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Rectangular bands in universal algebra: Two applications

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The concept of rectangular band, well known in semigroup theory (see, e.g., CLIFFORD and PRESTON [5]), has been utilized, implicitly or explicitly, in a few contexts in universal algebra, for example by CHANG, JÓNSSON, TARSKI [4], FAJTLOWICZ [7], GOULD [12], PŁONKA [23], NEUMANN [22], and TAYLOR [26]. In this note we employ rectangular bands to obtain two results concerning automorphism groups of universal algebras. The first of these results is a new proof of E.T. SCHMIDT's [24] theorem establishing the abstract independence of the concepts of automorphism group and subalgebra lattice. Although Schmidt's result has been re-proved and generalized in several ways (as in GOULD and PLATT [14], FRIED and GRÄTZER [9], LAMPE [21], and STONE [25]), the present proof is, in the author's view, simpler than the others and yields a stronger result in the finite case, namely the following:

Given a finite group G and a finite, non-trivial lattice L, there is a finite algebra $\mathfrak{A}$ of three binary operations, such that $G \cong \operatorname{Aut} \mathfrak{A}$ and $L \cong \operatorname{Sub} \mathfrak{A}$. Moreover, the three binary operations may be replaced by a single quaternary operation without altering the automorphisms or subalgebras.

Our second result is concerned with automorphism groups of direct products. It establishes, in a somewhat stronger form, the following statement:

Given a group G, there exist multi-unary algebras $\mathfrak{A}$ and $\mathfrak{B}$, each having a trivial automorphism group, such that $G \cong \operatorname{Aut}(\mathfrak{A} \times \mathfrak{B})$. Moreover, $\mathfrak{A}$ and $\mathfrak{B}$ are finite if G is finite.

Concepts and notations of universal algebra used here are taken from GRÄTZER [16], except for the notations $\operatorname{End} \mathfrak{A}$, $\operatorname{Aut} \mathfrak{A}$, $\operatorname{Sub} \mathfrak{A}$, $\operatorname{Con} \mathfrak{A}$, respectively denoting the endomorphism monoid, automorphism group, subalgebra lattice, and congruence lattice of a universal algebra $\mathfrak{A}$. Moreover, for sets A and B and an element $x \in A \times B$, the respective components of x will be denoted x_0 and x_1, that is, $x = \langle x_0, x_1 \rangle$. The projections $\pi_0 \colon A \times B \to A$ and $\pi_1 \colon A \times B \to B$ are then given by $x\pi_i = x_i$, for $i = 0, 1$.

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1. Automorphism groups and subalgebra lattices. Given sets A and B, define a binary operation on $A\times B$ by $x*y=\langle x_0, y_1\rangle$. The resulting groupoid is a semigroup known as the *rectangular band on* $A\times B$. In KIMURA [20] the following conditions on a semigroup S are proved to be equivalent:

(i) S satisfies the identities $xyz=xz$ and $x^2=x$.
(ii) S satisfies the identity $xyx=x$.
(iii) There exist sets A and B such that S is isomorphic to the rectangular band on $A\times B$.

A semigroup satisfying any of these equivalent conditions is called a *rectangular band.* (A generalization to Cartesian products of finitely many sets was introduced by PŁONKA [23] and termed *diagonal algebra;* see also FAJTLOWICZ [7].)

When A and B are endowed with operations of the same similarity type, the imposition of the rectangular band operation on the direct product of the algebras results in an algebra whose endomorphisms, subalgebras, and congruences readily decompose. Specifically, we have the following lemma, in which the notation $L_0\otimes L_1$ denotes, for algebraic lattices L_0 and L_1, the lattice obtained by adjoining a zero to the partial sublattice of $L_0\times L_1$ given by $\{x\in L_0\times L_1 \mid x_0\neq 0\neq x_1\}$. (The lemma contains a statement about congruence lattices that is included only for the sake of completeness, as it will not be used in the sequel. It was essentially noted by TAYLOR [26] and is a ready consequence of results of FRASER and HORN [8].)

