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Additive functions with regularity properties

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1. Recently J. L. MAUCLAIRE and LEO MURATA [1] proved that a multiplicative function $g(n)$, satisfying the conditions

$$|g(n)| = 1 \quad (n = 1, 2, \ldots),$$

and

$$\frac{1}{x}\sum_{n\le x}|g(n+1)-g(n)| \to 0 \quad (x\to\infty)$$

has to be completely multiplicative. For a real z let $\|z\|$ denote its distance from the nearest integer. Their theorem is equivalent with the following assertion: If f is additive and

$$\frac{1}{x}\sum_{n\le x}\|f(n+1)-f(n)\| \to 0 \quad (x\to\infty) \tag{1.1}$$

then f is completely additive.

I conjecture that the following assertion is true: If f is an additive function satisfying (1.1), then $f(n)=c\log n+g(n)$, where $g(n)$ is an integer valued additive function.

In [2] the following simple assertion was proved: If $f(n)$ is additive and $n\|f(n+1)-f(n)\|=O(1)$, then $f(n)=c\log n+g(n)$, where $g(n)$ is an integer valued additive function. Now we prove the following stronger

Theorem 1. *If $f(n)$ is additive and*

$$n\|f(n+1)-f(n)\| = O(n^{\gamma}) \tag{1.2}$$

with a constant $\gamma<1$, then $f(n)=c\log n+g(n)$, where $g(n)$ is integer valued.

Proof. By the cited result of Mauclaire and Murata, we may assume that f is completely additive. Let I_n be the nearest integer to $(f(n+1)-f(n))$, and $\sigma(n)= =(f(n+1)-f(n))-I_n$. Then we have $\sigma(n)\in\left[-\frac{1}{2},\frac{1}{2}\right]$, and from (1.2), $n|\sigma(n)|= =O(n^{\gamma})$. Let

$$T(x) = \max_{m\le x} m|\sigma(m)|.$$

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We shall prove step by step the following assertions:

(1) The assertion is true if $T(x)=O(1)$ $(x\to\infty)$.

(2) If $T(x)=O(x^\gamma)$, $\gamma<1$, then $T(x)=O(\log x)$.

(3) If $T(x)\to\infty$, then the fractional parts of $m\sigma(m)$ are everywhere dense in $[0, 1)$.

(4) Completion of proof.

We start from the identity

$$f\big((n+1)^2-1\big)-f\big((n+1)^2\big)=f(n)-f(n+1)+f(n+2)-f(n+1),$$

which by $\sigma(n)\to 0$ implies that

$$(*)\qquad \sigma(n+1)=\sigma(n)-\sigma\big((n+1)^2-1\big) \quad \text{if} \quad n>n_0.$$

Applying this identity for $n+1, \dots, n+H-1$ instead of n, we get

$$\sigma(n+H)-\sigma(n)=\sum_{j=0}^{H-1}\big(\sigma(n+j+1)-\sigma(n+j)\big)=-\sum_{j=1}^{H}\sigma\big((n+j)^2-1\big),$$

so that

$$\sum_{H=0}^{R-1}\sigma(n+H)-R\sigma(n)=-\sum_{l=1}^{R-1}\sigma\big((n+l)^2-1\big)(R-l) \quad (n>n_0).$$

Let $n=mR$ and observe that

$$(1.3)\qquad \sum_{H=0}^{R-1}\sigma(mR+H)=f(mR+R)-f(mR)-\sum_{H=0}^{R-1}I_{mR+H}= \\ =\sigma(m)+\big(I_m-\sum I_{mR+H}\big).$$

The absolute value of the left hand side of (1.3) is not greater than

$$\frac{RT\big((m+1)R\big)}{mR}=\frac{T\big((m+1)R\big)}{m},$$

that is, less than 1/2 provided $m>m_0$, and R is not too large. Consequently it is $\sigma(m)$; therefore

$$(1.4)\qquad \sigma(m)-R\sigma(mR)=-\sum_{l=1}^{R-1}\sigma\big((mR+1)^2-1\big)(R-1),$$

if $T\big((m+1)R\big)<\frac{m}{2}$. The right hand side of (1.4) is majorated by

$$\frac{R^2T\big((m+1)^2R^2\big)}{m^2R^2},$$

hence

$$(1.5)\qquad |m\sigma(m)-Rm\sigma(mR)|\leqq\frac{T\big((m+1)^2R^2\big)}{m}.$$

Assume now that $T(x)$ is bounded, $T(x)\leqq K$. Putting $m=N_1$, $r=N_2$, and $m=N_2$, $R=N_1$ into (1.5) and using the triangle inequality we get

$$|N_1\sigma(N_1)-N_2\sigma(N_2)| \leqq \frac{K}{N_1}+\frac{K}{N_2}$$

for every large N_1, N_2. This shows that $N\sigma(N)$ is a Cauchy sequence, consequently $N\sigma(N)\to A$.

