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The finite interpolation property for small sets of classical polynomials

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0. Introduction

LAGRANGE's [7] interpolation formula tells us that an arbitrary operation on the real numbers may be matched at a finite number of points by some polynomial, that is, by an operation composed solely from addition, multiplication, and constants. Let us abstract the essence of this definition. Following FOSTER [3] and PIXLEY [9], we shall say that such a collection F of operations over a set S has the finite interpolation property if any other arbitrary operation can be matched at an arbitrary finite set of arguments by some composition of the operations of F together with constants of S. In other words, any partial operation defined on a finite subset of S can be extended to a composition of operations of $F\cup S$, defined on all of S.

The formulation of this concept immediately provokes the question of whether there are apparently weaker sets of operations which nevertheless have the finite interpolation property. For example, in KNOEBEL [4], less was required when the finite interpolation property was established over the reals for the set $\{+, {}^2\}$, where "^{2}" is the operation of squaring. More generally, $\{+, {}^2\}$ has the finite interpolation property in any field not of characteristic 2. Similarly for multiplication, it was proven in KNOEBEL [5] that the set $\{\times, s\}$ has the finite interpolation property in any field where s is unit translation: $s(x)=x+1$.

The object of this article is to generalize these two results by replacing squaring and translation by rather arbitrary classical polynomials. We investigate in this paper four settings determined by two dichotomies: multiplication or addition over the complex or real numbers. In each of the four cases, we characterize those polynomials of one argument which together with the given binary operation yield the finite interpolation property over the given set.

The specific results are these. If p is a polynomial of degree at least two over the complex numbers $\mathbf{C}$, then the set $\{+, p\}$ has the finite interpolation property over $\mathbf{C}$, and conversely. Restricting the operations to the real numbers $\mathbf{R}$, we find that p

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must also be of even degree or of odd degree with the leading coefficient negative for the set $\{+, p\}$ to have the finite interpolation property over $\mathbf{R}$.

To describe the situation with multiplication, we shall say that a polynomial is *cyclic of order* k if it is of the form

$$p(x) = c_0(x^k - c_1)(x^k - c_2)\dots(x^k - c_j)$$

for some constants $c_0, c_1, \dots, c_j$; the polynomial p is *cyclic* if it is cyclic of some order $k \geqq 2$. Then the set $\{\times, p\}$ has the finite interpolation property over $\mathbf{C}$ if, and only if, the polynomial p is not cyclic and not constant, and $p(0) \neq 0$. Over $\mathbf{R}$, we need only to avoid p being cyclic of order 2, but p should cross the x-axis.

The order of presentation is in order of increasing difficulty of the proofs. The method of proof applies standard results about classical polynomials to the theorems found in KNOEBEL [5], [6]. Numerous examples will illustrate the tightness of the hypotheses. Three open problems close the paper.

Needed in the sequel are certain definitions. By *polynomial* we mean a polynomial function in one variable in the classical sense, that is, a one-place function

$$p(x) = \alpha_n x^n + \alpha_{n-1} x^{n-1} + \dots + \alpha_0$$

on $\mathbf{R}$ or $\mathbf{C}$, composed from addition, multiplication and constants. The degree of p is abbreviated 'deg p'. An *operation* ω on a set S is any n-place function $\omega: S^n \to S$ for some finite n. If F is a family of operations on a set S, then by an *F-polynomial* we understand a composition of operations from F together with constants from S. For example, over $\mathbf{R}$, a $\{+, \times\}$-polynomial is just a classical polynomial of $\mathbf{R}[x]$; a $\{+\}$-polynomial is a multilinear operation. With a little effort one can show that a $\{+, {}^2\}$-polynomial over $\mathbf{R}$ or $\mathbf{C}$ is any monic polynomial whose degree is a power of 2. We say that a family F of operations on a set S has the *finite interpolation property*, if, for every positive integer n, for every finite subset $T \subseteqq S^n$ and for every function $f: T \to S$, there is an F-polynomial ω such that f agrees with ω on T, that is, $f = \omega | T$. Briefly, we say that F has the f.i.p. over S.

