*Acta Sci. Math.* 44 (1982), 215—237

# Inner injective transextensions of semigroups

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## 1. Introduction

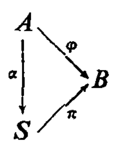
Consider an extension $A$ of a semigroup $B$, that is to say, a surjective homomorphism $\varphi\colon A\to B$. The multiplication in $A$ naturally appears as determined by $B$ up to the extension congruence $\operatorname{Ker}\varphi$. If we succeed in localizing elements in each of the blocks of $\operatorname{Ker}\varphi$ (e.g. by an assignment of some kind of coordinates to them) then we can refine the multiplication modulo $\operatorname{Ker}\varphi$ in $A$ up to the multiplication of individual elements. These may be represented as couples $(\varphi(a), \lambda(a))$, where $\varphi(a)$ is the label of the block containing $a$ and $\lambda(a)$ is the secondary label (coordinate) by which $a$ can be located within its block.

The secondary labels are taken from some suitable auxiliary set $X$ and assigned to the elements of $A$ by a function $\lambda\colon A\to X$, which we call a *localizer* and require (for this purpose) to be injective on each block of $\operatorname{Ker}\varphi$. Identifying $A$ with a subset $S\subseteq B\times X$ by $a\mapsto(\varphi(a), \lambda(a))$, we can determine a unique function $f\colon S\times S\to X$ such that

$$(b, m)(c, n) = \big(bc, f(b, m, c, n)\big) \tag{1}$$

for every $(b, m)$, $(c, n)\in S$.

The set $S$ together with the multiplication given by (1) is thus a semigroup isomorphic to $A$ by the isomorphism $\alpha\colon A\to S$ with $\alpha(a)=\big(\varphi(a), \lambda(a)\big)$, the first projection $\pi\colon S\to B\colon(b, m)\mapsto b$ is a surjective homomorphism onto $B$ and the diagram




commutes. For this reason, we say that $\pi\colon S\to B$ is a *semiproduct* right equivalent to the extension $\varphi\colon A\to B$, or a semiproduct representing $\varphi\colon A\to B$.

Received February 13, 1981, and in revised form December 10, 1981.

It is clear that every extension can be represented by a right equivalent semiproduct, thus we can construct various extensions of a given semigroup $B$ also in this form. The general problem of finding all extensions of $B$ can thus be reduced, from this point of view, to special kinds of completions of partial semigroups according to the following

Extension scheme: Let a partial semigroup $P$ be given together with a homomorphism $\varphi: P \to B$ onto a semigroup $B$. Complete $P$ to a semigroup $A$ with the same underlying set by turning it into a semiproduct in such a way that $\varphi: P \to B$ becomes the projection $\varphi: A \to B$ of the semiproduct.

In the special case when no product is defined in $P$ one has to choose, according to this extension scheme, a localizer $\lambda: P \to X$ and to find a function $f$ making the multiplication defined by (1) associative.

The classical group extensions easily come under the above extension scheme: $P$ appears here as a partial group divided into blocks of equal size by a homomorphism $\varphi$ of $P$ onto a group $B$, the block which is mapped onto the identity of $B$ is a subgroup $C$ of $P$, and all products $ca$, $ac$ for $c \in C$, $a \in P$, are defined in such a way that the action of $C$ on each block is simply transitive.

Namely, the simply transitive action of $C$ on the blocks of Ker $\varphi$ was used by Schreier to build up a most natural localizer: if we choose in a block a reference point $x_0$ then we get a bijection $\lambda$ between the block and the group $C$ by setting $\lambda(x)=c$ iff $cx_0=x$.

In this paper we want to carry over Schreier's idea to semigroup extension schemes, in which in the partial semigroup $P$ to be completed we have only the products $ax$ for a single left cancellable element $a \in P$ and for all $x \in P$ defined, and the surjective homomorphism $\varphi: P \to B$ takes $a$ to a left identity $\varphi(a)=e$ of $B$. In this case the blocks of Ker $\varphi$ are just the connected components of the graph with edges $(x, ax)$, $x \in P$, and since the action of $a$ is injective on each block it is only natural to use integers $Z$ (or integers modulo some $m$) to coordinatize the blocks.

Although the idea is very simple, the detailed elaboration which follows is far from being so. We would like to acknowledge our thanks to L. Márki who helped us to put right a number of technical items.

## 2. $\mathscr{S}$-transextension scheme

Our basic category will be the category $\mathscr{G}^\circ$ of *pointed groupoids* $(G, a)$, $a \in G$, with morphisms $h: (G, a) \to (H, b)$ respecting both multiplication and points (considered as nullary operations). The subcategory of $\mathscr{G}^\circ$ of *pointed semigroups* will be denoted by $\mathscr{S}^\circ$.

Given a pointed groupoid $(G, a)$, we can define its *left connectedness* to be the equivalence $\sim_a$ on $G$ defined by

$$x \sim_a y \Leftrightarrow \exists m, n \quad (a^m \cdot x = a^n \cdot y)$$

where $a^m = a \cdot a^{m-1}$ for $m \geqq 2$.

2.1. Definition. An extension $\varphi$: $(T, a) \to (S, e)$ in $\mathcal{G}^\circ$ is called an *inner injective transextension* or shortly an $\mathcal{I}$*-extension,* if $a$ is left cancellative and the left connectedness $\sim_a$ is a congruence on $T$ such that $\sim_a = \mathrm{Ker}\ \varphi$. Then $e$ is a left identity in $S$ called the left identity of the extension.

Each $\mathcal{I}$-extension $\varphi: (T, a) \to (S, e)$ determines an assignment $x \mapsto \mathcal{U}_x = (T_x, f_x)$, $T_x = \varphi^{-1}(x)$, of unary algebras $\mathcal{U}_x$ to elements $x$ of $S$, with one injective connected operation $f_x$ coinciding with the left inner translation by $a$ restricted to $\varphi^{-1}(x)$.

Let $\mathcal{N}$ denote the semiring $N \cup \{\infty\}$ of non-negative integers completed by a greatest element, where $m \cdot \infty = \infty \cdot m = \infty$ for all $m \neq 0$, $0 \cdot \infty = \infty \cdot 0 = 0$. We assign to every injective connected unar $(X, f)$ an element of $\mathcal{N}$, denoted Type $(f)$, as follows:

Type $(f)$ = min $\{n;\ 0 < n < \infty$ and $f^n(x) = x$ for all $x \in X\}$ if such $n$ exists,
Type $(f) = 0$ iff $f^{-1}(x) = \emptyset$ for some $x \in X$,
Type $(f) = \infty$ otherwise.

The semiring $\mathcal{N}$ is lattice ordered by the divisibility relation. We denote by $\vee$ and $\wedge$ the lattice operations of the least common multiple and the greatest common divisor, respectively.

We have the following easy statement readily obtained from the results of Novotný [5] on commuting transformations.

2.2. Statement. *Let* $(X, f)$ *and* $(Y, g)$ *be injective connected unars. There exists a homomorphism* $h$: $(X, f) \to (Y, g)$, $hf = gh$, *iff* Type $(g)$ *divides* Type $(f)$ *in* $\mathcal{N}$. *If* Type $(f) \neq 0$, *then* $h$ *must be surjective. The unars* $(X, f)$ *and* $(Y, g)$ *are isomorphic iff* Type $(f)$ = Type $(g)$.

Returning to the $\mathcal{I}$-extension $\varphi$: $(T, a) \to (S, e)$, we can describe the assignment $x \mapsto \mathcal{U}_x$ up to isomorphism by a *type function* $r$: $S \to \mathcal{N}$: $x \mapsto$ Type $(f_x)$. On the other hand, we can start with $(S, e)$, $e$ a left identity of $S$, and a function $r$: $S \to \mathcal{N}: x \mapsto r(x)$ as a sort of "plot" for the construction of an $\mathcal{I}$-extension of $(S, e)$ in the form of an $S$-semiproduct, using the ring $Z$ of integers as an auxiliary algebra. We form $S \times Z$ and identify $(x, m) \equiv (y, n)$ iff $x = y$ and $m \equiv n \pmod{r(x)}$ for $0 < r(x) < \infty$, $m = n$ otherwise. With the aid of an *initialization function* $i$: $S \to Z$: $x \mapsto i(x)$ we cut out of $S \times Z/\equiv$ a unar $(P, f)$ with $P = \{(m, n) \in S \times Z | m \geqq i(x)$ if $r(x) = 0\}/\equiv$

and $f(x, m) \equiv (x, m+1)$. To turn $P$ into a groupoid we introduce two additional functions $k: S \to Z$ and $l: S \times S \to Z$ and prescribe a *multiplication formula*

$$\text{(M)} \qquad (x, m)(y, n) \equiv (xy, m+k(x)n+l(x\mu, y))$$

where $\mu=\mu(y, n)=e$ if $r(y)\neq 0$, or $r(y)=0$ and $n\neq i(y)$, $\mu$ is the empty symbol if $r(y)=0$ and $n=i(y)$. The sixtuple $(S, e, r, i, k, l)$ sets up an *$\mathcal{I}$-extension construction scheme,* or shortly an *$\mathcal{I}$-scheme.* If (M) correctly defines a multiplication as a function $P\times P \to P$, we say that the $\mathcal{I}$-scheme is *$\mathcal{G}$-correct.* A $\mathcal{G}$-correct $\mathcal{I}$-scheme turns the unar $(P, f)$ into a groupoid satisfying $f(uv)=f(u)v$ for all $u, v\in P$. If moreover there exists an $a\in P$ of the form $a=(e, m)$, for some $m\in Z$, and such that $f(t)=at$ for all $t\in P$, then we call the $\mathcal{I}$-scheme *$\mathcal{G}^\circ$-correct.* A $\mathcal{G}^\circ$-correct $\mathcal{I}$-scheme $(S, e, r, i, k, l)$ determines a unique semiproduct

$$\pi: (P, a) \to (S, e): (x, m) \mapsto x$$

in $\mathcal{G}^\circ$ and we write in this case $(P, a)=\mathcal{I}(S, e, r, i, k, l)$.

