bar{l} = \bar{l}$. All we have to verify is that $s\bar{s}(s\bar{s})' = (i\cdot\bar{k}, \bar{j}_{r(s\bar{s})})$ and $(s\bar{s})'s\bar{s} = (\bar{i}_{l(s\bar{s})}, \bar{l}\cdot\bar{j})$. We will show the first equality. The second one follows dually. In the band $\mathscr{B}(\bar{I}, \bar{J}; \bar{A}, \bar{B})$ we have

$$(s's)(\bar{s}\bar{s}') = (\bar{i}_{l(s)}, j)(\bar{i}, \bar{j}_{r(\bar{s})}) = (\bar{i}_{l(s)} \cdot \bar{i}\bar{A}_j, j\bar{B}_{\bar{i}} \cdot \bar{j}_{r(\bar{s})}).$$

Taking into consideration the remark after (C4) we obtain that

$$(s's)(\bar{s}\bar{s}') = (\bar{i}\bar{A}_j\bar{A}_{\bar{j}_{l(s)}}, j\bar{B}_{\bar{i}}\bar{B}_{\bar{i}_{r(\bar{s})}}).$$

Since $j\in\bar{J}_{l(s)}$ and $\bar{i}\in\bar{I}_{r(\bar{s})}$ where $\bar{J}_{l(s)}$ is a right zero semigroup and $\bar{I}_{r(\bar{s})}$ a left zero semigroup we infer by (W1) (a) and (W2) (a) that

$$(s's)(\bar{s}\bar{s}') = (\bar{i}\bar{A}_j, j\bar{B}_{\bar{i}})\mathscr{R}(\bar{i}\bar{A}_j, \bar{j}_{l(s)r(\bar{s})}).$$

Similarly, by applying the remark after (C4) and the fact that $\bar{I}_{r(\bar{s})}$ is a left zero semigroup one sees that

$$(i, \bar{j}_{r(s)})(\bar{k}, \bar{j}_{r(s\bar{s})}) = (i\cdot\bar{k}\bar{A}_{\bar{j}_{r(s)}}, \bar{j}_{r(s)}\bar{B}_{\bar{k}}\cdot\bar{j}_{r(s\bar{s})}) =$$

$$= (i\cdot\bar{i}_{r(s)}\cdot\bar{k}, \bar{j}_{r(s)}\bar{B}_{\bar{k}}\cdot\bar{j}_{r(s\bar{s})}) = (i\cdot\bar{k}, \bar{j}_{r(s)}\bar{B}_{\bar{k}}\cdot\bar{j}_{r(s\bar{s})}) = (i\cdot\bar{k}, \bar{j}_{r(s\bar{s})}).$$

In the last step we have utilized that $J_{r(s)}\bar{B}_{\bar{k}} \in \bar{J}_{r(s\bar{s})}$ and $\bar{J}_{r(s\bar{s})}$ is a right zero semigroup. Now we can easily check that

$$(i \cdot \bar{k}, \bar{j}_{r(s\bar{s})}) = (ss')(s(\bar{i}\bar{A}_j, \bar{j}_{l(s)r(\bar{s})}))(s(\bar{i}\bar{A}_j, \bar{j}_{l(s)r(\bar{s})}))' =$$
$$= s(\bar{i}\bar{A}_j, \bar{j}_{l(s)r(\bar{s})})(s(\bar{i}\bar{A}_j, \bar{j}_{l(s)r(\bar{s})}))' \mathcal{R} s(\bar{i}\bar{A}_j, \bar{j}_{l(s)r(s)}) s' \mathcal{R} s(s's)(\bar{s}\bar{s}')s' = s\bar{s}\bar{s}'s' \mathcal{R} s\bar{s}(s\bar{s})'$$

hold in the semigroup S. On the other hand, $s\bar{s}(s\bar{s})' \mathcal{L} (i \cdot \bar{k}, \bar{j}_{r(s\bar{s})})$ by the definition of $(s\bar{s})'$. This completes the proof of the fact that $(s\bar{s})(s\bar{s})' = (i \cdot \bar{k}, \bar{j}_{r(s\bar{s})})$.

By applying the technique used in Lemmas 3.1, 3.2, 3.4, 3.5 and 3.6 one can prove the following lemmas.

Lemma 3.7. $\mathbf{S}$ *is a semigroup.*

Lemma 3.8. *Let* $(a, s, \sigma) \in \mathbf{S}$ *with* $\sigma \in J_\alpha$, $\sigma' \in J_{\alpha'}$. *Then* $(a', s', \sigma^* \chi_{s'} \cdot \varepsilon) \in \mathbf{S}$ *and the equality*

$$(a, s, \sigma)(a', s', \sigma^* \chi_{s'} \cdot (j_{\alpha\tau_s^{-1}} \cdot \gamma'_{s,s'}) \cdot \varepsilon)(a, s, \sigma) = (a, s, \sigma) \tag{12}$$

holds for every inverse σ^* *of* σ, *for every* a' *in* $I_\alpha^{\bar{1}_{l(s)}}$, *and for every idempotent* ε *in* $J_{\alpha\tau_s^{-1}}^{\bar{j}_{r(s)}}$. *Consequently,* $\mathbf{S}$ *is regular.*

Since the proofs need rather long and complicated calculations we left them to the reader.

Observe that the relation $\mathfrak{C}$ defined on $\mathbf{S}$ by

$$(a, s, \varrho) \mathfrak{C} (\bar{a}, \bar{s}, \bar{\varrho}) \quad \text{if and only if} \quad s = \bar{s} \tag{13}$$

is a congruence relation. The idempotent $\mathfrak{C}$-classes are

$$\mathbf{C}_e = \{(a, e, \varrho) : (a, e, \varrho) \in \mathbf{S}\}, \quad (e \in E).$$

Lemma 3.9. *The mapping* $\varphi\colon \bigcup \{\mathbf{C}_e : e \in E\} \to \mathfrak{B}(I, J; A, B)$ *defined by* $(a, e, \varrho)\varphi = (a, \varrho)$ *is an onto isomorphism.*

Proof. φ is one-to-one and onto since if $(a, \varrho) \in \mathfrak{B}(I, J; A, B)$ with $a \in I_\alpha^{\bar{i}}$, $\varrho \in J_\alpha^{\bar{j}}$ then $e = (\bar{i}, \bar{j})$ is the unique idempotent element in S such that $(a, e, \varrho) \in \mathbf{S}$ and, obviously, we have $(a, e, \varrho)\varphi = (a, \varrho)$. A straightforward calculation shows that $((a, e, \sigma)(\bar{a}, \bar{e}, \bar{\sigma}))\varphi = (a, e, \sigma)\varphi, (\bar{a}, \bar{e}, \bar{\sigma})\varphi$, that is, φ is an isomorphism.

Lemma 2.20 shows that $\mathfrak{B}(I, J; A, B)$ is an orthodox semigroup. Hence $\bigcup \{\mathbf{C}_e : e \in E\}$ is also an orthodox semigroup which implies that the idempotents in $\mathbf{S}$ form a subsemigroup. Since $\mathbf{S}$ is regular by Lemma 3.8 $\mathbf{S}$ is also orthodox. Moreover, Lemma 2.22 ensures by the proof of Proposition 1.1 that the congruence relation $\mathfrak{C}$ defined in (13) is $(\bar{\mathbf{C}}, \mathfrak{C}')$-parcelling where, by using the notations of (6), (9) and Lemma 2.22 we define $\bar{\mathbf{C}} = \bar{\mathbf{B}}\varphi^{-1}$ and $\mathfrak{C}'$ to be the congruence relation on the band of idempotents of $\mathbf{S}$ corresponding under φ^{-1} to the congruence $\mathcal{D} \cap \varkappa|\mathbf{B}$ on $\mathbf{B}$.