A highly restricted version of this concept is that of (m, n)-transitivity, where m and n are positive integers. We say our family F of operations is (m, n)-*transitive* over S if, for every subset $T_m \subseteqq S$ of m elements, for every subset $T_n \subseteqq S$ of n elements and for every function $f: T_m \to T_n$, there is a composition $\omega: A \to A$ of operations in F such that $f = \omega | T_m$. Oftentimes, we wish to obtain (m, n)-transitivity by means of constants as well; this is most easily accomplished by the phrase, '$F \cup S$ is (m, n)-transitive.'

To prove that in our theorems the conditions on polynomials are tight enough and really necessary, we introduce the idea of preservation of properties. Let P be a property, that is, a finitary relation on S, say $P \subseteqq S^m$. We say P is *preserved* by an operation $\omega: S^n \to S$ if, whenever

$$(s_j^1, s_j^2, \ldots, s_j^m) \in P \quad (\text{for all } j = 1, \ldots, n),$$

and

$$s_0^i = \omega(s_1^i, s_2^i, \ldots, s_n^i) \quad (\text{for all } i = 1, \ldots, m),$$

then

$$(s_0^1, s_0^2, \ldots, s_0^m) \in P.$$

Clearly, preservation passes through composition; that is, if P is preserved by $n+1$ operations $\omega_0, \omega_1, \ldots, \omega_n$ where ω_0 is n-ary and $\omega_1, \ldots, \omega_n$ are p-ary, then P is preserved by the composition $\omega_0(\omega_1, \ldots, \omega_n)$. Typically, we apply this to show that a particular constraint is necessary for the f.i.p. by finding a property P which is preserved by all operations of F (and constants) and yet is *not* preserved by every conceivable operation needed to establish the f.i.p..

Some historical comments are in order. To gain more information on the origins and subsequent development of Lagrange's interpolation formula, one should start by looking in the index of GOLDSTINE [4]. Robin McLeod has pointed out to me that the finite interpolation property has acquired a meaning in contemporary universal algebra different from that in numerical analysis; see DAVIS [2] for the classical definition. Infinite universal algebras with the finite interpolation property are called 'functionally complete in the small' by FOSTER [3]. Each such algebra generates, in a natural way by the subextension of identities, a class of algebras each of which is isomorphic to a bounded subdirect power of the generating algebra. This theorem of Foster (Theorem 19.1, loc. cit.) is an infinite analog of STONE's [12] representation theorem for Boolean algebras. For related work on the finite interpolation property, the interested reader should consult the recent surveys by KNOEBEL [6], PIXLEY [9] QUACKENBUSH [10] and ROSENBERG [11].

1. Multiplication

In this section we give necessary and sufficient conditions, over both the real and complex numbers, for the set $\{\times, p\}$ to have the f.i.p.. Needed to prove these is the following result from an earlier paper.

Theorem 1.0 (KNOEBEL [5]). *If $F \cup S$ is (2,2)-transitive over S, and there is a binomial $\times$ in F with a null element 0 and a unit element 1 in S, that is, $0 \times s = 0 = s \times 0$, $1 \times s = s = s \times 1$ $(s \in S)$, then F has the finite interpolation property over S.*

With this we can prove the next theorem.

Theorem 1.1. *Let $p \in \mathbf{C}[x]$. Then $\{\times, p\}$ has the finite interpolation property over $\mathbf{C}$ if, and only if,*

(i) *p is not constant,*
(ii) $p(0)\neq 0$, *and*
(iii) *p is not cyclic.*

Proof. $\Rightarrow$ Let $F=\{\times, p\}$. By way of contradiction first assume $p(0)=0$. Then any non-constant F-monomial composed from $\times, p$ and constants will have the same property, and thus we do not even have (1,1)-transitivity.

Similarly, if p is cyclic of order k, let P_k be the binary relation holding for pairs (a, b) in $\mathbf{C}^2$ whenever there is a c in $\mathbf{C}$ such that a and b are both roots of z^k-c, i.e., $b=ae^{2\pi ij/k}$ for some integer j. Clearly p, as well as multiplication and constants, preserve P_k, so F is not (2,2)-transitive.

