Acta Sci. Math. 44 (1982), 281—286

On derivations in generalized matroid lattices

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1. Introduction. The notion of a generalized matroid lattice (GML for short) was introduced in [8]. A GML is a finite lattice L which satisfies the property that for each join-irreducible element $u \in L$ and for each element $b \in L$ the transposed intervals $[u \wedge b, u]$ and $[b, u \vee b]$ are isomorphic. The name "generalized matroid lattice" stems from the fact that many properties of matroid lattices (=geometric lattices= finite atomistic lattices with covering condition) can be proved in the original form or in a somewhat modified form also for GMLs. For results in this direction we refer to [8] and [2]. Similar results in a somewhat more general context appear in [7].

In the present paper we shall have a closer look on the uniquely determined lower cover u' of a join-irreducible element $u(\neq 0)$ in a GML. This lower cover u' will be called the (first) derivation of the join-irreducible element u. Concerning derivations of join-irreducible elements, we generalize in Section 3 some results of [6]. In the following Section 2 we deal with some basic facts which will be used in the sequel.

Acknowledgment. The author thanks the referee for his helpful remarks.

2. Some basic facts. By $x \prec y$ we mean that x is a lower cover of y.

Dealing with finite lattices, we define an element u to be join-irreducible if it has exactly one lower cover $u' \prec u$. In this context the least element 0 is not considered as join-irreducible. The uniquely determined lower cover u' of a join-irreducible element u will also be called the (first) derivation of u. By the n-th derivation $u^{(n)}$ of a join-irreducible element u we mean the element $(u^{(n-1)})'$. The element $u^{(n)}$ exists exactly if the elemente $u, u', u'', \ldots, u^{(n-1)}$ are all join-irreducible. By $J(L)$ we mean the set of all join-irreducible elements of a finite lattice L.

We restrict ourselves now to the abovementioned class of generalized matroid lattices.

Definition. (cf. [8]) Let L be a finite lattice and denote by $J(L)$ the set of all joinirreducible elements of L. We call L a *generalized matroid lattice* (briefly: GML) if the following isomorphism property (I) is satisfied:

Received January 19, 1981, and in revised form March 16, 1982.

(I) For all $u \in J(L)$ and for all $b \in L$ the transposed intervals $[u \wedge b, u]$ and $[b, u \vee b]$ are isomorphic.

The isomorphism indicated by this definition will be denoted by $[u \wedge b, u] \cong [b, u \vee b]$. The isomorphism property (I) implies (upper) semimodularity in the sense of [1].

Lemma 1. [8, Theorem 3] *A generalized matroid lattice is upper semimodular, that is, the implication*

$$a \wedge b \prec a \Rightarrow b \prec a \vee b$$

holds true.

By Lemma 1 the Jordan-Dedekind chain condition holds in a GML, i.e. all maximal chains of an interval $[x, y]$ have the same length (or dimension) which will be denoted by $d[x, y]$. This uniquely determined nonnegative integer $d[x, y]$ is called the *length* (or dimension) of the interval $[x, y]$.

Using arguments concerning the lengths of intervals, we show now that in a GML arbitrary join-irreducible elements and arbitrary elements form modular pairs in the sense of the following

Definition. Let a, b be elements of a lattice L. We say that (a, b) is a *modular pair*, and we write $(a, b)M$, if the implication

$$c \leqq b \Rightarrow (c \vee a) \wedge b = c \vee (a \wedge b) \quad (c \in L)$$

holds in L.

Lemma 2. *Let L be a generalized matroid lattice. Then $(u, b)M$ holds for each join-irreducible element $u \in J(L)$ and for each $b \in L$.*

Proof. We show that the implication

$$c \leqq b \Rightarrow (c \vee u) \wedge b = c \vee (u \wedge b)$$

holds in L. By [3, Lemma 1.4] it is sufficient to show that $(u, b)M$ holds in the interval $[u \wedge b, u \vee b]$. We may therefore assume that $u \wedge b \leqq c \leqq b$. By the isomorphism property (I) we have

$$[u \wedge b, u] = [u \wedge c, u] \cong [c, c \vee u],$$

which implies

$$d[c, c \vee u] = d[u \wedge b, u]. \tag{1}$$

By the isomorphism property (I) we have moreover

$$[u \wedge b, u] = [u \wedge \{b \wedge (c \vee u)\}, u] \cong [b \wedge (c \vee u), u \vee \{b \wedge (c \vee u)\}]$$

which yields

$$d[b \wedge (c \vee u), u \vee \{b \wedge (c \vee u)\}] = d[u \wedge b, u]. \tag{2}$$

Furthermore we have

$$(3)\qquad c \leqq b\wedge(c\vee u) \leqq u\vee \{b\wedge(c\vee u)\} \leqq u\vee c.$$

The relations (1), (2), and (3) together imply now $u\vee\{b\wedge(c\vee u)\}=u\vee c$ and $c=b\wedge(c\vee u)$. This latter equality together with the trivial $c=c\vee(u\wedge b)$ implies the assertion.