Lemma 1.1. *Let* $\mathfrak{A}=\langle A; F\rangle$ *and* $\mathfrak{B}=\langle B; F\rangle$ *be universal algebras of the same similarity type, let* $\langle A\times B; F\rangle$ *denote their direct product, and let* $*$ *be the rectangular-band operation on* $A\times B$. *Then the algebra* $\mathfrak{C}=\langle A\times B;\ F\cup\{*\}\rangle$ *has the following properties.*

(1.1.1) $\mathrm{End}\,\mathfrak{C}\cong\mathrm{End}\,\mathfrak{A}\times\mathrm{End}\,\mathfrak{B}$, *and likewise for automorphisms.*
(1.1.2) $\mathrm{Sub}\,\mathfrak{C}\cong\mathrm{Sub}\,\mathfrak{A}\times\mathrm{Sub}\,\mathfrak{B}$ *if* F *contains nullary operations.*
(1.1.3) $\mathrm{Sub}\,\mathfrak{C}\cong\mathrm{Sub}\,\mathfrak{A}\otimes\mathrm{Sub}\,\mathfrak{B}$ *if* F *contains no nullary operations.*
(1.1.4) $\mathrm{Con}\,\mathfrak{C}\cong\mathrm{Con}\,\mathfrak{A}\times\mathrm{Con}\,\mathfrak{B}$.

Proof. As all parts of the lemma are proved in a very straightforward manner, we prove only (1.1.1) in detail and note that a common proof of (1.1.2) and (1.1.3) is achieved by verifying that $\mathrm{Sub}\,\mathfrak{C}=\{U\times V \mid U\in\mathrm{Sub}\,\mathfrak{A}$ and $V\in\mathrm{Sub}\,\mathfrak{B}\}$. The distinction between (1.1.2) and (1.1.3) arises from the convention that $\emptyset\in\mathrm{Sub}\,\mathfrak{A}$ if and only if $\mathfrak{A}$ has no nullary operations.

Given $\alpha\in\mathrm{End}\,\mathfrak{A}$ and $\beta\in\mathrm{End}\,\mathfrak{B}$, define $\gamma\colon A\times B\to A\times B$ pointwisely: $x\gamma=\langle x_0\alpha, x_1\beta\rangle$ for all $x\in A\times B$. Obviously $\gamma\in\mathrm{End}\,(\mathfrak{A}\times\mathfrak{B})$. Moreover, $(x*y)\gamma=\langle x_0, y_1\rangle\gamma=\langle x_0\alpha, y_1\beta\rangle=\langle x_0\alpha, x_1\beta\rangle*\langle y_0\alpha, y_1\beta\rangle=x\gamma*y\gamma$ for all $x, y\in A\times B$, and therefore $\gamma\in\mathrm{End}\,\mathfrak{C}$. Writing $\gamma=\alpha\times\beta$, we thus see that $\{\alpha\times\beta \mid \alpha\in\mathrm{End}\,\mathfrak{A}, \beta\in\mathrm{End}\,\mathfrak{B}\}$

is a submonoid of End $\mathfrak{C}$, and is obviously isomorphic to End $\mathfrak{A}\times$End $\mathfrak{B}$. Hence, to prove (1.1.1) it suffices to show that every endomorphism of $\mathfrak{C}$ belongs to this submonoid.

Let $\varphi\in\mathrm{End}\,\mathfrak{C}$. Fixing $u\in A\times B$, define maps φ_0: $A\to A$ and φ_1: $B\to B$ by $a\varphi_0=\langle a, u_1\rangle\varphi\pi_0$ and $b\varphi_1=\langle u_0, b\rangle\varphi\pi_1$ for all $\langle a, b\rangle\in A\times B$. If we can show that φ_0 and φ_1 are independent of the choice of u, it will follow that φ_0 and φ_1 are endomorphisms of $\mathfrak{A}$ and $\mathfrak{B}$ respectively and that $\varphi=\varphi_0\times\varphi_1$. Now, for $v\in A\times B$ and a, b, u as above, we have $a\varphi_0=\langle a, u_1\rangle\varphi\pi_0=(\langle a, v_1\rangle * u)\varphi\pi_0=(\langle a, v_1\rangle\varphi * u\varphi)\pi_0 = =\langle\langle a, v_1\rangle\varphi\pi_0, u\varphi\pi_1\rangle\pi_0=\langle a, v_1\rangle\varphi\pi_0$, and likewise $b\varphi_1=\langle v_0, b\rangle\varphi\pi_1$.

We can now prove the main result of this section.