It will be our immediate task to find the conditions for an $\mathcal{I}$-extension scheme $(S, e, r, i, k, l)$ to be $\mathcal{G}^\circ$-correct. This done, we shall next be most interested in the "associative" $\mathcal{I}$-schemes determining $\mathcal{I}$-extensions in $\mathcal{S}^\circ$ — the *$\mathcal{S}^\circ$-correct $\mathcal{I}$*-extension schemes. These found, we investigate how large a class of $\mathcal{I}$-extensions in $\mathcal{S}^\circ$ can be optained by the class of all $\mathcal{S}^\circ$-correct $\mathcal{I}$-schemes. We shall prove that all $\mathcal{I}$-extensions in $\mathcal{S}^\circ$ can thus be obtained. Then we shall clear up a technical point when two $\mathcal{I}$-schemes determine right equivalent $S$-semiproducts, in order to get possibly simple semiproduct representatives of $\mathcal{I}$-extensions in $\mathcal{S}^\circ$. We shall also state, in a number of statements, conditions under which an $\mathcal{S}^\circ$-correct $\mathcal{I}$-extension scheme determines an $\mathcal{I}$-extension in the category of pointed

— semigroups with identity (or "monoids"),
— commutative (=abelian) semigroups,
— right cancellative semigroups,
— left cancellative semigroups,
— right reductive semigroups,
— groups.

In particular, in the case of group $\mathcal{I}$-extensions our theory comes to a strong resemblance with the theory of extensions of P. A. Grillet [2].

Our final point will be to show the role of $\mathcal{I}$-extensions in $\mathcal{S}^\circ$ in a larger class of transextensions.

## 3. $\mathcal{S}^\circ$-correctness of $\mathcal{I}$-schemes

3.1. Statement. *An $\mathcal{I}$-scheme $(S, e, r, i, k, l)$ is $\mathcal{G}$-correct iff the following three "correctness conditions" hold for any $x, y \in S$:*

(C1) $r(xy)=0 \Rightarrow k(x) \geqq 0$, $k(x)r(y)=0$, *and* $i(x)+k(x)i(y)+\min\{k(x)+l(xe, y), l(x, y)\} \geqq i(xy)$, $\big(r(y) \neq 0 \Rightarrow i(x)+l(xe, y) \geqq i(xy)\big)$,

(C2) $r(xy)$ *divides* $r(x)$ *in* $\mathcal{N}$,

(C3) $r(y) \neq \infty \Rightarrow k(x)r(y) \equiv 0 \pmod{r(xy)}$.

Proof. Assume that $(x, m)(y, n) \in P$ for all $(x, m)$, $(y, n) \in P$. If $r(xy)=0$, then $r(x)=0$. Indeed, if $r(x) \neq 0$ we can, for a given $(y, n) \in P$, choose $(x, m) \in P$ for which $m+k(x)n+\max\{l(x, y), l(xe, y)\} < i(xy)$, thus $(x, m)(y, n) \notin P$. Likewise, if $r(xy)=0$ and at the same time $k(x) < 0$ or $k(x)r(y) \neq 0$, then for some $(y, n) \in P$, where $(y, n) \not\equiv (y, i(y))$ or $r(y) \neq 0$, we have $i(x)+k(x)n+l(xe, y) < i(xy)$, thus $(x, i(x))(y, n) \notin P$, a contradiction. So if $r(xy)=0$, then $k(x) \geqq 0$ and $k(x)r(y)=0$. Now if $r(xy)=0=r(y)$, we must have both

$$i(x)+k(x)\big(i(y)+1\big)+l(xe, y) \geqq i(xy)$$

and

$$i(x)+k(x)i(y)+l(x, y) \geqq i(xy).$$

If $r(xy)=0 \neq r(y)$, then $k(x)=0$ and it must be $i(x)+l(xe, y) \geqq i(xy)$. We have proved (C1) under the assumption that $(x, m)(y, n) \in P$ for any $(x, m)$, $(y, n) \in P$.

Assume now (C1) and that $r(xy)=0$ implies $r(x)=0$, and let $(x, m), (y, n) \in P$. If $r(xy) \neq 0$ then clearly $(x, m)(y, n) \in P$. Let $r(xy)=0$. Then $r(x)=0$, hence $m \geqq i(x)$. If further $r(y)=0$, then also $n \geqq i(y)$, and since $k(x) \geqq 0$, $k(x)n \geqq k(x)i(y)$. Therefore, for $n > i(y)$,

$$m+k(x)n+l(xe, y) \geqq i(x)+k(x)\big(i(y)+1\big)+l(xe, y) \geqq i(xy);$$

for $n=i(y)$,

$$m+k(x)n+l(x, y) \geqq i(x)+k(x)i(y)+l(x, y) \geqq i(xy),$$

hence $(x, m)(y, n) \in P$. If $r(y) \neq 0$, then $k(x)=0$,

$$m+k(x)n+l(xe, y) \geqq i(x)+l(xe, y) \geqq i(xy),$$

hence again $(x, m)(y, n) \in P$.

We conclude that (C1) holds and $r(xy)=0$ implies $r(x)=0$ iff $P$ is closed under the (multivalued) multiplication given by (M) for a given $\mathcal{I}$-scheme. Notice that (C2) and $r(xy)=0$ imply $r(x)=0$.

Assume next that

(1) $$(x, m)(y, n) \equiv (x, m')(y, n) \quad \text{whenever} \quad (x, m) \equiv (x, m').$$

Then for $(x, m)$ with $r(x) \neq \infty$ and any $(y, n)$,

$$(x, m+r(x))(y, n) \equiv (x, m)(y, n),$$

hence $r(x) \equiv 0 \pmod{r(xy)}$, which means that $r(xy)$ divides $r(x)$ in $\mathcal{N}$. If $r(x) = \infty$ then $r(xy) \neq 0$ by the above, hence again $r(xy)$ divides $r(x)$ in $\mathcal{N}$. We have proved (C2) under the assumption (1). Conversely, assume (C2) and let $(x, m) \equiv (x, m')$. Then for any $(y, n) \in P$, $(x, m)(y, n) \equiv (x, m')(y, n)$. Indeed, if $r(x) = 0$ or $\infty$ then $m = m'$. If $0 < r(x) < \infty$, then $m \equiv m' \pmod{r(x)}$, thus $m \equiv m' \pmod{r(xy)}$ by (C2), therefore

$$m+k(x)n+l(x\mu, y) \equiv m'+k(x)n+l(x\mu, y) \pmod{r(xy)}.$$

Assume finally that $(x, m)(y, n) \equiv (x, m)(y, n')$ whenever $(y, n) \equiv (y, n')$. Then for $(y, n)$ with $r(y) \neq \infty$ and any $(x, m) \in P$, $(x, m)(y, n+r(y)) \equiv (x, m)(y, n)$, hence $k(x)r(y) \equiv 0 \pmod{r(xy)}$. We have proved (C3) under the assumption. Conversely, assume (C3) and let $(y, n) \equiv (y, n')$. Then for any $(x, m) \in P$, $(x, m)(y, n) \equiv (x, m)(y, n')$. This is clear for $r(y) = 0$ or $\infty$, since then $n = n'$. For $0 < r(y) < \infty$ we have by (C3)

$$m+k(x)n+l(xe, y) \equiv m+k(x)n'+l(xe, y) \pmod{r(xy)}$$

since $n \equiv n' \pmod{r(y)}$.

We conclude that $(x, m)(y, n) \equiv (x, m')(y, n')$ whenever $(x, m) \equiv (x, m')$ and $(y, n) \equiv (y, n')$ iff both (C2) and (C3) hold.

3.2. Statement. *A $\mathcal{G}$-correct $\mathcal{I}$-scheme $(S, e, r, i, k, l)$ is $\mathcal{G}^\circ$-correct iff the following three „inner translativity" conditions hold for any $x \in S$:*

(IT1) $k(e) \equiv 1 \pmod{r(e)}$,
(IT2) $l(e, x) \equiv l(e, e) \pmod{r(x)}$,
(IT3) $r(e) = 0 \Rightarrow 1 - l(e, e) \geqq i(e)$.

*The unique $a$ in $P$ for which $\pi\colon (P, a) \to (S, e)\colon (x, m) \mapsto x$ is an $\mathcal{I}$-extension in $\mathcal{G}^\circ$ is then $a \equiv (e, 1-l(e, e))$.*

Proof. Call $a \in P$ an *admissible point* if the $\mathcal{I}$-scheme yields an $\mathcal{I}$-extension $\pi\colon (P, a) \to (S, e)\colon (x, m) \mapsto x$ in $\mathcal{G}^\circ$ with $a(x, m) \equiv f(x, m) \equiv (x, m+1)$. Assume that $a = (e, p) \in P$ is admissible. Then $(e, p+2) \equiv (e, p)(e, p+1) \equiv (e, p+k(e)+k(e)p+ +l(e, e))$, $(e, p+1) \equiv (e, p)(e, p) \equiv (e, p+k(e)p+l(e, e))$, hence

$$p+k(e)p+k(e)+l(e, e) \equiv p+2 \pmod{r(e)},$$

$$p+k(e)p \qquad +l(e, e) \equiv p+1 \pmod{r(e)}.$$

Subtracting the two congruences we get (IT1). Since $r(x)$ divides $r(e)$ by (C2), (IT1) is equivalent to $\forall x\ (k(e)\equiv 1 \pmod{r(x)})$, therefore for any $(x, m)\in P$,

$$(x, m+1) \equiv (e, p)(x, m) \equiv (x, p+m+l(e, x)),$$

hence

$$p+l(e, x) \equiv 1 \pmod{r(x)}.$$

In particular,

$$p+l(e, e) \equiv 1 \pmod{r(e)}$$

whence the uniqueness of an admissible $a\in P$. By (C2) again, the latest congruence implies

$$p+l(e, e) \equiv 1 \pmod{r(x)}$$

hence we get (IT2).