We remark that Lemma 2 has also been proved by other methods in [8] and [2].

Finally we shall need

Lemma 3. *Let L be a generalized matroid lattice, $u\in J(L)$, $a\in L$ and $u'\vee a< <u\vee a$. Then $u'\vee a\prec u\vee a$.*

Proof. From $u'\vee a<u\vee a$ it follows that $u\not\leqq u'\vee a$. Hence it follows that $u\wedge(u'\vee a)=u'\prec u$. By the isomorphism property (I) we have then

$$[u', u] = [u\wedge(u'\vee a), u] \cong [u'\vee a, u\vee(u'\vee a)] = [u'\vee a, u\vee a],$$

that is, $u'\vee a\prec u\vee a$.

3. Properties of derivations. In this section we generalize some results of [6]. For other properties of derivations in a generalized matroid lattice we refer to [8].

Theorem 4. *Let L be a generalized matroid lattice, and let $u, u_1, \ldots, u_k$ be join-irreducible elements of L. Moreover, let $u \leqq u_1\vee\ldots\vee u_k$ where the join $u_1\vee\ldots\vee u_k$ is (without loss of generality) assumed to be irredundant, and suppose*

$$u \not\leqq u_1\vee\ldots\vee u_{i-1}\vee u_i'\vee u_{i+1}\vee\ldots\vee u_k$$

holds for all i $(1\leqq i\leqq k)$. Then

$$u' \leqq u_i'\vee\ldots\vee u_k'.$$

Proof. If we had

$$u_1'\vee u_2\vee\ldots\vee u_k = u_1\vee u_2\vee\ldots\vee u_k$$

then it would follow that

$$u \leqq u_1\vee u_2\vee\ldots\vee u_k = u_1'\vee u_2\vee\ldots\vee u_k,$$

contradicting the assumptions of the theorem. Hence we have

$$u_1'\vee u_2\vee\ldots\vee u_k < u_1\vee u_2\vee\ldots\vee u_k.$$

By Lemma 3 we obtain now that

$$(4)\qquad u_1'\vee u_2\vee\ldots\vee u_k \prec u_1\vee u_2\vee\ldots\vee u_k.$$

Moreover, $u \not\leqq u_1'\vee u_2\vee\ldots\vee u_k$ implies by the isomorphism property (I) that

$$[u\wedge(u_1'\vee u_2\vee\ldots\vee u_k), u] \cong [u_1'\vee u_2\vee\ldots\vee u_k, u_1\vee u_2\vee\ldots\vee u_k].$$

Because of (4) we obtain from this that

$$u \wedge (u_1' \vee u_2 \vee \ldots \vee u_k) \prec u. \tag{5}$$

But u is join-irreducible and has therefore exactly one lower cover u'. Hence (5) yields

$$u' = u \wedge (u_1' \vee u_2 \vee \ldots \vee u_k) \leqq u_1' \vee u_2 \vee \ldots \vee u_k.$$

Similarly, one shows that also

$$u' \leqq u_1 \vee \ldots \vee u_{j-1} \vee u_j' \vee u_{j+1} \vee \ldots \vee u_k$$

holds for $j=2, \ldots, k$. Thus we get

$$u' \leqq \bigwedge_{j=1}^{k} (u_1 \vee \ldots \vee u_{j-1} \vee u_j' \vee u_{j+1} \vee \ldots \vee u_k). \tag{6}$$

We show now that

$$\bigwedge_{j=1}^{k} (u_1 \vee \ldots \vee u_{j-1} \vee u_j' \vee u_{j+1} \vee \ldots \vee u_k) = u_1' \vee \ldots \vee u_k' \tag{7}$$

holds. Because of (6) this already implies the assertion of the theorem. In order to prove (7) we define for the sake of brevity

$$\bar{u}_m \overset{\text{def}}{=} u_1 \vee \ldots \vee u_{m-1} \vee u_{m+1} \vee \ldots \vee u_k.$$

According to the assumptions of the theorem we have $\bar{u}_m \vee u_m' < \bar{u}_m \vee u_m$. Applying again Lemma 3 we get

$$\bar{u}_m \vee u_m' \prec u_1 \vee \ldots \vee u_k.$$

From this we obtain by the isomorphism property (I) that

$$u_m \wedge (\bar{u}_m \vee u_m') \prec u_m$$

holds. Since u_m is join-irreducible, we have therefore

$$u_m \wedge (u_1 \vee \ldots \vee u_{m-1} \vee u_m' \vee u_{m+1} \vee \ldots \vee u_k) = u_m'. \tag{8}$$