$\Leftarrow$ In view of Theorem 1.0, we need only to establish the (2,2)-transitivity of $F\cup\mathbf{C}$. To this end, let a and b in $\mathbf{C}$ be distinct with A and B also in $\mathbf{C}$; we are looking for a $\{\times, p\}$-polynomial q such that $q(a)=A$ and $q(b)=B$. Without loss of generality, assume $a\neq 0$. We claim such a q can be found in the form

$$q(z) = \gamma\times p\,(\beta\times p(\alpha\times z)),$$

where α, β, γ are constants to be determined. Let r be a root of p such that $\frac{br}{a}$ is not also a root. Such a root always exists, since otherwise $r, \frac{b}{a}r, \left(\frac{b}{a}\right)^2 r, \ldots$ must all be roots. Since a polynomial has a finite number of roots, this must imply that $\left(\frac{b}{a}\right)^k=1$ for some $k\geqq 2$, and therefore p is cyclic, a contradiction.

Now a goes into A and b goes into B by the following sequence of polynomial transformations:

$$\begin{array}{ccccccccccc}
a & \longrightarrow & r & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & p(0) & \longrightarrow & A \\
 & \frac{r}{a}\times(\;) & & p(\;) & & \beta\times(\;) & & p(\;) & & \frac{A}{p(0)}\times(\;) & \\
b & \longrightarrow & \frac{br}{a} & \longrightarrow & p\left(\frac{br}{a}\right) & \longrightarrow & p^{-1}\left(\frac{Bp(0)}{A}\right) & \longrightarrow & \frac{Bp(0)}{A} & \longrightarrow & B.
\end{array}$$

The last multiplication is possible since $p(0)\neq 0$. We may choose

$$\beta = \frac{p^{-1}\left(\frac{Bp(0)}{A}\right)}{p\left(\frac{br}{a}\right)}$$

since $\frac{br}{a}$ is not a root; by $p^{-1}\left(\frac{Bp(0)}{A}\right)$ we mean a fixed root z_0 of $p(z)-\frac{Bp(0)}{A}=0$, which always exists over the complex numbers. The foregoing does not work when

$A=0$. In this case choose

$$\beta = \frac{B}{p\left(\frac{b}{a}r\right)},$$

and finish two steps earlier.

For example, the set $\{\times, x^2+x+1\}$ has the f.i.p. over **C**, but $\{\times, x^2+1\}$ does not. Similarly, $\{\times, x+1\}$ has the f.i.p., whereas $\{\times, x^3+1\}$ does not.

When **C** is replaced by **R**, the conditions must be formulated differently, but the proof is similar. By the phrase '*f crosses the x-axis*' we mean that there are a and b in **R** such that

$$f(a) < 0 < f(b).$$

Note that this is a stronger condition than merely saying that f has a real root.

Theorem 1.2. *Let* $p\in\mathbf{R}[x]$. *Then* $\{\times, p\}$ *has the finite interpolation property over* **R** *if, and only if,*

(i) p *is not constant,*
(ii) $p(0)\neq 0$,
(iii) p *is not cyclic of order* 2, *and*
(iv) p *crosses the x-axis.*

Proof. $\Rightarrow$ We show the contrapositive. If $p(0)=0$, then 0 cannot be taken into a nonzero element by any $\{\times, p\}$-polynomial.

If p does not cross the x-axis, then it is all of one sign, and consequently no polynomial in $\times$ and p can take two numbers of the same sign into numbers of opposite sign. More precisely, letting $P=\{\langle a, b\rangle | ab\geqq 0\}$, we see that all operations of $\{\times, p\}\cup\mathbf{R}$ preserve P; hence, e.g., $x+1$ is not a $\{\times, p\}$-polynomial in this case.

If p is cyclic of order 2, that is,

$$p(x) = r_0(x^2-r_1)(x^2-r_2)\dots(x^2-r_j),$$

and also $a=-b\neq 0$, then any $\{\times, p\}$-polynomial q will give $|q(a)|=|q(b)|$. Thus, for example, 1 and -1 cannot be taken into 1 and 2 respectively. The preservation relation in this case is $P=\{\langle a, \pm a\rangle | a\in\mathbf{R}\}$.