If $r(e)=0$ then $p=1-l(e, e)\geqq i(e)$ gives (IT3).

Conversely, if the three conditions hold, then $(e, 1-l(e, e))\in P$ and

$$(e, 1-l(e, e))(x, m) \equiv (x, 1-l(e, e)+k(e)m+l(e, x)) \equiv (x, m+1)$$

for any $(x, m)\in P$.

3.3. Statement. *Let $(S, e, r, i, k, l)$ be a $\mathcal{G}$-correct $\mathcal{I}$-scheme with $S$ a semigroup. Then $P$ is a semigroup iff the following "associativity conditions" hold:*

(A0) $k(x)k(y) \equiv k(xy) \pmod{r(xye)}$,

(A1) $l(xe, y)+l(xye, z) \equiv k(x)l(ye, z)+l(xe, yz) \pmod{r(xyz)}$,

(A2) *if* $r(y) = 0$ *and* $(r(yz) = 0 = k(y) \Rightarrow i(y)+l(ye, z) > i(yz))$, *then* $l(xe, y) \equiv l(x, y) \pmod{r(xyz)}$,

(A3) *if* $r(y) = 0 = r(yz)$ *and* $i(y)+k(y)(i(z)+1)+l(ye, z) = i(yz)$, *then* $l(xe, y)-l(x, y) \equiv l(xe, yz)-l(x, yz) \pmod{r(xyz)}$,

(A4) *if* $r(z) = 0$, *then* $l(xye, z)-l(xy, z) \equiv k(x)(l(ye, z)-l(y, z)) \pmod{r(xyz)}$,

(A5) *if* $r(y) = r(z) = r(yz) = 0$ *and* $i(y)+k(y)i(z)+l(y, z)=i(yz)$, *then* $l(x, y)+l(xy, z) \equiv k(x)l(y, z)+l(x, yz) \pmod{r(xyz)}$,

(A6) *if* $r(y) = r(yz) = 0 = k(y)$ *and* $i(y)+l(ye, z) = i(yz)$ *then* $l(xe, yz) \equiv l(x, yz) \pmod{r(xyz)}$.

Proof. For arbitrary $(x, m), (y, n), (z, p)\in P$,

(1) $$[(x, m)(y, n)](z, p) \equiv$$

$$\equiv (xyz, m+k(x)n+k(xy)p+l(x\mu(y, n), y)+l(xy\mu(z, p), z)),$$

(2) $$(x, m)[(y, n)(z, p)] \equiv$$

$$\equiv (xyz, m+k(x)n+k(x)k(y)p+k(x)l(y\mu(z, p), z)+l(x\mu((y, n)(z, p)), yz)).$$

We see from these expressions that the consideration of the equality

$$[(x, m)(y, n)](z, p) \equiv (x, m)[(y, n)(z, p)] \tag{3}$$

will depend on the triple

$$P(y, n, z, p) \equiv \big(\mu(y, n), \mu(z, p), \mu((y, n)(z, p))\big)$$

which will be referred to as the *pattern* of the couple $\big((y, n), (z, p)\big) \in P \times P$.

From the correctness conditions (C1) and (C2) of 3.1 it follows immediately

$$\mu(y, n) = e \Rightarrow \mu\big((y, n)(z, p)\big) = e. \tag{4}$$

Indeed, if $r(y) \neq 0$ then by (C2) also $r(yz) \neq 0$. If $r(y)=0$ and $n > i(y)$, then

$$(y, n)(z, p) \equiv \big(yz, n+k(y)p+l(y\mu(z, p), z)\big).$$

If $r(yz) \neq 0$ then there is nothing to prove. Supposing $r(yz)=0$ we have by (C1)

$$n+k(y)p+l\big(y\mu(z, p), z\big) > i(y)+k(y)i(z)+\min\{k(y)+l(ye, z), l(y, z)\} \geqq i(yz).$$

By (4) the number of possible patterns is reduced from eight to six listed as

$$P_1 = (e, e, e), \quad P_2 = (1, e, e), \quad P_3 = (1, e, 1),$$

$$P_4 = (e, 1, e), \quad P_5 = (1, 1, 1), \quad P_6 = (1, 1, e).$$

To each triple $\big((x, m), (y, n), (z, p)\big)$ of elements of $P$ we associate an equation (modulo $\equiv$)

$$C_j(x, y, z)\colon\ l(x\mu(y, n), y)+l(xy\mu(z, p), z) \equiv$$

$$\equiv k(x)l\big(y\mu(z, p), z\big)+l\big(x\mu((y, n)(z, p)), yz\big) \pmod{r(xyz)}$$

if the corresponding pattern is $P(y, n, z, p)=P_j$, $j=1, \ldots, 6$. Written in full, the six equations are

$$C_1\colon\ l(xe, y)+l(xye, z) \equiv k(x)l(ye, z)+l(xe, yz) \pmod{r(xyz)}$$

$$C_2\colon\ l(x, y)+l(xye, z) \equiv k(x)l(ye, z)+l(xe, yz) \pmod{r(xyz)}$$

$$C_3\colon\ l(x, y)+l(xye, z) \equiv k(x)l(ye, z)+l(x, yz) \pmod{r(xyz)}$$

$$C_4\colon\ l(xe, y)+l(xy, z) \equiv k(x)l(y, z)+l(xe, yz) \pmod{r(xyz)}$$

$$C_5\colon\ l(x, y)+l(xy, z) \equiv k(x)l(y, z)+l(x, yz) \pmod{r(xyz)}$$

$$C_6\colon\ l(x, y)+l(xy, z) \equiv k(x)l(y, z)+l(xe, yz) \pmod{r(xyz)}$$

Assume that $P$ fulfils (A0). Then we see from (1) and (2) that (3) holds iff the equation corresponding to $P(y, n, z, p)$ is true. Since $ez=z$ we get by (C2) that $r(z)$ divides $r(e)$ and (A0) implies $k(x)k(y) \equiv k(xy) \pmod{r(xyz)}$ for any $z \in S$.

Thus if one shows that the condition (A) of associativity of $P$ implies (A0) then we can express it as

(5) $\quad$ $(\mathrm{A}) \Leftrightarrow (\mathrm{A0})$ and $\forall y, z, j\big(\exists n, p\big(P(y, n, z, p) = P_j\big) \Rightarrow \forall x\big(C_j(x, y, z)$ holds$\big)\big)$.

To show that (A) implies (A0), note that we can always choose $n, p \in Z$ to $y, z \in S$ so that $P(y, n, z, p) = P_1$, and that then also $P(y, n, z, p') = P_1$ for every $p' \geqq p$. Comparing (1) and (2) for $p$ and $p$ replaced by $p+1$ we get immediately (A0), as well as (A1). It remains to consider the situations under which the patterns $P_2, \ldots, P_6$ occur, in order to get (A2)—(A6). The scheme for (A$j$), $j=2, \ldots, 5$, is

$$\exists n, p\big(P(y, n, z, p) = P_j\big) \Rightarrow \forall x\big(C'_j(x, y, z) \text{ holds}\big)$$

where $C'_5 = C_5$ and $C'_j = C_1 - C_j$ (this is meant to symbolize that $C'_j$ is obtained by subtracting $C_j$ from the always true equation $C_1$, hence $C'_j$ is equivalent to $C_j$, but somewhat simpler) for $j=2, 3, 4$.

($j=2$): We have $P(y, n, z, p) = P_2$ iff $\big(r(y)=0$ and $n=i(y)\big)$ and $\big(r(z)=0 \Rightarrow p > i(z)\big)$ and $\big(r(yz)=0 \Rightarrow i(y)+k(y)p+l(ye, z) > i(yz)\big)$, therefore it follows that $\exists n, p\,\big(P(y, n, z, p)=P_2\big) \Leftrightarrow r(y)=0$ and $\big(r(yz)=0=k(y) \Rightarrow i(y)+l(ye, z) > i(yz)\big)$ since for $r(yz)=0 \neq k(y)$ we have by (C1) that $k(y)>0$ and $r(z)=0$, hence for some $p, p>i(z)$, it is $i(y)+k(y)p+l(ye, z) > i(yz)$.

($j=3$): $P(y, n, z, p) = P_3 \Leftrightarrow \big(r(y)=0$ and $n=i(y)\big)$ and $\big(r(z)=0 \Rightarrow p > i(z)\big)$ and $\big(r(yz)=0$ and $i(y)+k(y)p+l(ye, z) = i(yz)\big)$, therefore $\exists n, p\,\big(P(y, n, z, p) = P_3\big) \Leftrightarrow \big(r(y)=0=r(yz)\big)$ and $\big(k(y)=0 \Rightarrow i(y)+l(ye, z)=i(yz)\big)$ and $\big(k(y)>0 \Rightarrow i(y)+k(y)\big(i(z)+1\big)+l(ye, z)=i(yz)\big)$ since for $k(y) \neq 0$ it is $r(z)=0$ by (C1) and $p=i(z)+1$ is the least possible choice for $p$ to get the pattern. Putting together, $\exists n, p\,\big(P(y, n, z, p)=P_3\big) \Leftrightarrow r(y)=0=r(yz)$ and $i(y)+k(y)\big(i(z)+1\big)+l(ye, z)=i(yz)$.

($j=4$): $P(y, n, z, p)=P_4 \Leftrightarrow \big(r(y)=0 \Rightarrow n > i(y)\big)$ and $\big(r(z)=0$ and $p=i(z)\big)$ and $\big(r(yz)=0 \Rightarrow n+k(y)i(z)+l(y, z) > i(yz)$, therefore $\exists n, p\big(P(y, n, z, p)=P_4\big) \Leftrightarrow r(z)=0$.