Theorem 3.10. *Let $\bar{Y}, \bar{I}, \bar{J}, \bar{A}, \bar{B}, Y, I, J, A$ and B have the properties required in Theorem 2.23. Suppose that (C4) holds for A and B. Let S be an orthodox semigroup with band of idempotents $\mathscr{B}(\bar{I}, \bar{J}; \bar{A}, \bar{B})$. Moreover, let h, χ, γ be an (S, I, J)-triple. Then $\mathbf{S}=\mathfrak{S}(S, I, J; h, \chi, \gamma)$ is an orthodox semigroup and the relation $\mathfrak{C}$ defined in (13) is a strong subband-parcelling congruence on $\mathbf{S}$ such that the factor semigroup $\mathbf{S}/\mathfrak{C}$ is isomorphic to S.*

Proof. The reasoning carried out before stating the theorem shows that all we have to prove is that $\mathfrak{C}$ is strong. We verify that, for every s in S, there exists $(a, s, \sigma)\in\mathbf{S}$ with $a\in I_{r(s)}$ and $\sigma\in J_{l(s)}$ idempotent such that (a, s, σ) is $\mathscr{L}$- and $\mathscr{R}$-equivalent to idempotents in $\bar{\mathbf{C}}$. By Lemma 1.3 and Theorem 1.4, we can restrict ourselves to elements s with $(\bar{i}_{r(s)}, \bar{j}_{r(s)})\mathscr{R}s\mathscr{L}(\bar{i}_{l(s)}, \bar{j}_{l(s)})$. These are precisely those elements for which $s=s''$ holds. Suppose that s fulfils this property and $a\in I_{r(s)}^{\bar{i}_{r(s)}}$. Then $(a, s, j_{l(s)})\in\mathbf{S}$ and we have seen in Lemma 3.8 that $(a', s', j_{l(s)}\chi_{s'}\cdot(j_{r(s)}\cdot\gamma'_{s,s'}))\in\mathbf{S}$ and (16) holds with $\sigma=j_{l(s)}$ and $\varepsilon=\gamma_{s,s'}\cdot\gamma'_{s,s'}$ for any $a'\in I_{l(s)}^{\bar{i}_{l(s)}}$. Since $j_{l(s)}\chi_{s'}$ is an idempotent in $J_{r(s)}$ by Lemma 3.1 and the idempotents in $J_{r(s)}$ form a right zero semigroup we have $j_{l(s)}\chi_{s'}\cdot(j_{r(s)}\cdot\gamma'_{s,s'})=\gamma'_{s,s'}$. Thus

$$(a, s, j_{l(s)})(a', s', \gamma'_{s,s'})(a, s, j_{l(s)}) = (a, s, j_{l(s)})$$

where one can easily check that $(a, s, j_{l(s)})(a', s', \gamma'_{s,s'})=(a, ss', \gamma_{s,s'}\cdot\gamma'_{s,s'})\in\bar{\mathbf{C}}$. Since $s=s''$ a similar argument shows that $(a, s, \zeta)\in\mathbf{S}$ with $\zeta=\gamma_{s,s'}\chi_s\cdot\gamma'_{s',s}\cdot j_{l(s)}$ and

$$(a', s', \gamma'_{s,s'})(a, s, \zeta)(a', s', \gamma'_{s,s'}) = (a', s', \gamma'_{s,s'}).$$

Here $\gamma_{s,s'}\chi_s=\gamma_{ss',s}\cdot(j_{r(s)}\cdot\gamma_{s,s'})\chi_s\sim\gamma_{s,s's}\cdot\gamma_{s',s}$ by Lemma 3.6 and (D6). Thus we have $\zeta\sim\gamma_{s,s's}\cdot(\gamma_{s',s}\cdot\gamma'_{s',s})\cdot j_{l(s)}$ by Lemma 2.17. Since $\gamma_{s,s's}\in J_{l(s)}$ is idempotent by Lemma 3.6 and, clearly, both $(\gamma_{s',s}\cdot\gamma'_{s',s})$ and $j_{l(s)}$ are idempotents in $J_{l(s)}$ we obtain that $\zeta\sim j_{l(s)}$. This implies that ζ is also idempotent and, since $\zeta\cdot j_{l(s)}=\zeta$ we infer that $\zeta=j_{l(s)}$.

Since $(a', s', \gamma'_{s,s'})(a, s, j_{l(s)})=(a', s's, j_{l(s)})\in\bar{\mathbf{C}}$ we conclude that $(a, s, j_{l(s)})$ and $(a', s', \gamma'_{s,s'})$ are inverses of each other in $\mathbf{S}$ and therefore $(a, ss', \gamma_{s,s'}\cdot\gamma'_{s,s'})\mathscr{R}(a, s, j_{l(s)})\mathscr{L}$ $\mathscr{L}(a', s's, j_{l(s)})$. Thus we have proved the theorem.

4. The main result

In this section we prove that any strong subband-parcelling extension of an orthodox semigroup S is isomorphic to some semigroup $\mathfrak{S}(S, I, J; h, \chi, \gamma)$.

An orthodox semigroup T is said to be a *strong subband-parcelling extension* of the orthodox semigroup S if S is isomorphic to $T/\varkappa$ for some strong subband-parcelling congruence $\varkappa$ on T.

Before drawing up our main result we verify two lemmas which make the proof of the theorem easier.

Lemma 4.1. *Let T be an orthodox semigroup whose band of idempotents is a semilattice Y of rectangular bands E_α $(\alpha\in Y)$. Let t and u be elements in T such that, for some inverses t' and u' of t and u, respectively, we have $E(tt')\leqq E(u'u)$. If $x\in E(tt')$ and $z\in E(uxu')$ then $zuxt=zut$.*

Proof. By assumption, we have $E(z)=E(uxu')=E(uxtt'u')=E(utt'u')$. Moreover, $tt'u'utt'=tt'$ as $E(tt')\leqq E(u'u)$. Thus we obtain that

$$zuxt = zux(tt')t = zux(tt')u'u(tt')t = z(uxtt'u')(utt'u')ut = zutt'u'ut = zut.$$

Lemma 4.2. *Let T be an orthodox semigroup with band of idempotents B. Suppose $\bar{B}$, δ is an associated pair in B. Let $\varkappa$ be a strong $(\bar{B}, \delta)$-parcelling congruence on T. Let $T/\varkappa$ be denoted by S. Then there exists a cross-section $\{u_s: s\in S, u_s\varkappa=s\}$ of the $\varkappa$-classes contained in $S_{\bar{B}}$ such that $u_e\in\bar{B}$ whenever e is idempotent and, furthermore, u_s and u_{s^*} are inverses of each other and $u_su_{s^*}=u_{ss^*}$ provided s is idempotent or s is not an inverse of itself and s^* is an inverse of s in S.*

Proof. Let the band of idempotents in S be E which is a semilattice Y of rectangular bands E_α $(\alpha\in Y)$. Let us choose and fix an element e_α in E_α for every α in Y. Moreover, choose and fix an element i_α of $\bar{B}$ in each $\varkappa$-class e_α. If $e\mathcal{R}e_\alpha$ in S then we have $i_\alpha j\varkappa j$ and $i_\alpha j\mathcal{R}i_\alpha$ for every element j of the $\varkappa$-class e contained in $\bar{B}$. For $e_\alpha e=e$ implies $i_\alpha j\varkappa j$, the equality $i_\alpha(i_\alpha j)=i_\alpha j$ trivially holds and $(i_\alpha j)i_\alpha=i_\alpha$ follows from the fact that $\varkappa|\bar{B}=\delta|\bar{B}\subseteqq\mathcal{D}$ whence we infer $(i_\alpha j)i_\alpha\mathcal{D}i_\alpha$ by $((i_\alpha j)i_\alpha)\varkappa=e_\alpha$. Thus we have seen that every $\varkappa$-class e with $e\mathcal{R}e_\alpha$ contains an element $i\in\bar{B}$ such that $i\mathcal{R}i_\alpha$. The dual assertion holds for the $\varkappa$-classes e with $e\mathcal{L}e_\alpha$. Now let us choose and fix an element $i^*_e\in\bar{B}$ in every $\varkappa$-class e with $e\mathcal{R}e_\alpha$ or $e\mathcal{L}e_\alpha$ such that $i^*_e\mathcal{R}i_\alpha$ and $i^*_e\mathcal{L}i_\alpha$, respectively. In particular, it is clear that $i^*_{e_\alpha}=i_\alpha$. If $f\mathcal{D}e_\alpha$ then there exist uniquely determined elements e_1 and e_2 in E such that $e_1\mathcal{R}f\mathcal{L}e_2$ and $e_1\mathcal{L}e_\alpha\mathcal{R}e_2$. Then $e_1e_2=f$. Define i^*_f to be $i^*_{e_1}\cdot i^*_{e_2}$. Since, for each α in Y, $\bar{B}\cap\{j\in B: j\varkappa=f$ for some $f\in E$ with $f\mathcal{D}e_\alpha\}$ is a rectangular band the set $\{i^*_f: f\mathcal{D}e_\alpha\}$ forms a subband in it and therefore $\{i^*_f: f\mathcal{D}e_\alpha\}$ is also a rectangular band. Let us define u_e to be i^*_e for every e in E.