$\Leftarrow$ We need only modify the proof developed in the complex case. We assume that the reader now has before himself the sequence of polynomial transformations of the previous proof. Note that the only real roots of unity are $r=\pm 1$ and therefore, for the proof to work over **R**, we need only rule out polynomials which are cyclic of order 2 in the first transformation of multiplying by $\frac{r}{a}$.

The only other steps that might be different for the real case are the third and fourth, which depend on finding a root of

$$p(x) = \frac{Bp(0)}{A}.$$

Such a root may not exist when p is of even degree. In such an eventuality, certainly

$$p(x) = -\frac{Bp(0)}{A}$$

has a root, since p crosses the x-axis. Using this root instead, we will end up with $\langle a, b\rangle$ going to $\langle A, -B\rangle$. However, redefining the third transformation as multiplication by $-\beta$ will rectify this unwanted sign.

By way of example, notice that if $p(x)=x^2+1$ or x^2+x+1, then $\{\times, p\}$ does *not* have the f.i.p. over $\mathbf{R}$, but if $p(x)=x+1$ or x^3+1, then it does.

2. Addition

We now turn to addition to see which polynomials achieve the f.i.p.. The proofs now are more complicated; we do the complex case first since it is simpler than the real. Both depend on the following theorem.

Theorem 2.0 (KNOEBEL [6]). *If $F\cup S$ is (3,2)-transitive over S, and there is a binomial $+$ in F with a unit element 0 in S, that is, $0+s = s = s+0$ $(s\in S)$, and such that $s+s\neq 0$ for some $s\in S$, then F has the finite interpolation property over S.*

Theorem 2.1. *Let $p\in\mathbf{C}[z]$. Then $\{+, p\}$ has the finite interpolation property over $\mathbf{C}$ if, and only if, p is of degree at least two.*

Proof. $\Rightarrow$ Let $F=\{+, p\}$. On the contrary if $\deg p\leqq 1$, then p is constant or linear, in which case only linear operations are obtainable by composition from $+$, p and elements of $\mathbf{C}$.

$\Leftarrow$ Let us show (2,2)-transitivity first. Because we have sums and constants, translations are available for use anywhere. If we wish to take a to A and b to B, it suffices to find a z_0 such that

$$p(z_0+\delta)-p(z_0) = \Delta$$

where $\delta=a-b$ and $\Delta=A-B$. Since $\deg p\geqq 2$, the left side has degree at least 1. Among the complex numbers there is a solution z_0 to this difference equation. Hence $\{+, p\}\cup\mathbf{C}$ is (2,2)-transitive.

For (3, 2)-transitivity, let us prove that for any distinct a, b, and c in $\mathbf{C}$ there must be a $\{+, p\}$-polynomial q for which $q(a)=q(b)\neq q(c)$. By repeated addition, Nz is a $\{+\}$-polynomial for any positive integer N. Consider the family of polynomials q_λ^N indexed by $N\in\mathbf{N}$:

$$q_\lambda^N(z) = p(z+\lambda)+Nz.$$

We claim one of these will do the trick. For each positive integer N there is a root λ_N in $\mathbf{C}$ of the equation

$$p(a+\lambda_N)+Na = p(b+\lambda_N)+Nb$$

since $\deg p\geqq 2$. Thus $q_{\lambda_N}^N(a)=q_{\lambda_N}^N(b)$ for all positive integers N. If for some N, $q_{\lambda_N}^N(b)\neq q_{\lambda_N}^N(c)$, we are finished.