($j=5$): $P(y, n, z, p)=P_5 \Leftrightarrow \big(r(y)=0$ and $n=i(y)\big)$ and $\big(r(z)=0$ and $p=i(z)\big)$ and $\big(r(yz)=0$ and $i(y)+k(y)i(z)+l(y, z)=i(yz)\big)$, therefore $\exists n, p\,\big(P(y, n, z, p) = P_5\big) \Leftrightarrow r(y)=r(z)=r(yz)=0$ and $i(y)+k(y)i(z)+l(y, z)=i(yz)$.

To conclude the proof we show that (A1)—(A6) and $P(y, n, z, p)=P_6$ imply $C_6$, and that $C_6$ and (A1)—(A5) imply (A6). We have

$$P(y, n, z, p) = P_6 \Leftrightarrow \big(r(y) = 0 \text{ and } n = i(y)\big) \text{ and } \big(r(z) = 0 \text{ and }$$
$$p = i(z)\big) \text{ and } \big(r(yz) = 0 \Rightarrow i(y)+k(y)i(z)+l(y, z) > i(yz)\big).$$

If $r(yz)=0 \neq k(y)$ then $i(y)+l(ye, z) \geqq i(yz)$, thus by (C1), $P(y, n, z, p+2)=P_2$. Since $P(y, n+1, z, p)=P_4$, $C_2$ and $C_4$ hold by ($j=2$) and ($j=4$). Subtracting $C_1$ from $C_2+C_4$ we get $C_6$. If $r(yz)=0=k(y)$ and $i(y)+l(ye, z)=i(yz)$ then

$P(y, n, z, p+1)=P_3$, hence we get $C_3$ and $C_4$ by $(j=3)$ and $(j=4)$. Subtracting $C_1$ from $C_3+C_4$ we get

$C_6'$: $$l(x, y)+l(xy, z) \equiv k(x)l(y, z)+l(x, yz) \pmod{r(xyz)}$$

Now $C_6$ holds iff (A6) holds.

3.4. Corollary. *If $(S, e, r, i, k, l)$ is a $\mathcal{G}$-correct $\mathcal{I}$-scheme and $e$ is a two-sided identity of $S$ then $P$ is associative iff*

(i) $k(x)k(y) \equiv k(xy) \pmod{r(xy)}$,

(ii) $l(x, y)+l(xy, z) \equiv k(x)l(y, z)+l(x, yz) \pmod{r(xyz)}$.

Proof. Straightforward.

3.5. Remark. Statements 3.1, 3.2 and 3.3 jointly characterize $\mathcal{S}^\circ$-correct $\mathcal{I}$-schemes by conditions (C1)—(C3), (IT1)—(IT3), (A0)—(A5). It is easy to satisfy all these conditions by a particular choice, e.g., $k(x)\equiv 1 \pmod{r(e)}$ for all $x\in S$, $l(x, y)=0=i(x)$ for all $x, y\in S$ and $r(x)=r\in\mathcal{N}$ for all $x\in S$, however, we are far from being able to describe all $\mathcal{S}^\circ$-correct $\mathcal{I}$-schemes, even under the tremendous simplification indicated by 3.4.

## 4. Universality of $\mathcal{I}$-schemes for $\mathcal{S}^\circ$

Let $a$ be a left cancellable element of a semigroup $T$. In order to facilitate some of the further calculations, we introduce partial injections $T\to T$: $u\mapsto a^m\circ u$, $m\in Z$, as follows

$$a^m\circ u = \begin{cases} a^m u & \text{if } m>0,\\ u & \text{if } m=0,\\ v & \text{if } m<0 \text{ and } a^{-m}v=u,\\ \emptyset & \text{if } m<0 \text{ and there is no } v \text{ in } T \text{ with } a^{-m}v=u.\end{cases}$$

The following lemma states some easy calculation rules.

4.1. Lemma. *If $a^n\circ u\neq\emptyset$, then for any $v\in T$ and $m\in Z$ it holds*

(a) $a^n\circ(uv)=(a^n\circ u)v$,

(b) $a^m\circ(a^n\circ u)=a^{m+n}\circ u$.

Proof. (a) is clear for $n\geqq 0$. Assume $n<0$ and denote $w=a^n\circ u$. Then $a^{-n}w=u$, hence $a^{-n}wv=uv$, hence $a^n\circ(uv)=wv=(a^n\circ u)v$.

(b) clearly holds if $m\geqq 0$ and $n\geqq 0$, as well as in the case $m=0$ or $n=0$. We shall consider in detail the three remaining cases.

Assume $m>0$ and $n<0$, denote $w=a^n\circ u$. If $m+n>0$, then $a^m\circ(a^n\circ u)=$ $=a^m w=a^{m+n}a^{-n}w=a^{m+n}u=a^{m+n}\circ u$. If $m+n<0$, then $u=a^{-n}w=a^{-n-m}a^m w$, hence $a^{m+n}\circ u=a^m w=a^m\circ(a^n\circ u)$. If $m+n=0$, then $a^m\circ(a^n\circ u)=a^m w=a^{-n}w=$ $=u=a^{m+n}\circ u$.

Assume $m<0$ and $n>0$. If $m+n>0$, then $a^{-m}a^{m+n}u=a^n u$, therefore $a^m\circ(a^n\circ u)=a^m\circ(a^n u)=a^{m+n}u=a^{m+n}\circ u$. If $m+n<0$ and $a^{m+n}\circ u\neq\emptyset$, say $a^{m+n}\circ u=v$, then $a^{-m-n}v=u$, hence $a^{-m}v=a^n u$, hence $a^{m+n}\circ u=v=a^m\circ(a^n u)=$ $=a^m\circ(a^n\circ u)$. If $m+n<0$ and $a^m\circ(a^n\circ u)\neq\emptyset$, say $a^m\circ(a^n\circ u)=a^m\circ(a^n u)=w$, then $a^{-m}w=a^n u$, hence $a^{-m-n}w=u$ by the $n$-fold cancellation of $a$ on the left, therefore $a^{m+n}\circ u=w=a^m\circ(a^n\circ u)$. If $m+n=0$, then $a^{-m}u=a^n u$, therefore $a^m\circ(a^n\circ u)=a^m\circ(a^n u)=u=a^{m+n}\circ u$.

Assume $m<0$ and $n<0$. If $a^{m+n}\circ u\neq\emptyset$, say $a^{m+n}\circ u=v$, then $a^{-m-n}v=$ $=a^{-n}a^{-m}v=u$, hence $a^n\circ u=a^{-m}v$ and $a^m\circ(a^n\circ u)=v=a^{m+n}\circ u$. If $a^m\circ(a^n\circ u)\neq$ $\neq\emptyset$, say $a^m\circ(a^n\circ u)=w$, then $a^{-m}w=a^n\circ u$, hence $a^{-n}a^{-m}w=u=a^{-n-m}w$ and $a^{m+n}\circ u=w=a^m\circ(a^n\circ u)$.

4.2. Theorem. *Every $\mathcal{I}$-extension $\varphi\colon (T,a)\to(S,e)$ in $\mathcal{S}^\circ$ is right equivalent with some semiproduct $\pi\colon (P,a)\to(S,e)\colon (x,m)\mapsto x$ determined by a suitable $\mathcal{S}^\circ$-correct $\mathcal{I}$-scheme $(S,e,r,i,k,l)$.*

Proof. Let $\varphi\colon (T,a)\to(S,e)$ be an $\mathcal{I}$-extension in $\mathcal{S}^\circ$. Then we have, for any $u,v\in T$,

(1) $$\varphi(u)=\varphi(v)\Leftrightarrow \exists m\in Z\quad (a^m\circ u=v).$$

The type function $r\colon S\to\mathcal{N}\colon x\mapsto \mathrm{Type}(f_x)$, $f_x\colon \varphi^{-1}(x)\to\varphi^{-1}(x)\colon u\mapsto au$, associated with this $\mathcal{I}$-extension satisfies by 2.2,

(2) $$r(xy) \text{ divides } r(x) \text{ in } \mathcal{N} \text{ for any } x,y\in S$$

since $\mathcal{U}_x$ is taken homomorphically to $\mathcal{U}_{xy}$ by the multiplication on the right by any element $w\in\varphi^{-1}(y)$.

Further we construct a mapping $\psi\colon S\to T$ (not necessarily a homomorphism) selecting one point from each $a$-component, i.e., $\varphi\psi=1_S$, as follows:

(3) $$\text{if } r(x)=0, \text{ then } a^{-1}\circ\psi(x)=\emptyset,$$

(4) $$\text{if } r(x)\neq 0 \text{ and } xe=x, \text{ then } \psi(x)\in\varphi^{-1}(x) \text{ is arbitrary,}$$

(5) $$\text{if } r(x)\neq 0 \text{ and } xe\neq x, \text{ then } \psi(x)a=\psi(xe)a,$$

(this is possible since multiplication by $a$ on the right takes $\varphi^{-1}(x)$ onto $\varphi^{-1}(xe)$, as it follows, e.g., from 2.2).