Now let s be a non-idempotent element in S such that $e_\alpha\mathcal{R}s\mathcal{L}e_\beta$ for some α, β in Y. Let s' be the inverse of s with $e_\alpha\mathcal{L}s'\mathcal{R}e_\beta$. Proposition 1.5 ensures the existence of elements t, t' in $S_{\bar{B}}$ which are inverses of each other and $t\varkappa=s$, $t'\varkappa=s'$. If $s=s'$ define $u_s=i_\alpha ti_\beta$. Now consider the case when $s\neq s'$. Since both tt' and $t't$ belong to $\bar{B}$ and $(tt')\varkappa=e_\alpha$, $(t't)\varkappa=e_\beta$ the elements $u_s=i_\alpha ti_\beta$ and $u_{s'}=i_\beta t'i_\alpha$ are also in $S_{\bar{B}}$, they are inverses of each other and $u_su_{s'}=i_\alpha$, $u_{s'}u_s=i_\beta$. Clearly, we have $u_s\varkappa=s$ and $u_{s'}\varkappa=s'$. Thus we have defined u_s for those s in S for which $e_\alpha\mathcal{R}s\mathcal{L}e_\beta$ for some $\alpha, \beta \in Y$. Finally, let $\tilde{s}$ be any element in S. Assume that $e\mathcal{R}\tilde{s}\mathcal{L}f$ where $e \in E_\alpha$ and $f \in E_\beta$. Then $s = e_\alpha \tilde{s} e_\beta$ satisfies $e_\alpha \mathcal{R} s \mathcal{L} e_\beta$. Define $u_{\tilde{s}}$ to be $u_{\tilde{s}} = i_e^* u_s i_f^*$. Obviously, we have $u_{\tilde{s}}\varkappa = \tilde{s}$. The idempotents i_e^* $(e \in E)$ are chosen such that the definition of $u_{\tilde{s}}$ is independent of the choice of e and f. If $\tilde{s}$ and $\tilde{s}'$ are inverses of each other in S such that s is not an inverse of itself and $\tilde{s}\tilde{s}' = e \in E_\alpha$, $\tilde{s}'\tilde{s} = f \in E_\beta$ then $s = e_\alpha \tilde{s} e_\beta \neq \neq e_\beta \tilde{s}' e_\alpha = s'$. Thus $u_{\tilde{s}}$ and $u_{\tilde{s}'}$ are inverses of each other in S_B and we have $u_{\tilde{s}} u_{\tilde{s}'} = i_e^*$, $u_{\tilde{s}'} u_{\tilde{s}} = i_f^*$. Thus the required conditions are fufilled by the cross-section $\{u_s : s \in S, u_s\varkappa = s\}$ which completes the proof of the lemma.

Now we turn to the main theorem of the paper.

Theorem 4.3. *Suppose T is an orthodox semigroup and $\varkappa$ is a strong subband-parcelling congruence on T. Denote $T/\varkappa$ by S. Then there exist $\bar{Y}, \bar{I}, \bar{J}, \bar{A}, \bar{B}, Y, I, J, A, B$ satisfying the conditions of Theorem* 2.23 *and* (C4) *and there exists an (S, I, J)-triple h, χ, γ such that T is isomorphic to $\mathfrak{S}(S, I, J; h, \chi, \gamma)$.*

Proof. Assume that $\varkappa$ is a $(\bar{B}_0, \delta)$-parcelling congruence on T where $\bar{B}_0, \delta$ is an associated pair on the band of idempotents B_0 in T. Denote the semilattice $B_0/\mathcal{D}$ by Y. One can easily see by Theorem 1.2 that $\bar{Y} = \bar{B}_0/\mathcal{D}$ is a subsemilattice in Y and Y is a semilattice $\bar{Y}$ of semilattices $Y_{\bar{\alpha}}$ with identities $\bar{\alpha}$ in $\bar{Y}$. Since $\varkappa$ is a strong $(\bar{B}_0, \delta)$-parcelling congruence the band of idempotents E in S is isomorphic to $\bar{B}_0/\delta$. Thus E is a semilattice $\bar{Y}$ of rectangular bands $E_{\bar{\alpha}} (\bar{\alpha} \in \bar{Y})$. Let us choose and fix an element $e_{\bar{\alpha}}$ in every $\mathcal{D}$-class $E_{\bar{\alpha}}$. Moreover, for every α in Y, select an element i_α in the $\mathcal{D}$-class α such that $i_\alpha\varkappa = e_{\bar{\alpha}}$ provided $\alpha \in Y_{\bar{\alpha}}$. This can be done by Lemma 1.3. If $\bar{\alpha} \in \bar{Y}$ then let $\bar{I}_{\bar{\alpha}}$ and $\bar{J}_{\bar{\alpha}}$ stand for the $\mathcal{L}$-class and $\mathcal{R}$-class, respectively, in $E_{\bar{\alpha}}$ containing $e_{\bar{\alpha}}$. If $\alpha \in Y$ then denote by I_α the $\mathcal{L}$-class in B_0 containing i_α and let J_α be the set of all elements σ in T for which $\sigma\varkappa$ is idempotent and $i_\alpha \mathcal{R} \sigma$. Suppose the transformation "′" on $J = \bigcup \{J_\alpha : \alpha \in Y\}$ assigns an inverse to each element. Clearly, such a transformation exists on J. Define a partial operation "$\cdot$" on $\bar{I} = \bigcup \{\bar{I}_{\bar{\alpha}} : \bar{\alpha} \in \bar{Y}\}$ as follows: if $a \in \bar{I}_{\bar{\alpha}}$, $b \in \bar{I}_{\bar{\beta}}$ then $a \cdot b$ is defined if and only if $\bar{\alpha} \geqq \bar{\beta}$ and if this is the case then $a \cdot b$ means their product in E. It is clear that $a \cdot b \mathcal{L} e_{\bar{\beta}}$, that is, $a \cdot b \in \bar{I}_{\bar{\beta}}$. With respect to this partial operation $\bar{I}$ is a lower associative semilattice $\bar{Y}$ of the left zero semigroups $\bar{I}_{\bar{\alpha}}$ $(\bar{\alpha} \in \bar{Y})$. Analogously, one can define a multiplication on the set $I = \bigcup \{I_\alpha : \alpha \in Y\}$ with respect to which I becomes a lower associative semilattice Y of the left zero semigroups I_α $(\alpha \in Y)$. For every $\bar{\imath}$ in $\bar{I}$, denote by $I^{\bar{\imath}}$ the set $\{i \in I : i\varkappa = \bar{\imath}\}$. The elements i_α $(\alpha \in Y)$ are chosen such that I is a disjoint union of the subsets $I^{\bar{\imath}}$ $(\bar{\imath} \in \bar{I})$. Let $I_\alpha^{\bar{\imath}} = I^{\bar{\imath}} \cap I_\alpha$ provided $\alpha \in Y_{\bar{\alpha}}$ and $\bar{\imath} \in \bar{I}_{\bar{\alpha}}$. By Lemma 1.3, these subsets are non-void and, since $\varkappa$ is a congruence, one can immediately see that $I = \bigcup \{I_\alpha^{\bar{\imath}} : (\bar{\imath}, \alpha) \in \bar{I} \otimes_Y Y\}$ and I is a partial left band over $\bar{I} \otimes_Y Y$. Let us define $\bar{J}$ dually to $\bar{I}$. Obviously, $\bar{J}$ is an upper associative semilattice $\bar{Y}$ of the right zero semigroups $\bar{J}_{\bar{\alpha}}$ $(\bar{\alpha} \in Y)$. Finally, define a partial operation on J in the following way: if $\varrho \in J_\alpha$, $\varrho' \in J_{\alpha'}$, $\sigma \in J_\beta$, $\sigma' \in J_{\beta'}$ and $\alpha' \leqq \beta$ then let $\varrho \cdot \sigma$ mean the product of ϱ

and σ in T and, in the opposite case, let $\varrho\cdot\sigma$ be undefined. Clearly, we have $\varrho\cdot\sigma\mathcal{R}i_\alpha$ and hence $\varrho\cdot\sigma\in J_\alpha$. Denote by $J^{\bar{j}}$ the set $\{j\in J: j\varkappa=\bar{j}\}$. Similarly to the case of I one can see that $J=\bigcup\{J^{\bar{j}}:\bar{j}\in\bar{J}\}$. Moreover, if $(\bar{j},\alpha)\in J\otimes_Y Y$ then $J^{\bar{j}}_\alpha=J^{\bar{j}}\cap J_\alpha$ is non-void and, since $\varkappa$ is a congruence we have $\varrho\cdot\sigma\in J^{\bar{j}\cdot\bar{k}}$ provided $\varrho\in J^{\bar{j}}$, $\sigma\in J^{\bar{k}}$ and their product is defined in J. Since $(\varrho\cdot\sigma)'\sim\sigma'\varrho'$ in T where $\sim$ is used to mean the least inverse semigroup congruence on T and $(\sigma'\sigma)\sigma'\varrho'=\sigma'\varrho'$ we have $J((\varrho\cdot\sigma)')\leqq J(\sigma'\cdot\sigma)=J(\sigma')$. This proves (B2). Properties (B1), (B3), (B4) and (B5) trivially hold in J. As far as (B6) is concerned, if ϱ and σ are idempotents in J_α and J_β, respectively, and $\tau\in J_\gamma$, $\tau'\in J_{\gamma'}$ with $\alpha\leqq\gamma$, $\beta\leqq\gamma'$ then $\tau'\varrho\tau\mathcal{D}\sigma$ holds in B_0 if and only if $\tau\sigma\tau'\mathcal{D}\varrho$. Thus we have shown that J is a right orthodox semigroup over $\bar{J}\otimes_Y Y$.