Let $\sigma(m)=\frac{A}{m}+\frac{\varepsilon_m}{A}$, $\varepsilon_m\to 0$. Furthermore, let p and q be arbitrary integers satisfying the relations: $1<q/p$, $A\log q/p<1/2$. Consider the relation

$$f(q)-f(p) = \sum_{n=pU}^{qU-1}(f(n+1)-f(n)) = \sum_{n=pU}^{qU-1}\sigma(n)+J(U),$$

where $J(U)$ is an integer depending on U. The sum on the right is

$$A\sum_{m=pU}^{qU-1}\frac{1}{m}+\sum_{m=pU}^{qU-1}\frac{\varepsilon m}{m} = A\log\frac{p}{q}+o_U(1)$$

as $U\to\infty$. Hence

$$f(q)-f(p)-A\log\frac{q}{p} = J(U)+o_U(1),$$

which shows that $J(U)$ is constant for $U>U_0(p,q)$. Consequently for $U\to\infty$ we get that $f(q)-f(p)-A\log q/p$ is an integer, which immediately implies our assertion.

Assume now that $T(x)=O(x^\gamma)$, $T(x)>Kx^\gamma$. Using (1.4) with $R=2$ we get

$$2m\sigma(2m) = m\sigma(m)+m\sigma((2m+1)^2-1). \tag{1.6}$$

Furthermore, from (*) we get

$$(2m+1)\sigma(2m+1) = \left(m+\frac{1}{2}\right)\sigma(m)-\frac{2m+1}{2}\sigma((2m+1)^2-1). \tag{1.7}$$

Let $x>x_0$ and assume that $T(2x)>T(x)$. The maximum of $|n\sigma(n)|$ in $[1, 2x]$ is reached in $\left(\frac{x}{2}, x\right]$. If the maximum is taken for even n, then from (1.6),

$$T(2x) \leqq T(x)+\max_{m\in\left(\frac{x}{2},x\right]} m|\sigma((2m+1)^2-1)|.$$

Since $(2m+1)^2-1=(2m)(2m+2)\leqq 2x(2x+2)=4x(x+1)$, the last term is majorated by $x^{-1}T(4x(x+1))$, therefore

$$T(2x) \leqq T(x)+\frac{T(4x(x+1))}{x}.$$

Assume that the maximum is reached for $2m+1\in(x, 2x]$. Applying (1.7), as earlier, we deduce

$$T(2x)\leqq\left(1+\frac{1}{x-1}\right)T(x)+\frac{1}{2(x-1)}T(4x^2).$$

Since $4x(x+1)<8x^2$, $2(x-1)>x$ for large x and $T(x)/x\to0$, we have

$$T(8x)\leqq T(x)+\frac{T(8x^2)}{x}+\varepsilon_x,$$

where $\varepsilon_x=\dfrac{T(x)}{x}\to0$.

Assume that $\gamma>1/2$. Then

$$\frac{T(8x^2)}{x}<x^{2\gamma-1},\quad \varepsilon_x\ll x^{2\gamma-1},$$

so that

$$T(2x)\leqq T(x)+cx^{2\gamma-1} \tag{1.8}$$

for $x\geqq x_0$. Putting $x_k=2^kx_0$ $(k=0, 1, 2, \ldots, N-1)$, we deduce that

$$T(2^Nx_0)\leqq T(x_0)+c\sum x_k^{2\gamma-1}\ll(2^Nx_0)^{2\gamma-1}. \tag{1.9}$$

By the monotonity of T, we have $T(x)=O(x^{2\gamma-1})$. So we have proved the following assertion: If $\frac{1}{2}<\gamma<1$, and $T(x)\ll x^\gamma$, then $T(x)\ll x^{2\gamma-1}$. Repeating this argument for $\gamma=\gamma_1$, $2\gamma_1-1=\gamma_2$, ..., in finitely many steps we get an exponent $\gamma_i\in(0, 1/2)$ such that $T(x)\ll x^{\gamma_i}\ll x^{1/2}$. Assume now that $\gamma=1/2$. Then (1.8) holds, i.e. $T(2x)\leqq T(x)+c$, and instead of (1.9) we get

$$T(2^Nx_0)\leqq T(x_0)+O(N).$$

Consequently $T(x)=O(\log x)$. Since for $\gamma<1/2$ we have $T(x)\ll x^\gamma\ll x^{1/2}$, therefore we have $T(x)=O(\log x)$ whenever $T(x)\ll x^\gamma$, $\gamma<1$.

Now let m_1, m_2 be chosen so that

$$T\big((m_1+1)m_2\big)<\frac{1}{2}m_2,\quad T\big((m_2+1)m_1\big)<\frac{1}{2}m_1.$$

This implies (1.4). From (1.5) we deduce that

$$|m_1\sigma(m_1)-m_2\sigma(m_2)|\leqq K(\log m_1m_2)\left(\frac{1}{m_1}+\frac{1}{m_2}\right) \tag{1.10}$$

holds with a suitable constant K for every pair m_1, m_2 satisfying

$$(c_1<)\,m_1<m_2<e^{\sigma m_1} \tag{1.11}$$

with a small positive constant σ, and a positive constant c_1. From (1.10) we get

$$\sigma(m_2)-\frac{m_1}{m_2}\sigma(m_1) = B\frac{\log m_1 m_2}{m_1 m_2}$$

for m_1, m_2 satisfying (1.11), where B is a bounded variable.