Define a function $i\colon S\to Z$ by

(6) $$a^{i(x)}\circ\psi(xe)a=\psi(x)a \text{ and } \big(r(xe)\neq 0,\ \infty\Rightarrow 0\leqq i(x)<r(xe)\big).$$

It follows from (4) and (5) that

(7) $$i(x)=0 \quad \text{if} \quad r(x)\neq 0, \quad \text{or} \quad r(x)=0 \quad \text{and} \quad x=xe.$$

A localizer $\lambda\colon T\to Z$ is defined by

(8) $$a^{\lambda(u)-i(\varphi(u))}\circ\psi\varphi(u)=u \quad \text{and} \quad \big(r(\varphi(u))\neq 0, \infty \Rightarrow \Rightarrow 0\leqq \lambda(u)-i(\varphi(u))<r(\varphi(u))\big).$$

A function $k\colon S\to Z$ is defined by

(9) $$a^{k(x)}\circ\psi(xe)=\psi(xe)a \quad \text{and} \quad \big(r(xe)\neq 0, \infty \Rightarrow 0\leqq k(x)<r(xe)\big).$$

Replacing $x$ by $xe$ in (9), we get $a^{k(xe)}\circ\psi(xe)=\psi(xe)a$, therefore

(10) $$k(xe)\equiv k(x) \pmod{r(xe)}.$$

Finally, a function $l\colon S\times S\to Z$ is defined by

(11) $$l(x,y)=\lambda(\psi(x)\psi(y))-i(x)-k(x)i(y).$$

We shall show that

(12) $$\text{if} \quad a^{-1}\circ\psi(y)\neq\emptyset, \quad \text{then} \quad l(xe,y)\equiv l(x,y) \pmod{r(xy)}.$$

Indeed, we have then $\psi(y)=au$, for some $u\in T$, furthermore, by (a) and (6)

$$a^{i(x)}\circ\psi(xe)\psi(y)=a^{i(x)}\circ\psi(xe)au=(a^{i(x)}\circ\psi(xe)a)u=\psi(x)au=\psi(x)\psi(y),$$

therefore

$$\lambda(\psi(x)\psi(y))\equiv\lambda(\psi(xe)\psi(y))+i(x) \pmod{r(xy)},$$

hence by (11)

$$i(x)+k(x)i(y)+l(x,y)\equiv i(xe)+k(xe)i(y)+l(xe,y)+i(x) \pmod{r(xy)}.$$

By (2), $r(xy)=r(xey)$ divides $r(xe)$, hence by (10), $k(xe)\equiv k(x) \pmod{r(xy)}$. By (7), $i(xe)=0$, hence we get (12).

Let $u, v$ be arbitrary elements of $T$ with $\varphi(u)=x$, $\varphi(v)=y$. We shall show that

(13) $$\lambda(uv)=\begin{cases}\lambda(u)+k(x)\lambda(v)+l(xe,y) & \text{if} \quad a^{-1}\circ v\neq\emptyset,\\ \lambda(u)+k(x)\lambda(v)+l(x,y) & \text{if} \quad a^{-1}\circ v=\emptyset.\end{cases}$$

By (8),

(14) $$uv=\big(a^{\lambda(u)-i(x)}\circ\psi(x)\big)\big(a^{\lambda(v)-i(y)}\circ\psi(y)\big).$$

We split the consideration of this expression into three cases :

I. Assume $\lambda(v)>i(y)$. Then, taking into account (6) and (9) we have

$$uv=a^{\lambda(u)-i(x)}\circ\psi(x)a^{\lambda(v)-i(y)}\psi(y)=a^{\lambda(u)-i(x)}\circ\big(a^{i(x)}\circ\psi(xe)a^{\lambda(v)-i(y)}\psi(y)\big)=$$
$$=a^{\lambda(u)+k(x)(\lambda(v)-i(y))}\circ\psi(xe)\psi(y).$$

By (8), $\psi(xe)\psi(y) = a^{\lambda(\psi(xe)\psi(y))-i(xy)}\circ\psi(xy)$, hence

$$uv = a^{\lambda(u)+k(x)(\lambda(v)-i(y))+\lambda(\psi(xe)\psi(y))-i(xy)}\circ\psi(xy).$$

From this we get by (8),

$$\lambda(uv) \equiv \lambda(u)+k(x)\lambda(v)-k(x)i(y)+\lambda\big(\psi(xe)\psi(y)\big) \pmod{r(xy)},$$

hence by (11),

$$\lambda(uv) \equiv \lambda(u)+k(x)\lambda(v)-k(x)i(y)+i(xe)+k(xe)i(y)+l(xe, y) \pmod{r(xy)},$$

hence by (7) and (10) we get finally

$$\lambda(uv) \equiv \lambda(u)+k(x)\lambda(v)+l(xe, y) \pmod{r(xy)}.$$

Since $\lambda(v)>i(y)$ means that $a^{-1}\circ v\neq\emptyset$, we have proved (13) under the assumption.

II. Assume $\lambda(v)=i(y)$. Then (14) becomes

$$uv = a^{\lambda(u)-i(x)}\circ\psi(x)\psi(y) = a^{\lambda(u)-i(x)+\lambda(\psi(x)\psi(y))-i(xy)}\circ\psi(xy),$$

hence by (8) and (11),

$$\lambda(uv) \equiv \lambda(u)-i(x)+i(x)+k(x)i(y)+l(x, y) \pmod{r(xy)}, \tag{15}$$

which proves (13) in case $a^{-1}\circ v=\emptyset$. However, if $a^{-1}\circ v\neq\emptyset$, then by (12), $l(x, y)\equiv \equiv l(xe, y) \pmod{r(xy)}$ and (15) gives (13) also in this case.

III. Assume $\lambda(v)<i(y)$. Then by (3) and (8), $r(y)=\infty$, hence by (7), $i(y)=0$, and by (12), $l(x, y)\equiv l(xe, y) \pmod{r(xy)}$. By (9) we get

$$a^{k(x)}ua = a^{k(x)}a^{\lambda(ua)-i(xe)}\circ\psi(\varphi(ua))=$$
$$=a^{\lambda(ua)-i(xe)}a^{k(x)}\circ\psi(xe)=a^{\lambda(ua)-i(xe)}\circ\psi(xe)a = uaa,$$

hence for all $n\in Z$, $ua^n=a^{-k(x)\lambda(v)}ua^{n+\lambda(v)}$. Now an easy calculation yields

$$a^{-k(x)\lambda(v)}\circ uv = a^{-k(x)\lambda(v)}\circ u\big(a\circ(a^{-1}\circ v)\big) = ua^{-\lambda(v)}\circ v = u\psi(y) =$$
$$= a^{\lambda(u)-i(x)}\circ\psi(x)\psi(y) = a^{\lambda(u)-i(x)+\lambda(\psi(x)\psi(y))-i(xy)}\circ\psi(xy),$$

hence

$$\lambda(uv) \equiv \lambda(u)-i(x)+k(x)\lambda(v)+i(x)+k(x)i(y)+l(x, y) \equiv$$
$$\equiv \lambda(u)+k(x)\lambda(v)+l(xe, y) \pmod{r(xy)},$$

which proves (13) under $\lambda(v)<i(y)$.

We have $\mathrm{Type}(f_x)=r(x)$, hence by (3), (4), (5), (6), and (8), the assignment $u\mapsto(\varphi(u), \lambda(u))$ is a bijection establishing by (13) a right equivalence between $\varphi:(T, a)\to(S, e)$ and the semiproduct $\pi:(P, a)\to(S, e)$ determined by $(S, e, r, i, k, l)$, whence the latter must be in $\mathscr{S}^{\circ}$.

## 5. Morphisms of $\mathcal{I}$-extensions

Let $\pi\colon (P, a)\to(S, e)$ and $\pi'\colon (P', a')\to(S', e')$ be two $\mathcal{I}$-extensions in $\mathcal{S}^\circ$ determined by two $\mathcal{I}$-schemes $(S, e, r, i, k, l)$ and $(S', e', r', i', k', l')$, respectively. By a morphism of $\pi$ into $\pi'$ we understand a couple $(h, \chi)$ of $\mathcal{S}^\circ$-homomorphisms $h\colon (P, a)\to(P', a')$, $\chi\colon (S, e)\to(S', e')$ making the diagram

$$\begin{array}{ccc} (P,a) & \xrightarrow{\pi} & (S,e) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \chi} \\ (P',a') & \xrightarrow[\pi']{} & (S',e') \end{array}$$

commutative. A morphism $(h, \chi)\colon \pi\to\pi'$ is injective (surjective, bijective) iff both $h$ and $\chi$ are so.

For a given $\mathcal{S}^\circ$-homomorphism $\chi\colon (S, e)\to(S', e')$ we shall try to find "companion" $\mathcal{S}^\circ$-homomorphisms $h\colon (P, a)\to(P', a')$ such that $(h, \chi)$ is a morphism of $\pi$ into $\pi'$. If such an $h$ exists then it can always be expressed in the form $h=h_p$,

(1) $$h_p(x, m) \equiv \big(\chi(x), m+p(x)\big), \quad (x, m)\in P,$$

with the aid of a suitable "parameter" function $p\colon S\to Z$. The next theorem relates the properties of possible parameter functions to $\chi$ and the two $\mathcal{I}$-schemes.

5.1. Theorem. *Let $\chi\colon (S, e)\to(S', e')$ be an $\mathcal{S}^\circ$-morphism. Then $p\colon S\to Z$ determines a mapping $h_p\colon P\to P'$ by* (1), *such that $(h_p, \chi)$ is a morphism of the $\mathcal{I}$-extension $\pi\colon (P, a)\to(S, e)$ determined by $(S, e, r, i, k, l)$ into the extension $\pi'\colon (P', a')\to(S', e')$ determined by $(S', e', r', i', k', l')$, iff the following conditions are satisfied for all $x, y\in S$:*

(H1) $r'(\chi(x))$ *divides* $r(x)$ *in* $\mathcal{N}$,
(H2) $r'(\chi(x))\equiv 0\Rightarrow i(x)+p(x)\geqq i'(\chi(x))$,
(H3) $k(x)\equiv k'(\chi(x)) \pmod{r'(\chi(x))}$,
(H4) $l(xe, y)+p(xy)\equiv p(x)+k(x)p(y)+l'(\chi(xe), \chi(y)) \pmod{r'(\chi(xy))}$,
(H5) *if* $r(y)=0=r'(\chi(y))$ *and* $i(y)+p(y)=i'(\chi(y))$, *then* $l(xe, y)-l(x, y)\equiv$ $\equiv l'(\chi(xe), \chi(y))-l'(\chi(x), \chi(y)) \pmod{r'(\chi(xy))}$,
(H6) *if* $r(y)=0$ *and* $\big(r'(\chi(y))\neq 0$ *or* $i(y)+p(y)\neq i'(\chi(y))\big)$, *then* $l(xe, y)\equiv$ $\equiv l(x, y) \pmod{r'(\chi(xy))}$.