Theorem 1.2. *Let L be an algebraic lattice of at least two elements and let G be a group.*

(1.2.1) (E. T. Schmidt [24]) *There is an algebra whose subalgebra lattice is isomorphic to L and whose automorphism group is isomorphic to G.*

(1.2.2) *If G is at most countable and each compact element of L contains at most countably many compact elements, then there is an infinite groupoid meeting the requirements of* (1.2.1).

(1.2.3) *If G and L are finite, there exists a finite algebra of three binary operations meeting the requirements of* (1.2.1); *there is also such a finite algebra having only a single quaternary operation.*

Proof. By classical results of Birkhoff and Frink [3] and Birkhoff [2] there exist algebras $\mathfrak{A}$ and $\mathfrak{B}$ having no nullary operations, such that Sub $\mathfrak{A}\cong L$, Aut $\mathfrak{B}\cong G$, and Sub $\mathfrak{B}\cong\mathbf{2}$, the two-element lattice. (For the sake of completeness we may specify that $\mathfrak{A}=\langle A;\ \{f_a|\,a\in A\}\cup\{\vee\}\rangle$, where A is the set of non-zero compact elements of L, the operation $\vee$ is the join inherited from L, and the unary operations f_a are given by $f_a(x)=a$ if $a\leq x$ and $f_a(x)=x$ otherwise. Moreover, we may take $\mathfrak{B}=\langle G;\ \{\lambda_g|\,g\in G\}\rangle$, where $\lambda_g(x)=gx$ for all $x\in G$.)

Unfortunately, $\mathfrak{A}$ may have non-trivial automorphisms, and hence must be modified to eliminate such automorphisms without altering the subalgebras. To this end, let ϱ be a well-ordering of A and define a binary operation f_ϱ on A by setting $f_\varrho(x, y)=x$ if $x\varrho y$ and $f_\varrho(x, y)=y$ otherwise. Adjoining f_ϱ to the operations of $\mathfrak{A}$ we obtain an algebra $\mathfrak{A}_\varrho$ whose subalgebras are precisely those of $\mathfrak{A}$. Moreover, any automorphism of $\mathfrak{A}_\varrho$ is an automorphism of the well-ordered set $\langle A, \varrho\rangle$ and hence must be identity.

Although $\mathfrak{A}_\varrho$ and $\mathfrak{B}$ need not be of the same similarity type, the difference is, for our purposes, purely superficial. We can increase operational rank (e.g., by substituting for an n-ary operation f the $(n+1)$-ary operation f' given by $f'(x_1, \ldots, x_{n+1})=f(x_1, \ldots, x_n)$) and introduce projection operations (such as

$p(x_1, \ldots, x_n)=x_1)$ as needed in order to convert $\mathfrak{A}_\varrho$ and $\mathfrak{B}$ into algebras $\mathfrak{A}'$ and $\mathfrak{B}'$ sharing a common similarity type and maintaining the desired properties: $\operatorname{Sub}\mathfrak{A}'=\operatorname{Sub}\mathfrak{A}_\varrho=\operatorname{Sub}\mathfrak{A}\cong L$ and $\operatorname{Aut}\mathfrak{A}'=\operatorname{Aut}\mathfrak{A}_\varrho=\{1_A\}$; while $\operatorname{Sub}\mathfrak{B}'=\operatorname{Sub}\mathfrak{B}\cong\mathbf{2}$ and $\operatorname{Aut}\mathfrak{B}'=\operatorname{Aut}\mathfrak{B}\cong G$. As neither $\mathfrak{A}'$ nor $\mathfrak{B}'$ has nullary operations, the above lemma gives an algebra $\mathfrak{C}$ satisfying $\operatorname{Aut}\mathfrak{C}\cong G$ and $\operatorname{Sub}\mathfrak{C}\cong L\otimes\mathbf{2}\cong L$, whereupon (1.2.1) is proved.

To prove (1.2.2) and (1.2.3) we assume the hypotheses of (1.2.2) and begin by choosing the above $\mathfrak{A}$ and $\mathfrak{B}$ to be groupoids, of cardinality at most $|L|$ and $|G|$ respectively. By a result of HANF [17], such a groupoid $\mathfrak{A}=\langle A; f\rangle$ with $\operatorname{Sub}\mathfrak{A}\cong L$ indeed exists: essentially as noted by WHALEY [27] (see also JÓNSSON [19]) it suffices to take A to be the set of non-zero compact elements of L and to fix for each $x\in A$ an enumeration $\{x_0, \ldots, x_n, \ldots\}$ of the set $\{y\in A | y\leqq x\}$ in such a way that $x_i=x_j$ implies $x_{i+1}=x_{j+1}$. The required binary operation f can then be defined by $f(x, y)=x_{n+1}$ if $y=x_n$ for some n, and $f(x, y)=x\vee y$ otherwise.