For every element s in S, let s' stand for the inverse of s satisfying $e_{r(s)}\mathcal{L}s'\mathcal{R}e_{l(s)}$.

Let us choose a cross-section $\{u_s: s\in S,\ u_s\varkappa=s\}$ of the $\varkappa$-classes possessing the properties required in Lemma 4.2. The proof of Lemma 4.2 ensures that this cross-section can be chosen such that $u_{e_{\bar{\alpha}}}=i_{\bar{\alpha}}$ for every $\bar{\alpha}$ in $\bar{Y}$. Denote by u_s' the inverse of u_s fulfilling $i_{r(s)}\mathcal{L}u_s'\mathcal{R}i_{l(s)}$. Clearly, we have $u_s'\varkappa=s$.

Let τ_s be the isomorphism of $r(s)Y$ onto $l(s)Y$ which corresponds the element β with $u_s'i_\alpha u_s\mathcal{D}i_\beta$ to every α in $r(s)Y$. Clearly, τ_e is the identity automorphism of $r(e)Y=l(e)Y$ provided $e\in E$. Moreover, if s and s^* are inverses of each other in S then $\tau_{s^*}=\tau_s^{-1}$. If $\bar{\alpha}\in\bar{Y}$ and $\bar{\alpha}\leqq r(s)$ then $s'E_{\bar{\alpha}}s\subseteq E_{\bar{\alpha}\tau_s}$.

Now we verify that every element t in T is uniquely represented in the form $t=au_s\sigma$ where $s=t\varkappa\in S$, $a\in I_\alpha^{ss'}$ and $\sigma\in J_{\alpha\tau_s}^{s's}$ for some α in $Y_{r(s)}$. Let $t\in T$ and denote $t\varkappa$ by s. Let a be an element in I which is $\mathcal{R}$-related to t. Obviously, such an a exists and is unique. Suppose that $a\in I_\alpha$ and $\alpha\in Y_{\bar{\alpha}}$. Then $s=t\varkappa\mathcal{R}a\varkappa\mathcal{L}e_{\bar{\alpha}}$, that is, $\bar{\alpha}=r(s)$ and $a\in I_\alpha^{ss'}$. On the other hand, we have $\sigma=i_{\alpha\tau_s}u_s't\mathcal{R}i_{\alpha\tau_s}u_s'au_s\mathcal{R}i_{\alpha\tau_s}$ as $u_s'au_s\in\alpha\tau_s$. Since $\sigma\varkappa=e_{r(s)\tau_s}s's=s's$ we obtain that $\sigma\in J_{\alpha\tau_s}^{s's}$. We can easily see that $au_s\sigma=au_si_{\alpha\tau_s}u_s't=a(u_si_{\alpha\tau_s}u_s')at=at=t$ as $a\mathcal{R}t$ and $a,\ u_si_{\alpha\tau_s}u_s'\in\alpha$. As far as the uniqueness of this representation is concerned, observe that if $s\in S$ and $a\in I_\alpha^{ss'}$, $\sigma\in J_{\alpha\tau_s}^{s's}$ with $\alpha\in Y_{r(s)}$ then $(au_s\sigma)\varkappa=ss'ss's=s$ and $au_s\sigma\mathcal{R}au_s\sigma\sigma'u_s'a=a$. Therefore if an element t is represented in both of the forms $au_s\sigma$ and $\bar{a}u_{\bar{s}}\bar{\sigma}$ then $s=\bar{s}$ and $a=\bar{a}$. From the latter equality $\alpha=\bar{\alpha}$ follows where $a\in I_\alpha$ and $\bar{a}\in I_{\bar{\alpha}}$. Multiplying the equality $au_s\sigma=au_s\bar{\sigma}$ by $i_{\alpha\tau_s}u_s'$ on the left we conclude $\sigma=\bar{\sigma}$. Put

$$\mathrm{S}=\{(a,s,\sigma):\ s\in S,\ a\in I_\alpha^{ss'}\ \text{ and }\ \sigma\in J_{\alpha\tau_s}^{s's}\ \text{ for some } \alpha \text{ in }\ Y_{r(s)}\}.$$

We have shown in this paragraph that the mapping $\Phi\colon T\to\mathrm{S}$ which assigns (a,s,σ) to $t=au_s\sigma$ is one-to-one and onto. In the sequel we give an (S,I,J)-triple h,χ,γ such that Φ becomes an isomorphism of T onto $\mathfrak{S}(S,I,J;h,\chi,\gamma)$.

First we deal with the idempotent $\varkappa$-classes. If $t\varkappa=e\in E$ then $t=au_e\sigma=ai_\alpha u_ei_\alpha\sigma=ai_\alpha\sigma=a\sigma$ where $a\in I_\alpha$. For $\alpha\leqq r(e)=l(e)$ and u_e is an idempotent element in T with $u_e \mathscr{D} i_{l(e)}$. If $a \in I_\alpha^{\bar{\imath}}$ and $\sigma \in J_\alpha^{\bar{\jmath}}$ then $(a\sigma)\varkappa = \bar{\imath}\bar{\jmath}$. Therefore every element t in T with the property that $t\varkappa$ is idempotent is uniquely representable in the form $a\sigma$ where $a \in I_\alpha$ and $\sigma \in J_\alpha$ for some α in Y. Now we will use this representation for the elements of the idempotent $\varkappa$-classes. Let $\bar{\imath} \in \bar{I}_{\bar{\alpha}}, \bar{\jmath} \in \bar{J}_{\bar{\beta}}$ and $\alpha \in Y_{\bar{\alpha}}$, $\beta \in Y_{\bar{\beta}}$. Assume that $a \in I_\alpha^{\bar{\imath}}$ and $\sigma \in J_\beta^{\bar{\jmath}}, \sigma' \in J_{\beta'}$. Then $(\sigma a)\varkappa = \bar{\jmath}\bar{\imath}$ and thus σa is uniquely written in the form $a_1\sigma_1$ where $a_1 \in I_{\alpha_1}^{(\bar{\jmath}\bar{\imath})(\bar{\jmath}\bar{\imath})'}$ and $\sigma_1 \in J_{\alpha_1}^{(\bar{\jmath}\bar{\imath})'(\bar{\jmath}\bar{\imath})}$. Let us denote a_1 by aA_σ and σ_1 by σB_a and, moreover, $(\bar{\jmath}\bar{\imath})(\bar{\jmath}\bar{\imath})'$ by $\bar{\imath}\bar{A}_{\bar{\jmath}}$ and $(\bar{\jmath}\bar{\imath})'(\bar{\jmath}\bar{\imath})$ by $\bar{\jmath}\bar{B}_{\bar{\imath}}$. Clearly, $\bar{\imath}\bar{A}_{\bar{\jmath}} \in \bar{I}_{\bar{\alpha}\bar{\beta}}$ and $\bar{\jmath}\bar{B}_{\bar{\imath}} \in \bar{J}_{\bar{\alpha}\bar{\beta}}$. Observe that $\sigma a\sigma' \in \alpha_1$. Moreover, $(\sigma B_a)' \in J_{\alpha_1'}$ if and only if $a\sigma'\sigma a \in \alpha_1'$. Thus $\alpha_1 \leqq \beta$ and $\alpha_1' \leqq \alpha$ immediately follow. If σ is idempotent then $\alpha_1 = \alpha_1' = \alpha\beta$. Suppose $\alpha = \beta'$. Then $\sigma a\sigma' = \sigma\sigma'$ whence $\alpha_1 = \beta$. Furthermore, $\sigma B_a = i_{\alpha_1}\sigma a = i_\beta \sigma a = \sigma a = \sigma i_\alpha \sim \sigma$. However, if $\alpha < \beta'$ then we clearly have $(i_\alpha\sigma')' \in J_{\alpha_1}$ as $\sigma i_\alpha \sigma' \mathscr{D} \sigma a \sigma'$. Thus we have verified that the families $A = \{A_\sigma : \sigma \in J\}$ and $B = \{B_a : a \in I\}$ of transformations of I and J, respectively, satisfy (C1).