*Moreover, $(h_p, \chi)$ is injective iff $\chi$ is so and*

(H7) $r(x)=r'(\chi(x))$ *for* $r(x)\neq 0$, $r'(\chi(x))=0$ *or* $\infty$ *for* $r(x)=0$,

*and $(h_p, \chi)$ is surjective iff $\chi$ is so and*

(H8) *if* $r(x)=0$ *then* $r'(\chi(x))=0$ *and* $i(x)+p(x)=i'(\chi(x))$.

Proof. Clearly, the formula (1) defines a function $h_p$ from $P$ to $P'$ iff both (H1) and (H2) hold. Hence we shall further assume that $h_p\colon P\to P'$ is a mapping.

It is also clear from (1) that the couple $(h_p, \chi)$ is a morphism of $\pi$ into $\pi'$ iff $h_p\colon P\to P'$ is an $\mathcal{S}$-homomorphism, that is

$$h_p((x, m)(y, n)) \equiv h_p(x, m)h_p(y, n) \quad \text{for all} \quad (x, m), (y, n)\in P. \tag{2}$$

So (to prove the theorem) we have to prove (2) is equivalent with (H3)—(H6).

Assume (2). If $n>i(y)$, $n+p(y)>i'(\chi(y))$, then (2) becomes

$$\begin{aligned}(\chi(xy), m+k(x)n+l(xe, y)+p(xy)) \equiv \\ \equiv (\chi(x)\chi(y), m+p(x)+k'(\chi(x))(n+p(y))+l'(\chi(x)e', \chi(y))).\end{aligned} \tag{3}$$

If we compare (3) as it is and (3) with $n$ replaced by $n+1$, we get (H3); (H3) and (3) imply (H4). If $r(y)=0=r'(\chi(y))$ and $i(y)+p(y)=i'(\chi(y))$, then (2) becomes, for $n=i(y)$,

$$\begin{aligned}(\chi(xy), m+k(x)n+l(x, y)+p(xy)) \equiv \\ \equiv (\chi(x)\chi(y), m+p(x)+k'(\chi(x))(n+p(y))+l'(\chi(x), \chi(y))),\end{aligned} \tag{4}$$

hence by (H3),

$$l(x, y)+p(xy) \equiv p(x)+k(x)p(y)+l'(\chi(x), \chi(y)) \pmod{r'(\chi(xy))}.$$

Subtracting this equality from (H4) we get the equality of (H5). If $r(y)=0$, but $r'(\chi(y))\neq 0$ or $i(y)+p(y)\neq i'(\chi(y))$, then for $n=i(y)$, (2) becomes

$$\begin{aligned}(\chi(xy), m+k(x)n+l(x, y)+p(xy)) \equiv \\ \equiv (\chi(x)\chi(y), m+p(x)+k'(\chi(x))(n+p(y))+l'(\chi(x)e', \chi(y))),\end{aligned} \tag{5}$$

hence

$$l(x, y)+p(xy) \equiv p(x)+k(x)p(y)+l'(\chi(x)e', \chi(y)) \pmod{r'(\chi(xy))},$$

which together with (H4) yields the equation of (H6).

To prove (2) from (H3)—(H6), we have to consider four cases:

(6) $(r(y)\neq 0$ or $n>i(y))$ and $(r'(\chi(y))\neq 0$ or $n+p(y)>i'(\chi(y)))$,

(7) $(r(y)=0$ and $n=i(y))$ and $(r'(\chi(y))=0$ and $n+p(y)=i'(\chi(y)))$,

(8) $(r(y)=0$ and $n=i(y))$ and $(r'(\chi(y))\neq 0$ or $n+p(y)>i'(\chi(y)))$,

(9) $(r(y)\neq 0$ or $n>i(y))$ and $(r'(\chi(y))=0$ and $n+p(y)=i'(\chi(y)))$.

Now, under (6), (2) is equivalent to (3) and the latter follows from (H3) and (H4). Under (7), (2) is equivalent to (4), while this follows from (H3), (H4), and (H5). Under (8), (2) is equivalent to (5), and this follows from (H3), (H4), and (H6). Case

(9) cannot occur since by (H1), $r'(\chi(y))=0$ implies $r(y)=0$, hence $n>i(y)$, while by (H2), $n+p(y)>i(y)+p(y)\geqq i'(\chi(y))$.

The rest of the theorem concerning injectiveness and surjectiveness of $(h_p, \chi)$ is obvious.

Let us call two $\mathcal{I}$-schemes $(S, e, r, i, k, l)$ and $(S', e', r', i', k', l')$ *$\mathcal{S}^\circ$-equivalent* if they yield equivalent $\mathcal{I}$-extensions in $\mathcal{S}^\circ$. The above theorem has the following straightforward

5.2. Corollary. *Two $\mathcal{I}$-schemes $(S, e, r, i, k, l)$ and $(S', e', r', i', k', l')$ are $\mathcal{S}^\circ$-equivalent iff there exist an $\mathcal{S}^\circ$-isomorphism $\chi: (S, e)\to(S', e')$ and a function $p: S\to Z$, such that*

(E1) $r(x) = r'(\chi(x))$,
(E2) $r(x) = 0 \Rightarrow i(x)+p(x) = i'(\chi(x))$,
(E3) $k(x) \equiv k'(\chi(x)) \pmod{r(x)}$,
(E4) $l(xe, y)+p(xy) \equiv p(x)+k(x)p(y)+l'(\chi(xe), \chi(y)) \pmod{r(xy)}$,
(E5) $r(y) = 0 \Rightarrow l(xe, y)-l(x, y) \equiv l'(\chi(xe), \chi(y))-l'(\chi(x), \chi(y)) \pmod{r(xy)}$.

Proof. We get (E1) and (E2) by replacing (H1) and (H2) by the stronger (H7) and (H8), (E3)—(E5) are obvious modifications of (H3)—(H5), and a version of (H6) is omitted since its assumption cannot occur here.

## 6. Special properties of $\mathcal{I}$-extensions

6.1. Statement. *An $\mathcal{I}$-scheme $(S, e, r, i, k, l)$ determines an $\mathcal{I}$-extension $\pi: (P, a)\to(S, e)$ in the category of semigroups with identity iff*

(M1) *$S$ is a semigroup with identity $e$,*
(M2) *if $r(xy)=0$ then $k(x)\geqq 0$ and $k(x)r(y)=0$ and $i(x)+k(x)i(y)+l(x, y) \geqq i(xy)$,*
(M3) *$r(xy)$ divides $r(x)$ in $\mathcal{N}$,*
(M4) *if $r(y) \neq \infty$ then $k(x)r(y) \equiv 0 \pmod{r(xy)}$,*
(M5) $k(e) \equiv 1 \pmod{r(e)}$,
(M6) $l(e, x) \equiv l(e, e) \pmod{r(x)}$,
(M7) *if $r(e) = 0$, then $i(e)= -l(e, e)$,*
(M8) $k(x)k(y) \equiv k(xy) \pmod{r(xy)}$,
(M9) $l(x, y)+l(xy, z) \equiv k(x)l(y, z)+l(x, yz) \pmod{r(xyz)}$.

Proof. The condition (M1) is clearly necessary. If it is satisfied, then (M2)—(M4) restates (C1)—(C3) of 3.1 (slightly simplified by (M1)), (M5), (M6) are exactly (IT1), (IT2) of 3.2. The condition (M7) is stronger than (IT3) of 3.2, its necessity follows from the fact that the identity $(e, m)$ of $P$ must be $(e, i(e))$ in the case $r(e)=0$. Indeed, we have $a(e, m)\equiv a$, where $a\equiv(e, 1-l(e, e))$ by 3.2, whence $m=-l(e, e)$, thus $i(e)\leqq -l(e, e)$. Now, if $(e, -l(e, e))$ is not the "endpoint" $(e, i(e))$, then $i(e)< -l(e, e)$ and there is an element $b=(e, -l(e, e)-1)$ in $P$. But then also $b^n=(e, -l(e, e)-n)$ is in $P$ for all $n\geqq 1$, which is a contradiction with $r(e)=0$.

The conditions (M8), (M9) are exactly (i), (ii) of 3.4. We have proved the necessity of (M1)—(M9).

Assume now (M1)—(M9) fulfilled. Our $\mathcal{I}$-scheme is then clearly $\mathcal{S}^0$-correct and the only thing we have to show is that $(e, -l(e, e))$ is the identity of $P$:

$$(e, -l(e, e))(x, m) \equiv (x, -l(e, e)+k(e)m+l(e, x)) \equiv (x, m)$$

by (M5) and (M6),

$$(x, m)\,(e, -l(e, e)) \equiv (x, m-k(x)l(e, e)+l(x, e)) \equiv (x, m)$$

since $l(x, e)\equiv k(x)l(e, e)\ (\mathrm{mod}\ r(x))$ by (M9).

6.2. Remark. An $\mathcal{S}^0$-correct $\mathcal{I}$-scheme $(S, e, r, i, k, l)$ determines $\pi\colon (P, a)\to(S, e)$ with $(P, a)$ finite iff $S$ is finite and $0<r(x)<\infty$ for all $x\in S$. If $P$ has an identity, then each $a$-component $\mathcal{U}_x$ of type $r(x)$, $x\in S$, decomposes into cycles of length $q(x)=\dfrac{r(x)}{k(x)\wedge r(x)}$ of the right inner translation of $P$ by $a$. The integer $r(x)$ divides $k(x)q(x)$, hence $p(x)=\dfrac{k(x)q(x)}{r(x)}=\dfrac{k(x)}{k(x)\wedge r(x)}$ is an integer and $p(x)\wedge q(x)=\dfrac{k(x)}{k(x)\wedge r(x)}\wedge\dfrac{r(x)}{k(x)\wedge r(x)}=1$. If, for some $y\in S$, we have $q(y)=q(x)$ and $xy=y$ or $yx=y$, then $k(x)k(y)\equiv k(y)\ (\mathrm{mod}\ r(y))$, and using $k(x)=\dfrac{p(x)r(x)}{q(x)}$, we get $k(x)k(y)-k(y)=\dfrac{p(x)r(x)}{q(x)}\cdot\dfrac{p(y)r(y)}{q(y)}-\dfrac{p(y)r(y)}{q(y)}=p(y)r(y)\times \times\left[\dfrac{p(x)r(x)-q(x)}{q^2(x)}\right]\equiv 0\ (\mathrm{mod}\ r(y))$. It follows that $p(y)\left[\dfrac{p(x)r(x)-q(x)}{q(x)}\right]$ is an integer, and since $p(y)\wedge q(y)=1$, $\dfrac{p(x)r(x)-q(x)}{q^2(x)}$ is an integer, too.