In GOULD [10], a groupoid $\mathfrak{B}$ satisfying $\operatorname{Aut}\mathfrak{B}\cong G$ was defined on the set G as follows. First, choose an enumeration $\{g_0, \ldots, g_n, \ldots\}$ of G satisfying $g_0=1$ and $g_{i+1}=g_{j+i}$ whenever $g_i=g_j$. Then define the binary operation f by $f(x, y)=g_{n+1}x$, where $g_n=yx^{-1}$. Setting $\mathfrak{B}=\langle G; f\rangle$ we note that $\operatorname{Sub}\mathfrak{B}\cong\mathbf{2}$. Indeed, it suffices to show that each $x\in G$ generates (in $\mathfrak{B}$) the entire set G. Since $g_0x=x$ and $f(x, g_nx)=g_{n+1}x$ for all n, it follows by induction that x generates $\{g_nx | n<\omega\}=Gx=G$.

Starting with these groupoids $\mathfrak{A}$ and $\mathfrak{B}$, we obtain as in the proof of (1.2.1) an algebra $\mathfrak{C}$ with $\operatorname{Sub}\mathfrak{C}\cong L$ and $\operatorname{Aut}\mathfrak{C}\cong G$. Note that $\mathfrak{C}$ has precisely three operations, all of them binary, and that the cardinality of $\mathfrak{C}$ is at most $|L|\cdot|G|$. By a result of JEŽEK [18], given any algebra $\mathfrak{C}=\langle C; F\rangle$ having at most countably many operations and no nullary operations, there is a groupoid $\mathfrak{C}'$ of cardinality equal to $\aleph_0\cdot|C|$, such that $\operatorname{End}\mathfrak{C}'\cong\operatorname{End}\mathfrak{C}$ and $\operatorname{Sub}\mathfrak{C}'\cong\operatorname{Sub}\mathfrak{C}$. Applying this result to the $\mathfrak{C}$ given above, we have proved (1.2.2).

If L and G are finite, the above $\mathfrak{C}$ suffices to establish the first statement of (1.2.3). To prove the second statement, we distinguish two cases.

Case 1. Supposing $|G|=1$, we in fact obtain a ternary operation with the required properties. Recall the algebra $\mathfrak{A}_\varrho=\langle A; f, f_\varrho\rangle$, and define a ternary operation t by setting $t(x, y, z)=f(x, y)$ if $y=z$ and $t(x, y, z)=f_\varrho(x, y)$ if $y\neq z$. It is easily verified that $\operatorname{Sub}\langle A; t\rangle=\operatorname{Sub}\langle A; f\rangle\cong L$ and $\operatorname{Aut}\langle A; t\rangle=\operatorname{Aut}\langle A; f_\varrho\rangle=\{1_A\}\cong G$.

Case 2. Supposing $|G|\neq 1$, we note that the groupoid $\mathfrak{B}=\langle B; f\rangle$ defined above has no one-element subgroupoids. It follows that in the direct product $\mathfrak{A}\times\mathfrak{B}=\langle A\times B; f\rangle$ there are no one-element subgroupoids. Passing to the algebra

$\mathfrak{C}$ above, we may write $\mathfrak{C}=\langle C; f_1, f_2, f_3\rangle$ where $C=A\times B$ and $f_1=f$. Now define a quaternary operation q on C by

$$q(x,y,z,w)=\begin{cases} f_1(x,y) & \text{if } y=z=w,\\ f_2(x,y) & \text{if } y=z\neq w,\\ f_3(x,y) & \text{if } y\neq z.\end{cases}$$

It is readily verified that $\operatorname{Aut}\langle C; q\rangle=\operatorname{Aut}\mathfrak{C}\cong G$ and that $\operatorname{Sub}\mathfrak{C}\subseteq\operatorname{Sub}\langle C; q\rangle$. To verify the reverse inclusion, let $X\in\operatorname{Sub}\langle C; q\rangle$. As the empty set belongs to $\operatorname{Sub}\mathfrak{C}$, we assume $X\neq\emptyset$. Since f_1 is a polynomial of $\langle C; q\rangle$ it follows that X is closed under f_1 and therefore contains more than one element. Thus, given $x, y\in X$ we may choose $w\in X$ with $w\neq y$, whereupon $f_2(x,y)=q(x,y,y,w)\in X$ and $f_3(x,y)=$ $=q(x,y,w,w)\in X$. Hence $\operatorname{Sub}\langle C; q\rangle=\operatorname{Sub}\mathfrak{C}\cong L$ and the theorem is proved.