Now let $a \in I_\alpha$, $b \in I_\beta$ with $\alpha \geqq \beta$ and $\sigma \in J$. By definition, we have

$$\sigma ab = \sigma(a \cdot b) = (a \cdot b)A_\sigma \cdot \sigma B_{a \cdot b}, \tag{14}$$

where $(a \cdot b)A_\sigma \in I_{\alpha_1}$ and $\sigma B_{a \cdot b} \in J_{\alpha_1}$. On the other hand,

$$\sigma ab = (\sigma a)b = (aA_\sigma \cdot \sigma B_a)b = aA_\sigma(\sigma B_a \cdot b) = (aA_\sigma \cdot bA_{\sigma B_a}) \cdot \sigma B_a B_b, \tag{14'}$$

where the product $aA_\sigma \cdot bA_{\sigma B_a}$ is defined in I as it was noted after the definition of an (I, J)-pair. Since σab is uniquely representable in the form $a_0\sigma_0$ with $a_0 \in I_{\alpha_0}$, $\sigma_0 \in J_{\alpha_0}$ for some $\alpha_0 \in Y$ we infer that (14) and (14′) imply (C2) to be valid. Dually, one can prove that (C3) also holds. Since $\varkappa$ is a congruence relation (C2) and (C3) show by the definition of $\bar{A}$ and $\bar{B}$ that (W1) and (W2) are fulfilled by $\bar{A}, \bar{B}$. This completes the proof of the facts that $\bar{A}, \bar{B}$ is an $(\bar{I}, \bar{J})$-pair and A, B is an (I, J)-pair over $\bar{A}, \bar{B}$. (C4) trivially holds for A, B with $i_\alpha = j_\alpha$, $\alpha \in Y$.

Let s be an element in S. Define the mapping $h_s\colon \bigcup \{I_\alpha : \alpha \leqq l(s)\} \to \bigcup \{I_\alpha : \alpha \leqq r(s)\}$ by

$$ah_s = u_s a u_s' i_{\alpha\tau_s^{-1}}$$

provided $a \in I_\alpha^{\bar{\imath}}$ and $\alpha \leqq l(s)$. Similarly, let $\chi_s\colon \bigcup \{J_\alpha : \alpha \leqq r(s)\} \to \bigcup \{J_\alpha : \alpha \leqq l(s)\}$ be the mapping for which

$$\sigma\chi_s = i_{\beta\tau_s} u_s' \sigma u_s$$

whenever $\sigma \in J_\beta^{\bar{\jmath}}$ with $\beta \leqq r(s)$. Clearly, $ah_s \in I_{\alpha\tau_s^{-1}}^{s\bar{\imath}(s\bar{\imath})'}$ and $\sigma\chi_s \in J_{\beta\tau_s}^{(\bar{\jmath}s)'\bar{\jmath}s}$ as $u_s a u_s' \in \alpha\tau_s^{-1}$, $(ah_s)\varkappa = s\bar{\imath}s'e_{\bar{\alpha}\tau_s^{-1}} = s\bar{\imath}(s\bar{\imath})'$ and $u_s'\sigma u_s \mathscr{R} u_s' i_\beta u_s \in \beta\tau_s$, $(\sigma\chi_s)\varkappa = e_{\bar{\beta}\tau_s} s'\bar{\jmath}s = (\bar{\jmath}s)'\bar{\jmath}s$, respectively. Here $\alpha \in Y_{\bar{\alpha}}$ and $\beta \in Y_{\bar{\beta}}$. It is obvious that $\bar{\alpha} \leqq l(s)$ and $\bar{\beta} \leqq r(s)$. Therefore (D1) (a) and (b) are satisfied by h and χ, respectively. After the properties (D2)—(D8) we have noted that both sides of the respective equalities are defined. Thus we must check only that the equalities are valid.

In order to prove (D2) (a) assume that $a\in I_\alpha$, $b\in I_\beta$ and $l(s)\geqq\alpha\geqq\beta$. By definition, we have

$$ah_s\cdot bh_s\mathcal{R}u_sa(u_s'i_{\alpha\tau_s^{-1}}u_s)bu_s' = u_sa(i_\alpha u_s'i_{\alpha\tau_s^{-1}}u_s)bu_s' =$$
$$=u_sa(i_{\alpha\tau_s^{-1}}\chi_s)bu_s' = u_s(a\cdot bA_{i_{\alpha\tau_s^{-1}}\chi_s})\cdot i_{\alpha\tau_s^{-1}}\chi_sB_bu_s'.$$

Here $bA_{i_{\alpha\tau_s^{-1}}\chi_s}\mathcal{D}i_{\alpha\tau_s^{-1}}\chi_sB_b$ in the band B_0 whence we obtain that

$$a\cdot bA_{i_{\alpha\tau_s^{-1}}\chi_s}\mathcal{R}a\cdot bA_{i_{\alpha\tau_s^{-1}}\chi_s}\cdot i_{\alpha\tau_s^{-1}}\chi_sB_b.$$

Thus

$$ah_s\cdot bh_s\mathcal{R}u_sa\cdot bA_{i_{\alpha\tau_s^{-1}}\chi_s}u_s'\mathcal{R}(a\cdot bA_{i_{\alpha\tau_s^{-1}}\chi_s})h_s$$

and, since both $ah_s\cdot bh_s$ and $(a\cdot bA_{i_{\alpha\tau_s^{-1}}\chi_s})h_s$ belong to $I_{\beta\tau_s^{-1}}$, they are equal. For (D2) (b), suppose that $\varrho\in J_\alpha$, $\varrho'\in J_{\alpha'}$, $\sigma\in J_\beta$ and $\alpha'\leqq\beta\leqq r(s)$. Then we have

$$\varrho\chi_s\cdot\sigma\chi_s = i_{\alpha\tau_s}u_s'\varrho u_si_{\beta\tau_s}u_s'\sigma u_s = i_{\alpha\tau_s}u_s'\varrho(u_si_{\beta\tau_s}u_s'i_\beta)\sigma u_s =$$
$$= i_{\alpha\tau_s}u_s'\varrho(i_{\beta\tau_s}h_s)\sigma u_s = i_{\alpha\tau_s}u_s'(i_{\beta\tau_s}h_sA_\varrho)(\varrho B_{i_{\beta\tau_s}h_s})\sigma u_s.$$

Here $i_{\beta\tau_s}h_sA_\varrho\mathcal{L}i_\alpha\mathcal{R}\varrho B_{i_{\beta\tau_s}h_s}$ whence it follows by Lemma 4.1 that

$$\varrho\chi_s\cdot\sigma\chi_s = i_{\alpha\tau_s}u_s'(\varrho B_{i_{\beta\tau_s}h_s}\cdot\sigma)u_s = (\varrho B_{i_{\beta\tau_s}h_s}\cdot\sigma)\chi_s.$$

Now we check that property (D3) is satisfied. Assume that $a\in I_\alpha$ with $\alpha\leqq l(s)$ and $\sigma\in J_\beta$ with $\beta\leqq r(s)$. Suppose that $aA_{\sigma\chi_s}\in I_{\alpha_1}$ and $\sigma B_{ah_s}\in J_{\beta_1}$. Then, by definition, we have

$$aA_{\sigma\chi_s}h_s = u_s(aA_{\sigma\chi_s})u_s'i_{\alpha_1\tau_s^{-1}} = u_s(\sigma\chi_s)a(\sigma\chi_s)^*i_{\alpha_1}u_s'i_{\alpha_1\tau_s^{-1}} = \tag{15}$$
$$= u_si_{\beta\tau_s}u_s'\sigma u_sau_s'\sigma^*u_si_{\alpha_1}u_s'i_{\alpha_1\tau_s^{-1}},$$

where σ^* and $(\sigma\chi_s)^*$ are arbitrary inverses of σ and $\sigma\chi_s$, respectively. Since $u_si_{\beta\tau_s}u_{s'}\in\beta$ and $\sigma\in J_\beta$ we have

$$u_si_{\beta\tau_s}u_s'\sigma u_s = u_si_{\beta\tau_s}u_s'i_\beta\sigma u_s = i_{\beta\tau_s}h_s\cdot\sigma u_s. \tag{16}$$

On the other hand, $aA_{\sigma\chi_s}\in\alpha_1$ which implies $\sigma u_sau_s'\sigma^*\in\alpha_1\tau_s^{-1}$. Thus

$$(\sigma u_sau_s'\sigma^*)(u_si_{\alpha_1}u_s')i_{\alpha_1\tau_s^{-1}} = (\sigma u_sau_s'\sigma^*)i_{\alpha_1\tau_s^{-1}}. \tag{17}$$