Conversely, if $\dfrac{p(x)r(x)-q(x)}{q^2(x)}$ is an integer, then for arbitrary $r(y)$, $p(y)$ with $p(y)\wedge q(y)=1$, and $k(y)=\dfrac{p(y)r(y)}{q(x)}$ it holds

$$k(x)k(y)-k(y) = p(y)r(y)\left[\frac{p(x)r(x)-q(x)}{q^2(x)}\right]\equiv 0\ (\mathrm{mod}\ r(y)).$$

This property has been used in [3] and [4] for a special kind of constructions of monoids, starting with $S$ a left zero semigroup with an identity 1 adjoined, $q\in N^+$ and $r, p\colon S\to N^+$ ($N^+$ means the positive integers) such that $r(x)$ divides $r(1)$, $q$ divides $r(x)$, $q\wedge p(x)=1$, and $\dfrac{p(x)r(x)-q}{q^2}$ is an integer, for all $x\in S$. These constructions amount to those of $\mathcal{I}$-extensions determined by $(S, e, r, i, k, l)$ with $e=1$, $i(x)=0$, $k(x)=\dfrac{p(x)r(x)}{q}$ and $l(x, y)=0$, for all $x, y\in S$.

6.3. Statement. *An $\mathcal{S}^{\circ}$-correct $\mathcal{I}$-scheme $(S, e, r, i, k, l)$ determines an $\mathcal{I}$-extension $\pi\colon (P, a)\to(S, e)$ with $(P, a)$ commutative iff*

(AB1) *$S$ is commutative,*
(AB2) $l(x, y)\equiv l(y, x) \pmod{r(xy)}$,
(AB3) $k(x)\equiv 1 \pmod{r(x)}$.

Proof. If (AB1)—(AB3) hold, then $e$ is an identity in $S$ and the two products

$$(1)\qquad (x, m)(y, n)\equiv \big(xy, m+k(x)n+l(x, y)\big)$$

$$(2)\qquad (y, n)(x, m)\equiv \big(yx, n+k(y)m+l(y, x)\big)$$

are equal for any $(x, m), (y, n)\in P$.

Conversely, if (1) and (2) are equal for any $(x, m), (y, n)\in P$, then (AB1) holds,

$$(3)\qquad m+k(x)n+l(x, y)\equiv n+k(y)m+l(y, x) \pmod{r(xy)},$$

and also, replacing $n$ by $n+1$,

$$(4)\qquad m+k(x)n+k(x)+l(x, y)\equiv n+1+k(y)m+l(y, x) \pmod{r(xy)},$$

hence subtracting (3) from (4) we get $k(x)\equiv 1 \pmod{r(xy)}$, for any $y\in S$, which is equivalent to (AB3). By (3) and (AB3) we get (AB2).

6.4. Statement. *Let $(S, e, r, i, k, l)$ be an $\mathcal{S}^{\circ}$-correct $\mathcal{I}$-scheme. The semigroup $(P, a)=\mathcal{I}(S, e, r, i, k, l)$ is right cancellative iff*

(RC1) *$S$ is right cancellative,*
(RC2) $\big(r(x)\neq 0\Rightarrow r(xy)=r(x)\big)$ *and* $\big(r(x)=0\Rightarrow r(xy)=0$ *or* $\infty\big)$.

Proof. (RC2) means that right inner translations of $P$ take each $a$-component $\mathcal{U}_x$ into another component injectively, (RC1) ensures that distinct $a$-components are taken to distinct $a$-components.

6.5. Statement. *For an $\mathcal{S}^\circ$-correct $\mathcal{I}$-scheme $(S, e, r, i, k, l)$, the semigroup $(P, a)=\mathcal{I}(S, e, r, i, k, l)$ is left cancellative iff*

(LC1) *$S$ is left cancellative,*
(LC2) *if some $r(y) \neq 1$, then $k(x) \neq 0$ for all $x \in S$,*
(LC3) *if $0 < r(xy) < \infty$, then $k(x)r(y) = k(x) \vee r(xy)$,*
(LC4) *if $r(y) = 0$, then for every $n \neq i(y)$,*

$$k(x)i(y)+l(x, y) \not\equiv k(x)n+l(xe, y) \pmod{r(xy)}.$$

Proof. If $P$ is left cancellative, then so is its quotient $S$. Under (LC1), $P$ is left cancellative iff

(5) $$(x, m)(y, n_1) \equiv (x, m)(y, n_2) \Rightarrow (y, n_1) \equiv (y, n_2).$$

Under the assumption that

(6) $$r(y) \neq 0 \quad \text{or} \quad n_1 \neq i(y) \neq n_2,$$

(5) is equivalent to

(7) $$k(x)(n_1-n_2) \equiv 0 \pmod{r(xy)} \Rightarrow (n_1-n_2) \equiv 0 \pmod{r(y)}.$$

Since the difference $n_1-n_2$ ranges over the whole $Z$, (5) is equivalent to

(8) $$k(x)n \equiv 0 \pmod{r(xy)} \Rightarrow n \equiv 0 \pmod{r(y)},$$

for all $n \in Z$.

We shall prove (8) to be equivalent to the conjunction of (LC2) and (LC3).

Assume (8). If $k(x)=0$ for some $x \in S$, then from (8) it follows that $r(y)=1$ for all $y \in S$, thus (LC2) holds.

Of course, if $r(y)=1$ for all $y \in S$, the $\mathcal{I}$-extension is improper, $P \cong S$, so we further exclude this case from our consideration. If $0<r(xy)<\infty$, then also $0<r(y)<\infty$, since under $r(y)=0$ or $\infty$ it would be $n \equiv 0 \pmod{r(y)}$ iff $n=0$, while $k(x)n \equiv 0 \pmod{r(xy)}$ for $n=r(xy)\neq 0$, a contradiction to (8). By (C3) of 3.1, $r(xy)$ divides $k(x)r(y)$, hence $k(x)\vee r(xy)$ divides $k(x)r(y)$. If $\frac{k(x)r(y)}{k(x)\vee r(xy)}=p\neq 1$, then $r(y)$ does not divide $n=\frac{r(y)}{p}=\frac{r(xy)}{k(x)\wedge r(xy)}$, while $r(xy)$ divides $k(x)n=\frac{k(x)r(xy)}{k(x)\wedge r(xy)}$, again a contradiction to (8). Thus (8) implies (LC3).

Conversely, assume (LC2) and (LC3). If $k(x)=0$ then by (LC2) we have $r(y)=1$ for every $y \in S$ and $(P, a)$ is isomorphic to $(S, e)$, and by (LC1), $P$ is left-cancellative. Let us therefore assume that $k(x)\neq 0$ for every $x \in S$. If $r(xy)=0$ or $\infty$ then $k(x)n\equiv 0 \pmod{r(xy)}$ iff $n=0$, hence we get $n\equiv 0 \pmod{r(y)}$ trivially. If $0<r(xy)<\infty$ then by (LC3), $k(x)r(y)=\frac{k(x)r(xy)}{k(x)\wedge r(xy)}$, whence $r(xy)=$

$=r(y)(k(x)\wedge r(xy))$. Thus $r(xy)$ divides $k(x)n$ iff $r(y)$ divides $\frac{k(x)}{k(x)\wedge r(xy)}n$. Since $r(y)\wedge\frac{k(x)}{k(x)\wedge r(xy)}=\frac{r(xy)}{k(x)\wedge r(xy)}\wedge\frac{k(x)}{k(x)\wedge r(xy)}=1$, $r(y)$ dividing $\frac{k(x)n}{k(x)r(xy)}$ must divide $n$. Assuming $r(y)=0$ and $i(y)=n_1$ or $n_2$, we have that (5) is equivalent to $n\neq i(y)\Rightarrow(x,m)(y,n)\not\equiv(x,m)(y,i(y))$, which is equivalent to (LC4).

Recall that a semigroup $P$ is called *right reductive* if, for any $x,y\in P$,

(RR) $$\forall z\in P\ (xz=yz)\Rightarrow x=y,$$

i.e., the family of the right inner translations of $P$ separates points.

6.6. Statement. *For an $\mathcal{S}^\circ$-correct $\mathcal{I}$-scheme $(S,e,r,i,k,l)$, the semigroup $(P,a)=\mathcal{I}(S,e,r,i,k,l)$ is right reductive iff*

(RR1) $(r(x)\neq 0\Rightarrow r(xe)=r(x))$ *and* $(r(x)=0\Rightarrow r(xe)=0$ or $\infty)$,

(RR2) *if* $x\neq y$, $xe=ye$, *and* $k(x)\equiv k(y)\pmod{r(x)}$ *for* $x,y\in S$, *then for every* $q\in Z$ *there exists* $z\in S$ *such that either*

$$l(x,z)-l(y,z)\not\equiv q\pmod{r(xz)}\quad\textit{and}\quad r(z)=0,$$

*or*

$$l(xe,z)-l(ye,z)\not\equiv q\pmod{r(xz)}.$$

Proof. Assume $P$ right reductive. If $(x,m_1)\not\equiv(x,m_2)$ then, for some $(z,p)\in P$, $(x,m_1)(z,p)\not\equiv(x,m_2)(z,p)$. Hence if $m_1\not\equiv m_2\pmod{r(x)}$, then also $m_1\not\equiv m_2\pmod{r(xz)}$, and since $r(xz)$ divides $r(xe)$, $m_1\not\equiv m_2\pmod{r(xe)}$. The condition (RR1) follows.