We now define

$$x_m \overset{\text{def}}{=} u_1 \vee \ldots \vee u_m \vee u_{m+1}' \vee u_{m+2} \vee \ldots \vee u_k, \quad z_m \overset{\text{def}}{=} u_{m+1},$$

$$y_m \overset{\text{def}}{=} u_1' \vee \ldots \vee u_m' \vee u_{m+2} \vee \ldots \vee u_k$$

with $m=1, \ldots, k-1$. Since we have $y_m \leqq x_m$, Lemma 2 implies $(z_m, x_m)M$. This means that we have

$$(y_m \vee z_m) \wedge x_m = y_m \vee (z_m \wedge x_m). \tag{9}$$

According to the definitions of z_m and x_m we get from (8) the equation

$$z_m \wedge x_m = u'_{m+1}. \tag{10}$$

Substituting (10) in (9), by the definition of y_m we obtain the relation

$$(y_m \vee z_m) \wedge x_m = u'_1 \vee \dots \vee u'_m \vee u'_{m+1} \vee u_{m+2} \vee \dots \vee u_k.$$

This implies in particular

$$(y_1 \vee z_1) \wedge x_1 = (u'_1 \vee u_2 \vee u_3 \vee \dots \vee u_k) \wedge (u_1 \vee u'_2 \vee u_3 \vee \dots \vee u_k) =$$
$$= u'_1 \vee u'_2 \vee {}_3 \vee \dots \vee u_k = y_2 \vee z_2.$$

Hence we obtain

$$(y_2 \vee z_2) \wedge x_2 = (u'_1 \vee u'_2 \vee u_3 \vee \dots \vee u_k) \wedge (u_1 \vee u_2 \vee u'_3 \vee u_4 \vee \dots \vee u_k) =$$
$$= u'_1 \vee u'_2 \vee u'_3 \vee u_4 \vee \dots \vee u_k.$$

Continuing this procedure, we obtain finally (7) and the theorem is proved.

Now we consider as a special case those GMLs in which every join-irreducible element is a cycle. (By a *cycle* we mean a join-irreducible element z for which the interval $[0, z]$ is a chain). To these lattices we apply the preceding theorem in order to obtain an estimation for the dimension (=length) of a cycle. We remark that other properties of these special GMLs are considered in [4] and [5] in a slightly more general setting.

Corollary 5. *Let L be a generalized matroid lattice in which every join-irreducible element is a cycle. If $z, z_1, \dots, z_k$ are cycles of L and if $z \leqq z_1 \vee \dots \vee z_k$, then*

$$d[0, z] \leqq \max d[0, z_j] \quad (1 \leqq j \leqq k).$$

Proof. By Lemma 1 the lattice L is upper semimodular. Hence the interval $[0, z_1 \vee \dots \vee z_k]$ is also upper semimodular. The function d introduced in the preceding section is therefore a dimension function on the interval $[0, z_1 \vee \dots \vee z_k]$. Thus we have

$$d[0, z] \leqq d[0, z_1 \vee \dots \vee z_k] \leqq \sum_{j=1}^{n} d[0, z_j].$$

Without loss of generality we may assume that no z_j $(1 \leqq j \leqq k)$ can be replaced by a cycle $y_j < z_j$ in such a way that

$$z \leqq z_1 \vee \dots \vee z_{j-1} \vee y_j \vee z_{j+1} \vee \dots \vee z_k$$

also holds. This yields in particular that every derivation $z_j^{(n)}$ $(n=1, 2, \dots)$ of z_j satisfies the relation

$$z \not\leqq z_1 \vee \dots \vee z_{j-1} \vee z_j^{(n)} \vee z_{j+1} \vee \dots \vee z_k. \tag{11}$$

Moreover, the join representation $z_1 \vee \ldots \vee z_k$ must be irredundant. Otherwise we would have a relation

$$z \leqq z_1 \vee \ldots \vee z_{j-1} \vee z_{j+1} \vee \ldots \vee z_k$$

contradicting (11) since there exists a natural number m with $z_j^{(m)}=0$.

Hence all assumptions of Theorem 4 are satisfied. Since all join-irreducible elements are cycles, Theorem 4 can be applied repeatedly. This yields for the n-th derivation

$$z^{(n)} \leqq z_1^{(n)} \vee \ldots \vee z_k^{(n)}. \qquad (12)$$

Putting $t = \max d[0, z_j]$ $(1 \leqq j \leqq k)$ we have $z_j^{(t)}=0$ $(1 \leqq j \leqq k)$. Because of (12) we obtain therefore

$$z^{(t)} \leqq z_1^{(t)} \vee \ldots \vee z_k^{(t)} = 0,$$

that is, $z^{(t)}=0$. This implies $d[0, z] \leqq t$ which proves the corollary.

We remark that Theorem 4 and Corollary 5 were proved in [6] for the special case of modular lattices (of finite length) in which every join-irreducible element is a cycle. It should also be remarked that our proofs carry over without alteration to the case of lattices of finite length with isomorphism property (I).

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