If not, we will reach a contradiction. For in this worst case, we would have for all positive integers N,

$$q_{\lambda_N}^N(a) = q_{\lambda_N}^N(b) = q_{\lambda_N}^N(c).$$

In terms of p, this is the infinite family of equations

$$p(a+\lambda_N)+Na = p(b+\lambda_N)+Nb = p(c+\lambda_N)+Nc.$$

From these, upon eliminating N, we derive

$$\frac{p(a+\lambda_N)-p(b+\lambda_N)}{a-b} = \frac{p(b+\lambda_N)-p(c+\lambda_N)}{b-c}$$

for all positive integers N. But each side is a polynomial agreeing with the other side at an infinite number of points. Since a nonzero polynomial has a finite number of zeros, they must agree at all points:

$$\frac{p(a+z)-p(b+z)}{a-b} = \frac{p(b+z)-p(c+z)}{b-c}.$$

Hence the coefficients of z^{n-2} on each side are equal:

$$\alpha_n n(n-1)(a+b)+(n-1)\alpha_{n-1} = \alpha_n n(n-1)(b+c)+(n-1)\alpha_{n-1},$$

where the α_n come from $p(z)=\alpha_n z^n+\alpha_{n-1}z^{n-1}+\dots$. Therefore, $a=c$, a contradiction to the distinctness of a and c.

The preceding paragraph, together with the (2,2)-transitivity proven earlier, establishes the (3,2)-transitivity of $\{+, p\}\cup\mathbf{C}$. From Theorem 2.0 follows the f.i.p. of $\{+, p\}$.

By the way of illustration, note that $\{+,p\}$ has the f.i.p. over $\mathbf{C}$ when $p(x)=$ $=x^2+1$, x^2+x+1 or x^3+1, but not when $p(x)=x+1$.

Theorem 2.2. *Let* $p\in\mathbf{R}[x]$. *Then* $\{+,p\}$ *has the finite interpolation property over* $\mathbf{R}$ *if, and only if,*

(i) $\deg p\geqq 2$, *and*

(ii) $\deg p$ *is even; or when* $\deg p$ *is odd, the leading coefficient* α_n *of* p *is negative.*

Proof. $\Rightarrow$ As in Theorem 2.1, condition (i) cannot be dropped.

Now if p satisfies the first condition but fails the second, it must be that p is a nonlinear polynomial of odd degree with positive leading coefficient. For sufficiently large differences between a and b, p will magnify this difference, and thus there is no way in which such a large difference may be decreased. In more detail, one can find a positive real number m such that $p(a)-p(b)>m$ whenever $a-b>m$. Let $P=\{\langle a,b\rangle | b-a>m\}$. Then P is preserved by both $+$ and p. Hence $\{+,p\}\cup\mathbf{R}$ is not (2,2)-transitive, and consequently $\{+,p\}$ does not have the f.i.p..

$\Leftarrow$ Let us consider only the case of a polynomial of odd degree at least 3 with negative leading coefficient:

$$p(x) = \alpha_n x^n+\ldots+\alpha_0$$

with $\alpha_n<0$ and n odd and $n\geqq 3$. The case of a nonconstant polynomial of even degree is similar but less complicated, and so the necessary modifications will be left to the reader. Notice that, with addition and constants appropriately composed, all translations and their inverses are available. By suitable translations on both the x- and y-coordinates, we may thus, without loss of generality, safely assume that

(i) $p(0) = 0$,

(ii) $p(x) > 0$ if $x < 0$,

(iii) $p''(x) > 0$ if $x \leqq 0$.

We first show the (2,2)-transitivity of $\{+,p\}\cup\mathbf{R}$. Then later the argument will be modified to accommodate (3,2)-transitivity by showing that in any triple the first two numbers may be identified by some polynomial which keeps the third one distinct.

Assume $a,b,A,B\in S$ and $a>b$; we will try to find a $\{+,p\}$-polynomial f such that $f(a)=A, f(b)=B$. Set $\delta=a-b$ and $\Delta=A-B$. Again by the use of translations, we would be finished if we could find a λ in $\mathbf{R}$ such that

$$p(\lambda+\delta)-p(\lambda) = \Delta.$$

The required polynomial f would be $f(x)=p(x+\lambda-b)-p(\lambda+\delta)+A$. Such a root λ would exist if p were of even degree, since then the difference is of odd degree and always has a root in $\mathbf{R}$.