We close this section with some remarks concerning the above theorem. First, the hypothesis $|L|>1$ is obviously justified by the fact that an algebra with only one subalgebra can have only one automorphism. Second, the hypothesis in (1.2.2) concerning the compact elements is necessary because in the subalgebra lattice of any algebra of at most countably many operations, the compact elements are the finitely generated subalgebras, each of which is at most countable. The other hypothesis in (1.2.2), namely $|G|\leqq\aleph_0$, cannot in general be dispensed with. For example, if the unit element of L is compact, then every algebra having L as its subalgebra lattice must be finitely generated. Given an algebra $\mathfrak{A}=\langle A; F\rangle$ generated by a finite set S, the mapping that associates with each automorphism of $\mathfrak{A}$ its restriction to S is a one-to-one function of $\operatorname{Aut}\mathfrak{A}$ into A^S. As above, if F is at most countable, the fact that $\mathfrak{A}$ is finitely generated implies that A is at most countable, whereupon the same holds for A^S and hence for $\operatorname{Aut}\mathfrak{A}$ as well.

Finally, we note that the second statement in (1.2.3) improves a result of the author [11] establishing the existence of a finite algebra with the desired properties that has only one operation. We ask whether the rank of this operation can be reduced to two, thereby combining, in the finite case, the nicest features of (1.2.2) and (1.2.3). Precisely stated: *given a finite group G and a finite lattice L, does there exist a finite groupoid $\mathfrak{A}$ satisfying $G\cong\operatorname{Aut}\mathfrak{A}$ and $L\cong\operatorname{Sub}\mathfrak{A}$?* We conjecture that the answer is affirmative. (In this conjecture and in (1.2.2), "groupoid" is best possible in the sense that an algebra whose operations all have rank less than two must have a distributive subalgebra lattice.)

2. Automorphism groups of direct products. Given sets A and B, a binary operation $*$ is readily defined on the set of all functions of $A\times B$ into itself: for two such functions α and β, simply define $\alpha*\beta\colon A\times B\to A\times B$ to be the map that sends each $x\in A\times B$ to $\langle x\alpha\pi_0, x\beta\pi_1\rangle$. Straightforward calculation shows that $*$ is an associative operation satisfying the rectangular-band identities (i) cited in the previous section.

Moreover, it is readily observed that composition of mappings is left-distributive over $*$ in the sense that $\alpha(\beta*\gamma)=(\alpha\beta)*(\alpha\gamma)$ for all transformations α, β, γ of $A\times B$.

If now A and B are the carrier-sets of algebras $\mathfrak{A}$ and $\mathfrak{B}$ respectively, of the same similarity type, it is readily observed that End $(\mathfrak{A}\times\mathfrak{B})$ is closed under $*$. Thus, we enrich the endomorphism monoid of $\mathfrak{A}\times\mathfrak{B}$ to form the *endomorphism system* $\mathcal{M}(\mathfrak{A}\times\mathfrak{B})=\langle \mathrm{End}\,(\mathfrak{A}\times\mathfrak{B});\ \cdot,\ *,\ 1\rangle$, an algebraic system of type $\langle 2, 2, 0\rangle$ in which $\cdot$ denotes composition of mappings and 1 is the identity endomorphism, here regarded as a nullary operation on End $(\mathfrak{A}\times\mathfrak{B})$.

The following lemma shows that the aforementioned equational properties of $\mathcal{M}(\mathfrak{A}\times\mathfrak{B})$ actually characterize such endomorphism systems. Here and in the sequel, expressions of the form $x\cdot y$ will be written as xy, and $(xy)*(xz)$ will be written as $xy*xz$.