Since $\sigma u_sau_s'\sigma^*\mathcal{R}\sigma\cdot ah_s\cdot\sigma^*$ the equality

$$(\sigma u_sau_s'\sigma^*)i_{\alpha_1\tau_s^{-1}} = \sigma\cdot ah_s\cdot\sigma^*i_{\alpha_1\tau_s^{-1}} = ah_sA_\sigma \tag{18}$$

yields. The equality in (D3) (a) follows from (15) by applying (16), (17) and (18). Moreover, observe that this equality ensures $\beta_1=\alpha_1\tau_s^{-1}$. As far as the dual property

(D3) (b) is concerned, one can see by definition that

$$\sigma B_{ah_s}\chi_s = i_{\beta_1\tau_s} u_s'(\sigma B_{ah_s}) u_s = i_{\beta_1\tau_s} u_s' i_{\beta_1}\sigma(ah_s) u_s.$$

Since $\sigma B_{ah_s} \in J_{\beta_1}$ we have $\sigma(ah_s)\sigma^* \in \beta_1$. Therefore Lemma 4.1 implies

$$\sigma B_{ah_s}\chi_s = i_{\beta_1\tau_s} u_s'\sigma(ah_s) u_s.$$

Furthermore, $u_s'\sigma(ah_s)\sigma^* u_s \in \beta_1\tau_s = \alpha_1$ whence we obtain that $\alpha_1 \leqq \beta\tau_s$ and hence

$$\sigma B_{ah_s}\chi_s = i_{\alpha_1} i_{\beta\tau_s} u_s'\sigma(ah_s) u_s = i_{\alpha_1} i_{\beta\tau_s} u_s' \sigma u_s a u_s' i_{\alpha\tau_s^{-1}} u_s = i_{\alpha_1}\cdot\sigma\chi_s\cdot a u_s' i_{\alpha\tau_s^{-1}} u_s.$$

Utilizing that both a and $u_s' i_{\alpha\tau_s^{-1}} u_s$ are contained in α it follows that

$$\sigma B_{ah_s}\chi_s = i_{\alpha_1}\cdot\sigma\chi_s\cdot a i_\alpha u_s' i_{\alpha\tau_s^{-1}} u_s = i_{\alpha_1}\cdot\sigma\chi_s\cdot a\cdot i_{\alpha\tau_s^{-1}}\chi_s = \sigma\chi_s B_a\cdot i_{\alpha\tau_s^{-1}}\chi_s$$

which was to be proved.

Now we define constants $c_{s,\bar{s}}$ and $\gamma_{s,\bar{s}}$ for each pair of elements $s, \bar{s}$ in S. Since $(u_s u_{\bar{s}})\varkappa = s\bar{s} = u_{s\bar{s}}\varkappa$ there exist uniquely determined elements $c_{s,\bar{s}}$ in $I_\alpha^{s\bar{s}(s\bar{s})'}$ and $\gamma_{s,\bar{s}}$ in $J_{\alpha\tau_{s\bar{s}}}^{(s\bar{s})'s\bar{s}}$ such that $\alpha \in Y_{r(s\bar{s})}$ and

$$u_s u_{\bar{s}} = c_{s,\bar{s}} u_{s\bar{s}} \gamma_{s,\bar{s}}.$$

Here $u_s u_{\bar{s}} \in S_{B_0}$ whence we infer that $\alpha = r(s\bar{s})$ and $\alpha\tau_{s\bar{s}} = l(s\bar{s})$. This implies $\gamma_{s,\bar{s}} = u_{s\bar{s}}' u_s u_{\bar{s}}$ as $u_{s\bar{s}}' \mathcal{R} i_{l(s\bar{s})}$. Thus $\gamma_{s,\bar{s}} \in S_{B_0}$, that is, $\gamma_{s,\bar{s}}$ is also contained in $J_{l(s\bar{s})}$. If $\alpha \leqq l(s\bar{s})$ then

$$\gamma_{s,\bar{s}}' i_\alpha \gamma_{s,\bar{s}} \sim u_{\bar{s}}' u_s' u_{s\bar{s}} i_\alpha u_{s\bar{s}}' u_s u_{\bar{s}} \in \alpha\tau_{s\bar{s}}^{-1}\tau_s\tau_{\bar{s}}$$

which proves (D5). If $s, \bar{s}, \bar{\bar{s}} \in S$ then we have

$$\xi = \gamma_{s\bar{s},\bar{\bar{s}}}\cdot(i_{r(s\bar{s}\bar{\bar{s}})\tau_{s\bar{s}}}\cdot\gamma_{s,\bar{s}})\chi_{\bar{\bar{s}}} = \gamma_{s\bar{s},\bar{\bar{s}}} i_{r(s\bar{s}\bar{\bar{s}})\tau_{s\bar{s}}\tau_{\bar{\bar{s}}}} u_{\bar{\bar{s}}}' (i_{r(s\bar{s}\bar{\bar{s}})\tau_{s\bar{s}}}\cdot\gamma_{s,\bar{s}}) u_{\bar{\bar{s}}} =$$

$$= u_{s\bar{s}\bar{\bar{s}}}' u_{s\bar{s}} u_{\bar{\bar{s}}} i_{l(s\bar{s}\bar{\bar{s}})} u_{\bar{\bar{s}}}' i_{r(s\bar{s}\bar{\bar{s}})\tau_{s\bar{s}}} u_{s\bar{s}}' u_s u_{\bar{s}} u_{\bar{\bar{s}}}.$$

Here $u_{\bar{\bar{s}}} i_{l(s\bar{s}\bar{\bar{s}})} u_{\bar{\bar{s}}}' \in r(s\bar{s}\bar{\bar{s}})\tau_{s\bar{s}}$ and hence $u_{s\bar{s}}(u_{\bar{\bar{s}}} i_{l(s\bar{s}\bar{\bar{s}})} u_{\bar{\bar{s}}}')\cdot i_{r(s\bar{s}\bar{\bar{s}})\tau_{s\bar{s}}} u_{s\bar{s}}' \in r(s\bar{s}\bar{\bar{s}})$. Therefore we obtain that $\xi = u_{s\bar{s}\bar{\bar{s}}}' u_s u_{\bar{s}} u_{\bar{\bar{s}}}$. On the other hand, $\gamma_{s,\bar{s}\bar{\bar{s}}}\cdot\gamma_{\bar{s},\bar{\bar{s}}} = u_{s\bar{s}\bar{\bar{s}}}' u_s u_{\bar{s}\bar{\bar{s}}} u_{\bar{s}\bar{\bar{s}}}' u_{\bar{s}} u_{\bar{\bar{s}}} \sim \xi$ in the semigroup T and thus in J, too. This verifies (D6). If s is idempotent or s is not an inverse of itself and s^* is an inverse of s in S then, by definition, $\gamma_{s,s^*} = u_{ss^*}' u_s u_{s^*} = u_{ss^*}' u_{ss^*}$. Hence γ_{s,s^*} is idempotent which shows (D8).

In order to verify (D7) suppose that $e \in E$, $\alpha \leqq r(e)$ and $a \in I_\alpha$, $\varrho \in J_\alpha$. Then $u_e = c_{e,e}\gamma_{e,e}$ as u_e, $c_{e,e}$ and $\gamma_{e,e}$ belong to $r(e) = l(e)$. Thus, on the one hand, we have

$$ah_e = u_e a u_e' i_{\alpha\tau_e^{-1}} = u_e a i_\alpha = u_e a \gamma_{e,e}' i_\alpha = c_{e,e}\gamma_{e,e} a \gamma_{e,e}' i_\alpha = c_{e,e} a A_{\gamma_{e,e}}.$$

On the other hand,

$$\varrho\chi_e = i_{\alpha\tau_e} u_e' \varrho u_e = i_\alpha \varrho u_e = i_\alpha \varrho c_{e,e}\cdot\gamma_{e,e} = \varrho B_{c_{e,e}}\cdot\gamma_{e,e}$$

which shows that (D7) also holds.