Let now $x\neq y$, $xe=ye$, and $k(x)\equiv k(y)\pmod{r(x)}$. It is $r(x)=r(y)$, by $xe=ye$ and (RR1). For any $q\in Z$, there are $(x,m)$, $(y,n)\in P$ with $n-m=q$. Since $(x,m)\not\equiv(y,n)$, there exists $(z,p)\in P$ such that $(x,m)(z,p)\not\equiv(y,n)(z,p)$. This means that either

$$q=n-m\not\equiv l(x,z)-l(y,z)\pmod{r(xz)}$$

or

$$q=n-m\not\equiv l(xe,z)-l(ye,z)\pmod{r(xz)},$$

according to whether or not, $r(z)=0$ and $p=i(z)$. This proves (RR2) necessary.

Assume now (RR1) and (RR2). Let $(x,m)\not\equiv(y,n)$. If $x=y$, then by (RR1), $(x,m)a\not\equiv(y,n)a$, where $a\equiv(e,1-l(e,e))$. If $x\neq y$ and $xe\neq ye$, then again $(x,m)a\not\equiv(y,n)a$. Let $x\neq y$, $xe=ye$, and assume that $(x,m)(z,p)\equiv(y,n)(z,p)$ for all $(z,p)\in P$. It follows that $n-m\equiv(k(x)-k(y))p+l(xe,z)-l(ye,z)\pmod{r(xz)}$ for all $p>i(z)$, hence $k(x)\equiv k(y)\pmod{r(xz)}$ for all $z\in S$. Therefore

$$n-m\equiv l(x,z)-l(y,z)\pmod{r(xz)},\quad\text{for}\quad r(z)=0\quad\text{and}\quad p=i(z),$$

and

$$n-m \equiv l(xe, z)-l(ye, z) \pmod{r(xz)} \quad \text{otherwise},$$

a contradiction to (RR2).

6.7. Statement. *For an $\mathcal{S}^\circ$-correct $\mathcal{I}$-scheme $(S, e, r, i, k, l)$, the semigroup $(P, a)=\mathcal{I}(S, e, r, i, k, l)$ is a group iff*

(G1) *$S$ is group,*
(G2) $r(e)\neq 0$.

Proof. The necessity of the conditions is obvious. Assume next (G1) and (G2). We show that $(e, -l(e, e))$ is an identity of $P$ in the same way as in the proof of 6.1. By (G1) and (C2) of 3.1, $r(x)=r(e)$ for all $x\in S$. The proof will be completed by showing that

$$(x, m)^{-1} \equiv (x^{-1}, -l(e, e)-k(x^{-1})m-l(x^{-1}, x)).$$

First, since by (G2) and (C2) we have $r(x)\neq 0$, there exists an element of this form in $P$. By (A0) and (IT1), $k(x)k(x^{-1})\equiv k(e)\equiv 1 \pmod{r(e)}$, and

$$(x, m)(x, m)^{-1} \equiv (e, m-k(x)l(e, e)-k(x)k(x^{-1})m-k(x)l(x^{-1}, x)+l(x, x^{-1})) \equiv$$
$$\equiv (e, -k(x)l(e, e)-k(x)l(x^{-1}, x)+l(x, x^{-1})) \equiv (e, -l(e, e)),$$

since by (A1) of 3.3,

$$l(x, e)\equiv k(x)l(e, e), \quad l(x, x^{-1})+l(e, x) \equiv k(x)l(x^{-1}, x)+l(x, e),$$

and thus by (IT2) of 3.2,

$$-l(x, e)-k(x)l(x^{-1}, x)+l(x, x^{-1}) \equiv -l(e, e).$$

Finally,

$$(x, m)^{-1}(x, m) \equiv (e, -l(e, e)-k(x^{-1})m-l(x^{-1}, x)+k(x^{-1})m+l(x^{-1}, x)) \equiv$$
$$\equiv (e, -l(e, e)).$$

## 7. Transextension in monoids factorize through $\mathcal{I}$-extensions

The aim of this concluding section is to show that every transextension in the category $\mathcal{M}^\circ$ of pointed semigroups with identity factorizes through some $\mathcal{I}$-extension and that there is a biggest one among such $\mathcal{I}$-extensions. Thus $\mathcal{I}$-extensions form, in this category, an important intermediary step in the constructions of transextensions, to be followed by an extension of a different kind using the decomposition of connected components of a translation into levels.

Unfortunately, we cannot do the same in $\mathscr{S}^\circ$, the assumption of the presence of an identity in $T$ cannot be relaxed as is shown by the example following the statement.

7.1. Statement. *Let $T$ be a semigroup with identity 1. If for some $a \in T$, the $a$-connectedness*

$$x \sim y \Leftrightarrow \exists m, n \in \mathcal{N} (a^m x = a^n y)$$

*is a congruence on $T$, then so is its refinement*

$$x \approx y \Leftrightarrow \exists m \in \mathcal{N} (a^m x = a^m y),$$

*and the quotient $(T/\approx, a/\approx)$ is the biggest $\mathscr{I}$-extension of $(T/\sim, a/\sim)$ extendable to $T$.*

Proof. Let $\sim$ be a congruence on $T$, let $x \in T$. From $1 \sim a$ we get $x \sim xa$, hence for some $k = k(x) \geqq 0$, we have either $a^k x \approx xa$ or $x \approx a^k xa$. We next show that if $a^k x \approx xa$ and $x \sim y$, then also $a^k y \approx ya$. Indeed, let us have, say, $a^p a^k x = a^p xa$ and $a^m x = a^n y$. Then $a^p a^n ya = a^p a^m xa = a^m a^p xa = a^m a^p a^k x = a^p a^k a^m x = a^p a^k a^n y$, hence $a^k y \approx ya$. Similarly we show that if $x \approx a^k xa$, then also $y \approx a^k ya$ for any $y \sim x$. For, if $a^p x = a^p a^k xa$ and $a^m x = a^n y$, then $a^p a^n y = a^p a^m x = a^m a^p x = a^m a^p a^k xa = a^p a^k a^n ya = a^p a^n a^k ya$, hence $y \approx a^k ya$.

The equivalence $\approx$, as well as $\sim$, is clearly a right congruence. We prove $\approx$ to be a left congruence as follows. Take some $z \in T$ and assume first $a^k z \approx za$, for some $k \geqq 0$. Then we have $a^{pk} z \approx za^p$ for all $p \geqq 0$. If $x \approx y$, say $a^p x = a^p y$, then $a^{pk} zx \approx za^p x = za^p y \approx a^{pk} zy$, hence $zx \approx zy$. Assume now $z \approx a^k za$, for some $k \geqq 0$. Then $z \approx a^{pk} za^p$ for all $p \geqq 0$. Again, if $x \approx y$, say $a^p x = a^p y$, then $zx \approx a^{pk} za^p x = a^{pk} za^p y \approx zy$.

The last assertion of the statement follows from the fact that $\approx$ is obviously the least congruence on $T$ making $a/\approx$ left cancellable in $T/\approx$.

7.2. Example. Consider the semigroup $S$

| | $a$ | $b$ | $c$ | $d$ | $f$ | $g$ | $h$ | $k$ | $m$ | $n$ | $p$ | $q$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $a$ | $b$ | $a$ | $d$ | $c$ | $g$ | $f$ | $k$ | $m$ | $k$ | $p$ | $n$ | $q$ |
| $b$ | $a$ | $b$ | $c$ | $d$ | $f$ | $g$ | $m$ | $k$ | $m$ | $n$ | $p$ | $q$ |
| $c$ | $g$ | $f$ | $q$ | $q$ | $q$ | $q$ | $n$ | $n$ | $p$ | $q$ | $q$ | $q$ |
| $d$ | $f$ | $g$ | $q$ | $q$ | $q$ | $q$ | $p$ | $p$ | $n$ | $q$ | $q$ | $q$ |
| $f$ | $g$ | $f$ | $q$ | $q$ | $q$ | $q$ | $p$ | $n$ | $p$ | $q$ | $q$ | $q$ |
| $g$ | $f$ | $g$ | $q$ | $q$ | $q$ | $q$ | $n$ | $p$ | $n$ | $q$ | $q$ | $q$ |
| $h$ | $q$ | $q$ | $q$ | $q$ | $q$ | $q$ | $q$ | $q$ | $q$ | $q$ | $q$ | $q$ |
| $k$ | $q$ | $q$ | $q$ | $q$ | $q$ | $q$ | $q$ | $q$ | $q$ | $q$ | $q$ | $q$ |
| $m$ | $q$ | $q$ | $q$ | $q$ | $q$ | $q$ | $q$ | $q$ | $q$ | $q$ | $q$ | $q$ |
| $n$ | $q$ | $q$ | $q$ | $q$ | $q$ | $q$ | $q$ | $q$ | $q$ | $q$ | $q$ | $q$ |
| $p$ | $q$ | $q$ | $q$ | $q$ | $q$ | $q$ | $q$ | $q$ | $q$ | $q$ | $q$ | $q$ |
| $q$ | $q$ | $q$ | $q$ | $q$ | $q$ | $q$ | $q$ | $q$ | $q$ | $q$ | $q$ | $q$ |

An easy computation gives that $a$-connectedness is a congruence on $S$. On the other hand, $x \approx y$ iff either $x=y$ or $\{x, y\}=\{h, m\}$. Further $c \cdot h=n$, $c \cdot m=p$ and $n \not\approx p$. Thus $\approx$ is not a congruence on $S$.

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