Lemma 2.1. *Let* $\mathcal{M}=\langle M;\ \cdot,\ *,\ 1\rangle$ *be an algebraic system of type* $\langle 2, 2, 0\rangle$. *The following conditions are jointly equivalent to the existence of algebras* $\mathfrak{A}$ *and* $\mathfrak{B}$ *such that* $\mathcal{M}\cong\mathcal{M}(\mathfrak{A}\times\mathfrak{B})$.

(2.1.1) $\langle M;\ \cdot,\ 1\rangle$ *is a monoid;*
(2.1.2) $\langle M;\ *\rangle$ *is a rectangular band;*
(2.1.3) $x(y*z)=xy*xz$ *for all* $x, y, z\in M$.

Moreover, given (2.1.1)—(2.1.3) *the algebras* $\mathfrak{A}$ *and* $\mathfrak{B}$ *can be chosen to have unary operations only and to be finite if* M *is finite.*

Proof. Having noted the converse, let us suppose that (2.1.1)—(2.1.3) hold. By the result of KIMURA [20] quoted in the previous section, there exist sets A and B such that $\langle M;\ *\rangle$ is isomorphic to $\langle A\times B;\ *\rangle$, the rectangular band on $A\times B$. As the other operations in $\mathcal{M}$ can be transferred to $A\times B$ by means of this isomorphism, we have a system $\mathcal{M}'=\langle A\times B;\ \cdot,\ *,\ 1\rangle$ satisfying (2.1.1)—(2.1.3) and isomorphic to $\mathcal{M}$.

Now define a multi-unary algebra $\mathfrak{C}=\langle A\times B;\ \{f_u|u\in A\times B\}\rangle$ in which f_u denotes left multiplication by u, that is, f_u maps each $x\in A\times B$ to ux. It readily follows (as in BIRKHOFF [2] or ARMBRUST and SCHMIDT [1]) that End $\mathfrak{C}=\{\varrho_u|u\in A\times B\}$, where $x\varrho_u=xu$, for all $x\in A\times B$. The map $u\to\varrho_u$ is an isomorphism of $\langle A\times B;\ \cdot,\ 1\rangle$ onto End $\mathfrak{C}$.

By (2.1.3) each f_u is in fact an endomorphism of $\langle A\times B;\ *\rangle$, whence it follows as in Lemma 1.1 that for each u there exist maps $f_u^A\colon A\to A$ and $f_u^B\colon B\to B$ such that $f_u(\langle a, b\rangle)=\langle f_u^A(a), f_u^B(b)\rangle$ for all $\langle a, b\rangle\in A\times B$. Hence $\mathfrak{C}=\mathfrak{A}\times\mathfrak{B}$, where $\mathfrak{A}=\langle A;\ \{f_u^A|u\in A\times B\}\rangle$ and $\mathfrak{B}=\langle B;\ \{f_u^B|u\in A\times B\}\rangle$. To conclude that the map $u\to\varrho_u$ is an isomorphism of $\mathcal{M}'$ onto $\mathcal{M}(\mathfrak{A}\times\mathfrak{B})$, it remains only to note that $\varrho_{u*v}=\varrho_u*\varrho_v$ for all $u, v\in A\times B$. Indeed, $x\varrho_{u*v}=x(u*v)=xu*xv=\langle (xu)\pi_0, (xv)\pi_1\rangle=\langle x\varrho_u\pi_0, x\varrho_v\pi_1\rangle=x(\varrho_u*\varrho_v)$ for all $x\in A\times B$, whereupon the lemma is proved.

Before stating the main result of this section, we borrow from EHRENFEUCHT and GRZEGOREK [6] the following definition. Given sets A and B, a function $\alpha\colon A\times B\to A\times B$ is said to be *axial* if either $\alpha\pi_0=\pi_0$ *(left axial)* or $\alpha\pi_1=\pi_1$ *(right axial)*. Clearly α is left axial if and only if $1*\alpha=\alpha$, and right axial if and only if $\alpha*1=\alpha$.

We now show that an arbitrary group can be realized as the automorphism group of the direct product of algebras whose automorphism groups are trivial. In fact, we have the following stronger result.