Finally, we prove (D4). Let $s, \bar{s} \in S$ and $a \in I_\alpha$ with $\alpha \leqq l(s\bar{s})$. Utilizing that both $u_s u_{\bar{s}} a u'_{\bar{s}} u'_s$ and $u_s i_{\alpha\tau_{\bar{s}}^{-1}} u'_s$ are contained in $\alpha\tau_{\bar{s}}^{-1}\tau_s^{-1}$ one sees that

$$ah_{\bar{s}} h_s = u_s(u_{\bar{s}} a u'_{\bar{s}} i_{\alpha\tau_{\bar{s}}^{-1}}) u'_s i_{\alpha\tau_{\bar{s}}^{-1}\tau_{\bar{s}}^{-1}} = u_s u_{\bar{s}} a u'_{\bar{s}} u'_s u_s i_{\alpha\tau_{\bar{s}}^{-1}} u'_s i_{\alpha\tau_{\bar{s}}^{-1}\tau_s^{-1}} =$$

$$= u_s u_{\bar{s}} a u'_{\bar{s}} u'_s i_{\alpha\tau_{\bar{s}}^{-1}\tau_s^{-1}} \mathscr{R} c_{s,\bar{s}} u_{s\bar{s}} \gamma_{s,\bar{s}} a \gamma'_{s,\bar{s}} u'_{s\bar{s}} i_{\alpha\tau_{\bar{s}}^{-1}\tau_s^{-1}} \mathscr{R} c_{s,\bar{s}} u_{s\bar{s}} \gamma_{s,\bar{s}} a \gamma'_{s,\bar{s}} i_\beta u'_{s\bar{s}} i_{\alpha\tau_{\bar{s}}^{-1}\tau_s^{-1}}$$

where $\gamma_{s,\bar{s}} a \gamma'_{s,\bar{s}} \in \beta$. However, the latter element is $\mathscr{L}$-related to $i_{\alpha\tau_{\bar{s}}^{-1}\tau_s^{-1}}$ as well as $ah_{\bar{s}}h_s$. Thus we obtain that

$$ah_{\bar{s}} h_s = c_{s,\bar{s}} u_{s\bar{s}} (\gamma_{s,\bar{s}} a \gamma'_{s,\bar{s}} i_\beta) u'_{s\bar{s}} i_{\alpha\tau_{\bar{s}}^{-1}\tau_s^{-1}} = c_{s,\bar{s}} \cdot a A_{\gamma_{s,\bar{s}}} h_{s\bar{s}},$$

that is, (D4) (a) is fulfilled. Now let $s, \bar{s} \in S$ and $\varrho \in J_\alpha$ with $\alpha \leqq r(s\bar{s})$. Applying Lemma 4.1 we infer that

$$\varrho \chi_s \chi_{\bar{s}} = i_{\alpha\tau_s\tau_{\bar{s}}} u'_{\bar{s}} i_{\alpha\tau_s} u'_s \varrho u_s u_{\bar{s}} = i_{\alpha\tau_s\tau_{\bar{s}}} u'_{\bar{s}} u'_s \varrho u_s u_{\bar{s}}.$$

Here $u_s u_{\bar{s}} = c_{s,\bar{s}} u_{s\bar{s}} \gamma_{s,\bar{s}}$ and $u'_{\bar{s}} u'_s \sim \gamma'_{s,\bar{s}} u'_{s\bar{s}}$ whence it follows that $u'_{\bar{s}} u'_s \varrho u_s u_{\bar{s}} \mathscr{R}$ $\mathscr{R} u'_{\bar{s}} u'_s i_\alpha u_s u_{\bar{s}} \mathscr{L} \gamma'_{s,\bar{s}} u'_{s\bar{s}} i_\alpha c_{s,\bar{s}} u_{s\bar{s}} \gamma_{s,\bar{s}} \mathscr{R} \gamma'_{s,\bar{s}} u'_{s\bar{s}} \varrho c_{s,\bar{s}} u_{s\bar{s}} \gamma_{s,\bar{s}}$. Consequently, we have

$$\varrho \chi_s \chi_{\bar{s}} = i_{\alpha\tau_s\tau_{\bar{s}}} \gamma'_{s,\bar{s}} u'_{s\bar{s}} \varrho c_{s,\bar{s}} u_{s\bar{s}} \gamma_{s,\bar{s}}.$$

(D5) ensures $(i_{\alpha\tau_{s\bar{s}}} \cdot \gamma_{s,\bar{s}})' \in J_{\alpha\tau_s\tau_{\bar{s}}}$ and therefore (B6) implies that $(i_{\alpha\tau_s\tau_{\bar{s}}} \cdot \gamma'_{s,\bar{s}})' \in J_{\alpha\tau_{s\bar{s}}}$. Thus we can deduce from the last equality that

$$(i_{\alpha\tau_{s\bar{s}}} \cdot \gamma_{s,\bar{s}}) \cdot \varrho \chi_s \chi_{\bar{s}} = i_{\alpha\tau_{s\bar{s}}} \cdot \gamma_{s,\bar{s}} i_{\alpha\tau_s\tau_{\bar{s}}} \gamma'_{s,\bar{s}} u'_{s\bar{s}} \varrho c_{s,\bar{s}} u_{s\bar{s}} \gamma_{s,\bar{s}} =$$

$$= i_{\alpha\tau_{s\bar{s}}} u'_{s\bar{s}} i_\alpha \varrho c_{s,\bar{s}} u_{s\bar{s}} \gamma_{s,\bar{s}} = \varrho B_{c_{s,\bar{s}}} \chi_{s\bar{s}} \cdot \gamma_{s,\bar{s}}$$

as $\varrho c_{s,\bar{s}} \mathscr{R} \varrho \mathscr{R} i_\alpha$. This shows (D4) (b) which completes the proof of the fact that h, χ, γ is an (S, I, J)-triple.

All that remained to be proved is that the one-to-one and onto mapping Φ defined above is an isomorphism of T onto $\mathfrak{S}(S, I, J; h, \chi, \gamma)$. Let t and $\bar{t}$ be elements in T with $t\Phi = (a, s, \sigma)$ and $\bar{t}\Phi = (\bar{a}, \bar{s}, \bar{\sigma})$, respectively. This means that $t = au_s\sigma$ and $\bar{t} = \bar{a} u_{\bar{s}} \bar{\sigma}$ where $a \in I_\alpha^{ss'}$, $\sigma \in J_{\alpha\tau_s}^{s's}$ and $\bar{a} \in I_{\bar{\alpha}}^{\bar{s}\bar{s}'}$, $\bar{\sigma} \in J_{\bar{\alpha}\tau_{\bar{s}}}^{\bar{s}'\bar{s}}$, for some α in $Y_{r(s)}$ and $\bar{\alpha}$ in $Y_{r(\bar{s})}$, respectively. Suppose that $\bar{a} A_\sigma \in I_{\alpha_1}$ and $\sigma B_{\bar{a}} \in J_{\alpha_1}$. Here $\alpha_1 \in Y_{r(\bar{s}) l(s)}$ by (C1) (a). Thus we have $\alpha_1 \tau_s^{-1} \in Y_{r(s\bar{s})}$ and $\alpha_1 \tau_{\bar{s}} \in Y_{l(s\bar{s})}$. (D5) and (B6) ensure that $\gamma_{s,\bar{s}} i_{\alpha_1\tau_{\bar{s}}} \gamma'_{s,\bar{s}} \in \alpha_1 \tau_s^{-1} \tau_{s\bar{s}}$ and $u_{s\bar{s}} \gamma_{s,\bar{s}} i_{\alpha_1\tau_{\bar{s}}} \gamma'_{s,\bar{s}} u_{(s\bar{s})'} \in \alpha_1 \tau_s^{-1}$. Hence it follows that

$$i_{\alpha_1\tau_s^{-1}} c_{s,\bar{s}} u_{s\bar{s}} \gamma_{s,\bar{s}} i_{\alpha_1\tau_{\bar{s}}} = i_{\alpha_1\tau_s^{-1}} u_{s\bar{s}} \gamma_{s,\bar{s}} i_{\alpha_1\tau_{\bar{s}}} = i_{\alpha_1\tau_s^{-1}} u_{s\bar{s}} i_{\alpha_1\tau_s^{-1}\tau_{s\bar{s}}} \gamma_{s,\bar{s}} i_{\alpha_1\tau_{\bar{s}}}.$$