However, when p is of odd degree, this won't work in general, but it can be made to work with the following modifications. Remember that Nx is a $\{+\}$-polynomial for any positive integer N. Set

$$q_\lambda^N(x) = p(x+\lambda)+Nx.$$

To obtain (2,2)-transitivity now with q_λ^N instead of p, the constant λ should be chosen to be a root of

$$q_\lambda^N(a)-q_\lambda^N(b) = \Delta.$$

In terms of p, this is

$$(*) \qquad p(a+\lambda)-p(b+\lambda)+N\delta = \Delta.$$

Whether such a λ exists depends on the value of N. Now $p(a+\lambda)-p(b+\lambda)$ has leading term $n\alpha_n(a-b)\lambda^{n-1}$ with negative coefficient and even exponent. Since $a>b$, the equation $(*)$ will have a root λ if N is chosen sufficiently large. But N can be any positive integer, so this is always possible. Hence $\{+,p\}\cup\mathbf{R}$ is (2,2)-transitive.

To establish (3,2)-transitivity, we argue as follows. Still assuming p to be a polynomial of odd degree satisfying conditions (i) to (iii), we will take a and b both to 0 (possible by the (2,2)-transitivity just established), but we will do it carefully enough so c goes to a nonzero real number. First of all, the ordering on a, b, and c may be reversed by translating to the left beyong 0 and applying p. Therefore, without loss of generality, we may assume $a>b>c$ or $a>c>b$. Secondly, in the next paragraph we need $\lambda+a$ to be negative. By choosing N sufficiently large — perhaps larger than before — we may guarantee $\lambda+a$ to be negative.

Proceeding as before with (2,2)-transitivity, we transform both a and b into 0 by the $\{+,p\}$-polynomial

$$q(x) = q_\lambda^N(x)-q_\lambda^N(a)$$

where $\lambda+a<0$, and $N>0$. Set $C=q(c)$ and $\gamma=q_\lambda^N(a)$. Thus

$$\begin{aligned} 0 &= q(a) = p(a+\lambda)+Na-\gamma, \\ 0 &= q(b) = p(b+\lambda)+Nb-\gamma, \\ C &= q(c) = p(c+\lambda)+Nc-\gamma. \end{aligned}$$

Notice that the arguments of p are all negative. Recall that $p''(x)>0$ when $x<0$, and hence this is also true for q. If by some fluke $C=0$, we would have a concave segment agreeing with a straight line at three points, which is nonsense. Thus $\{+,p\}\cup\mathbf{R}$ is (3,2)-transitive.

We make two comments. This convexity argument for (3,2)-transitivity using three points in the real case could be replaced by the argument using an infinite number of points in the complex case. Secondly, as an illustration, notice that both

$\{+, x+1\}$ and $\{+, x^3+1\}$ can be shown not to have the f.i.p. over **R** by directly observing that both functions are monotonically increasing, and so all compositions must have this property. On the other hand, the sets $\{+, x^2+1\}$, $\{+, x^2+x+1\}$ and $\{+, -x^3+1\}$ all have the f.i.p..

3. Open problems

We close with three open problems suggested by the theorems of this paper.

1. The set $\{+, \cosh\}$ can be shown to have the f.i.p. over **R** by arguments similar to those used in this paper since the hyperbolic cosine is shaped like a parabola. The problem is to find an appropriate definition of 'polynomial-like' so that the results of this paper are still true for functions which are not polynomials but similar to them in behavior.

2. Replace addition or multiplication by an arbitrary polynomial in k variables, and give necessary and sufficient conditions for the set $\{p, q\}$ to have the f.i.p. when q is a polynomial in one variable. More generally, which sets F of polynomials in any number of variables have the f.i.p. over **C** or **R**? Probably for most polynomials p of two or more arguments, $\{p\}$ by itself has the f.i.p. The evidence for this is Murskiĭ's [8] theorem that on a finite set, the proportion of two-place operations with the f.i.p. to all two-place operations approaches 1 as the cardinality of the set increases without bound. Compare this with the result of Davies [1] that the proportion of two-place Sheffer operations to all two-place operations approaches $1/e$ as the size of the finite set increases without bound. (An operation is *Sheffer* if all other operations are obtainable from it by composition without the help of constants.) Most likely, algebraic geometry will be needed to settle the exceptional cases.

3. Extend these results beyond **R** and **C** to more general structures, say, all fields.

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