Theorem 2.2. *Given a group G, there exist multi-unary algebras $\mathfrak{A}$ and $\mathfrak{B}$, both finite if G is finite, such that $G\cong\operatorname{Aut}(\mathfrak{A}\times\mathfrak{B})$ and $\mathfrak{A}\times\mathfrak{B}$ has no axial automorphisms other than the identity; thus* Aut $\mathfrak{A}$ *and* Aut $\mathfrak{B}$ *are both trivial.*

Proof. By the above lemma, it suffices to construct a system $\langle M;\ \cdot,\ *,\ \bar{1}\rangle$ satisfying (2.1.1)—(2.1.3) and an isomorphism $g\to\bar{g}$ of G onto the group of units of $\langle M;\ \cdot,\ \bar{1}\rangle$, such that no $g\in G\setminus\{1\}$ satisfies $\bar{1}*\bar{g}=\bar{g}$ or $\bar{g}*\bar{1}=\bar{g}$. To this end, set $M=G\times G$, and for each $g\in G$ set $\bar{g}=\langle g, g\rangle$. Define multiplication in M by:

$$xy=\begin{cases}\langle x_0y_0,\ x_0y_1\rangle & \text{if}\quad x_0=x_1,\\ x \text{ otherwise}\end{cases}$$

and let $\langle M;\ *\rangle$ be the rectangular band on $G\times G$. It is evident that $\bar{1}$ is an identity element with respect to multiplication, and that the map $g\to\bar{g}$ is an isomorphism of G onto the group of invertible elements of $\langle M;\ \cdot,\ \bar{1}\rangle$. Moreover, if an element $g\in G$ satisfies $\bar{g}=\bar{1}*\bar{g}$, it follows that $\langle g, g\rangle=\langle 1, g\rangle$, whence $g=1$; likewise $\bar{g}=\bar{g}*\bar{1}$ implies $g=1$. Thus, (2.1.3) and the associativity of multiplication are all that remains to be proved.

Let $x, y, z\in M$. If $x_0=x_1$ we have $x(y*z)=x\cdot\langle y_0, z_1\rangle=\langle x_0y_0, x_0z_1\rangle=$ $=\langle x_0y_0, x_0y_1\rangle*\langle x_0z_0, x_0z_1\rangle=xy*xz$, while if $x_0\neq x_1$ we have simply $x(y*z)=$ $=x=x*x=xy*xz$, whence (2.1.3) is proved. As for associativity, first note that $x_0\neq x_1$ implies $x(yz)=x=xz=(xy)z$. Thus, we now assume $x_0=x_1$. If $y_0=y_1$ we then have $x(yz)=x\cdot\langle y_0z_0, y_0z_1\rangle=\langle x_0y_0z_0, x_0y_0z_1\rangle=(xy)z$. If $y_0\neq y_1$ it follows that $x_0y_0\neq x_0y_1$, whence $x(yz)=xy=\langle x_0y_0, x_0y_1\rangle=\langle x_0y_0, x_0y_1\rangle\cdot z=(xy)z$. Thus associativity is proved and Lemma 2.1 now gives the desired direct product having no non-trivial axial automorphisms. As a non-trivial automorphism of either factor would obviously give rise to a non-trivial axial automorphism of the direct product, the theorem is proved.

We close this section by remarking that the finiteness of the algebra constructed above, in the case where G is finite, stands in striking contrast to the author's construction in [12] establishing the fact that any group having an element of order two is isomorphic to Aut $(\mathfrak{A}\times\mathfrak{A})$ for some multi-unary algebra $\mathfrak{A}$ having only the trivial endomorphism: if the group has more than two elements the algebra $\mathfrak{A}$ in that construction will be infinite. However, a different construction given in [12] (and subsequently generalized to Aut $(\mathfrak{A}^n)$ by the author and H. H. JAMES in [15]) produces a finite $\mathfrak{A}$ in the case where the given finite group retracts onto a two-element subgroup. One can easily verify that both constructions are free of non-trivial axial automorphisms.

The fact that finiteness is preserved in Theorem 2.3 makes it plausible that a finiteness-preserving construction can be found for Aut $(\mathfrak{A}\times\mathfrak{A})$ as well. By a result of the author in [13], representing a group as Aut $(\mathfrak{A}\times\mathfrak{A})$ for finite $\mathfrak{A}$ is equivalent to representing it as Aut $\mathfrak{B}$ for a finite algebra $\mathfrak{B}$ that is free on a two-element basis. Moreover, it is sufficient to allow only unary operations in the former case and binary in the latter. In view of the retraction theorem quoted above, the cyclic group of order four is the first group for which the question of such a representation is open.

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