Applying this equality one can see that

$$\begin{aligned}
(au_s\sigma)(\bar{a}u_{\bar{s}}\bar{\sigma}) &= au_s\cdot\bar{a}A_\sigma\cdot\sigma B_{\bar{a}}\cdot u_{\bar{s}}\bar{\sigma} = au_s\cdot\bar{a}A_\sigma\cdot i_{\alpha_1}\cdot\sigma B_{\bar{a}}\cdot u_{\bar{s}}\bar{\sigma} = \\
&= au_s\cdot\bar{a}A_\sigma\cdot i_{\alpha_1}\big(u'_s\, i_{\alpha_1\tau_s^{-1}}u_s\big)\big(u_{\bar{s}}i_{\alpha_1\tau_{\bar{s}}}u'_{\bar{s}}\big)i_{\alpha_1}\cdot\sigma B_{\bar{a}}\cdot u_{\bar{s}}\bar{\sigma} = \\
&= a\big(u_s\cdot\bar{a}A_\sigma u'_s\, i_{\alpha_1\tau_{\bar{s}}^{-1}}\big)u_s u_{\bar{s}}\big(i_{\alpha_1\tau_{\bar{s}}}u'_{\bar{s}}\cdot\sigma B_{\bar{a}}\cdot u_{\bar{s}}\big)\bar{\sigma} = \\
&= a\cdot\bar{a}A_\sigma h_s\cdot c_{s,\bar{s}}u_{s\bar{s}}\gamma_{s,\bar{s}}\cdot\sigma B_{\bar{a}}\chi_{\bar{s}}\cdot\bar{\sigma} = \qquad (19)\\
&= a\cdot\bar{a}A_\sigma h_s\big(i_{\alpha_1\tau_s^{-1}}c_{s,\bar{s}}u_{s\bar{s}}\gamma_{s,\bar{s}}i_{\alpha_1\tau_{\bar{s}}}\big)\sigma B_{\bar{a}}\chi_{\bar{s}}\cdot\bar{\sigma} = \\
&= a\cdot\bar{a}A_\sigma h_s\big(i_{\alpha_1\tau_s^{-1}}u_{s\bar{s}}i_{\alpha_1\tau_s^{-1}\tau_{s\bar{s}}}\gamma_{s,\bar{s}}i_{\alpha_1\tau_{\bar{s}}}\big)\sigma B_{\bar{a}}\chi_{\bar{s}}\cdot\bar{\sigma} = \\
&= (a\cdot\bar{a}A_\sigma h_s)u_{s\bar{s}}\big((i_{\alpha_1\tau_s^{-1}\tau_{s\bar{s}}}\cdot\gamma_{s,\bar{s}})\cdot\sigma B_{\bar{a}}\chi_{\bar{s}}\cdot\bar{\sigma}\big).
\end{aligned}$$

This proves that Φ is a homomorphism and therefore an isomorphism. The proof of the theorem is complete.

As an application of Theorem 3.10 and 4.3 we describe the structure of orthodox semigroups by means of their bands of idempotents and greatest inverse semigroup homomorphic images. An alternative structure theorem was given by YAMADA [7]. However, our construction is more economic as it makes use of structure mappings in a single variable only.

Let S be an inverse semigroup with semilattice of idempotents Y. For every α in Y, let I_α and J_α be a left zero semigroup and a right zero semigroup with distinguished elements i_α and j_α, respectively. Let I be a lower associative semilattice Y of the left zero semigroups I_α ($\alpha\in Y$) and J an upper associative semilattice Y of the right zero semigroups J_α ($\alpha\in Y$). Assume that A, B is an (I, J)-pair satisfying the property that

(C4)' $aA_{j_\beta}=i_\beta\cdot a$ and $\sigma B_{i_\beta}=\sigma\cdot j_\beta$ provided $\alpha,\beta\in Y$ with $\alpha\leqq\beta$ and $a\in I_\alpha$, $\sigma\in J_\alpha$.

Let h_s: $\bigcup\{I_\alpha:\alpha\leqq s^{-1}s\}\to\bigcup\{I_\alpha:\ \alpha\leqq ss^{-1}\}$ and χ_s: $\bigcup\{J_\alpha:\alpha\leqq ss^{-1}\}\to\bigcup\{J_\alpha:\alpha\leqq s^{-1}s\}$ be mappings such that $I_\alpha h_s\subseteqq I_{s\alpha s^{-1}}$ for $\alpha\leqq s^{-1}s$, $J_\alpha\chi_s\subseteqq J_{s^{-1}\alpha s}$ for $\alpha\leqq ss^{-1}$ and the following conditions are fulfilled:

(D2)' (a) if $a\in I_\alpha$, $b\in I_\beta$ with $s^{-1}s\geqq\alpha\geqq\beta$ then $ah_s\cdot bh_s=(a\cdot bA_{j_{s\alpha s^{-1}}\chi_s})h_s$,

(b) if $\varrho\in J_\alpha$, $\sigma\in J_\beta$ with $\alpha\leqq\beta\leqq ss^{-1}$ then $\varrho\chi_s\cdot\sigma\chi_s=(\varrho B_{i_{s^{-1}\beta s}h_s}\cdot\sigma)\chi_s$;

(D3)' if $a\in I_\alpha$ with $\alpha\leqq s^{-1}s$ and $\sigma\in J_\beta$ with $\beta\leqq ss^{-1}$ then

(a) $aA_{\sigma\chi_s}h_s=i_{s^{-1}\beta s}h_s\cdot ah_sA_\sigma$,

(b) $\sigma B_{ah_s}\chi_s=\sigma\chi_sB_a\cdot j_{s\alpha s^{-1}}\chi_s$;

(D4)' (a) if $a\in I_\alpha$ with $\alpha\leqq(s\bar{s})^{-1}s\bar{s}$ then $ah_{\bar{s}}h_s=c\cdot aA_\gamma h_{s\bar{s}}$ for some c in $I_{s\bar{s}(s\bar{s})^{-1}}$ and γ in $J_{(s\bar{s})^{-1}s\bar{s}}$,

(b) if $\varrho \in J_\alpha$ with $\alpha \leqq s\bar{s}(s\bar{s})^{-1}$ then $\varrho\chi_s\chi_{\bar{s}} = \varrho B_c \chi_{s\bar{s}} \cdot \gamma$ for some c in $I_{s\bar{s}(s\bar{s})^{-1}}$ and γ in $J_{(s\bar{s})^{-1}s\bar{s}}$;

(D7)' (a) if $a \in I_\alpha$ and $\alpha \leqq \beta$ then $ah_\beta = c \cdot aA_\gamma$ for some c in I_β and γ in J_β,
(b) if $\varrho \in J_\alpha$ and $\alpha \leqq \beta$ then $\varrho\chi_\beta = \varrho B_c \cdot \gamma$ for some c in I_β and γ in J_β.

A pair of mappings h, χ possessing these properties is called a *reduced* (S, I, J)*-pair.*

Let us define a multiplication on the set

$$\mathbf{S} = \{(a, s, \sigma)\colon s \in S, a \in I_{ss^{-1}}, \sigma \in J_{s^{-1}s}\}$$

as follows:

$$(a, s, \sigma)(\bar{a}, \bar{s}, \bar{\sigma}) = (a \cdot \bar{a} A_\sigma h_s, s\bar{s}, \sigma B_{\bar{a}} \chi_{\bar{s}}).$$

If the constant γ in (D4)' can be chosen such that it depends on s and $\bar{s}$ only and, moreover, (D7)' holds with the γ corresponding to the pair (β, β) then the reduced (S, I, J)-pair h, χ can be easily extended to an (S, I, J)-triple. Then Theorem 3.10 immediately implies that S forms an orthodox semigroup with respect to this multiplication whose band of idempotents is isomorphic to $\mathscr{B}(I, J; A, B)$ and whose greatest inverse semigroup homomorphic image is isomorphic to S. However, as c is not needed to be independent of the choice of a and ϱ in properties (D4) and (D7) since I_α is a left zero semigroup for every α in Y, we need not assume this property for γ if J_α is a right zero semigroup for each α in Y. Therefore the conclusion for S drawn up above holds for any reduced (S, I, J)-pair. The orthodox semigroup obtained in this way will be denoted by $\mathfrak{S}(S, I, J; h, \chi)$.

Conversely, if T is an orthodox semigroup with band of idempotents E then its least inverse semigroup congruence is clearly a strong $(E, \mathscr{D})$-parcelling congruence. The second part of the following theorem immediately follows from Theorem 4.3, one has to observe only that, in this special case, the constants $\gamma_{s,\bar{s}}$ can be eliminated in (D4) (b) and (19) as well as the contants $c_{s,\bar{s}}$ can in (D4) (a) and (19).

Theorem 4.4. *Let S be an inverse semigroup with semilattice of idempotents Y. For every α in Y, let I_α be a left zero semigroup and J_α a right zero semigroup with distinguished elements i_α and j_α, respectively. Suppose I to be a lower associative semilattice Y of the left zero semigroups I_α $(\alpha \in Y)$ and J an upper associative semilattice Y of the right zero semigroups J_α $(\alpha \in Y)$. Let A, B be an (I, J)-pair satisfying* (C4)'. *Assume that h, χ is a reduced (S, I, J)-pair. Then $\mathfrak{S}(S, I, J; h, \chi)$ is an orthodox semigroup with band of idempotents isomorphic to $\mathscr{B}(I, J; A, B)$ and with greatest inverse semigroup homomorphic image isomorphic to S.*

Conversely, if T is an orthodox semigroup with band of idempotents E which is a semilattice Y of rectangular bands then E is isomoprhic to $\mathscr{B}(I, J; A, B)$ for some I, J, A and B which fulfil the conditions required above. Moreover, denoting by S the greatest inverse semigroup homomorphic image of T, there exists a reduced (S, I, J)-pair h, χ such that T is isomorphic to $\mathfrak{S}(S, I, J; h, \chi)$.

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