Now let $m_1=U$ and m_2 run over the interval $[U, 2U-1]$. Then we have

$$f(2) = J(U)+\sigma(U)U\sum_{m_2=U}^{2U-1}\frac{1}{m_2}+O\left(\frac{\log U}{U}\right). \tag{1.12}$$

From (1.12) we get immediately that $m\sigma(m)$ varies slowly. Consequently, if $m\sigma(m)$ is not bounded then the set of the fractional parts of $m\sigma(m)$ is a dense subset in [0, 1). Let $\alpha\in[0, 1)$ be chosen so that $\{f(2)\}\neq\{\alpha\log 2\}$. Let U_j be an infinite sequence such that $\{\sigma(U_j)U_j\}\to\alpha$. Putting $U=U_j$ into (1.12), and taking into account that

$$\sum_{U_j}^{2U_j-1} 1/m_2 \to \log 2,$$

we get that $\{f(2)\}=\{\alpha\log 2\}$, which contradicts our assumption.

The proof of our theorem is complete.

2. Let $f(n)$ be a completely additive function, $N_1<N_2<\ldots$ an infinite sequence of integers, $J_N=[N, N+(2+\varepsilon)\sqrt{N}]$, and $\varepsilon>0$ a constant.

Theorem 2. *If*

$$f(n)\equiv\alpha_j \pmod 1 \quad \textit{for} \quad n\in J_{N_j} \quad (j=1, 2, \ldots) \tag{2.1}$$

where $\alpha_1, \alpha_2, \ldots$ are arbitrary real numbers, then $\alpha_1=\alpha_2=\ldots=0$ and $f(n)$ takes on integer values only.

Proof. The method of proof is almost the same as that used in [3]. First we prove the following

Lemma. *Let $1<v_1<v$, v_1, v be constants. Assume that $f(x)\equiv\alpha \pmod 1$ in the interval $J_N=[N, vN]$. Then for every $N\geqq N_0(v_1, v)$ we have*

$$f(n)\equiv 0 \pmod 1, \quad n<(v-v_1)N.$$

Let p, q be arbitrary integers satisfying the conditions

$$p<q<v_1p, \quad q<(v-v_1)N. \tag{2.2}$$

For $m=\left[\frac{N}{p}\right]+1$ we get

$$N<pm<qm<q\left(\frac{N}{p}+1\right)<\frac{q}{p}N+q<v_1N+(v-v_1)N=vN,$$

consequently for every pair p, q satisfying (2.2),

$$f(p) \equiv f(n+1) \pmod 1, \quad n \in \left(\frac{1}{v_1-1}, (v-v_1)N-1\right).$$

Let $f(n) \equiv \gamma \pmod 1$. Let n be chosen so that $(v_1-1)^{-2} < n^2 < (v-v_1)N-1$. Then $f(n^2) \equiv \gamma \pmod 1$, and so $\gamma = 0$. Hence

$$f(n) \equiv 0 \pmod 1, \quad \frac{1}{v_1-1} \leqq n \leqq (v-v_1)N-1.$$

It remains to prove that $f(k) \equiv 0 \pmod 1$ for $k \leqq (v_1-1)^{-1}$. Putting $m = \left[\frac{1}{v_1-1}\right]+1$, and letting N to be large, we have $f(km) \equiv 0 \pmod 1$, and $f(m) \equiv 0 \pmod 1$, implying that $f(k) \equiv 0 \pmod 1$. This proves the Lemma.

Now we prove the theorem. Let $N_j = N$ be temporarily fixed. For an integer k let

$$I_k = \left[\frac{N}{k}, \frac{N}{k} + (2+\varepsilon)\frac{\sqrt{N}}{k}\right].$$

If the intervals I_k, I_{k+1} contain a common integer element m, then $f(k) \equiv f(k+1) \pmod 1$. Indeed, $mk, m(k+1) \in J_N$, and (2.1) holds.

There is a common element m, if

$$\frac{N+(2+\varepsilon)\sqrt{N}}{k+1} - \frac{N}{k} \geqq 0,$$

i.e., if $k^2 - ((2+\varepsilon)\sqrt{N}-1)k + N \leqq 0$. This inequality holds in the interval $k \in [k_1, k_2]$, where

$$k_1 = \frac{1}{2}\{(2+\varepsilon)\sqrt{N}-1\} - \frac{1}{2}\sqrt{\{(2+\varepsilon)\sqrt{N}-1\}^2 - 4N}, \tag{2.3}$$

$$k_2 = \frac{1}{2}\{(2+\varepsilon)\sqrt{N}-1\} + \frac{1}{2}\sqrt{\{(2+\varepsilon)\sqrt{N}-1\}^2 - 4N}, \tag{2.4}$$

and so

$$f(k) \equiv \gamma_j \pmod 1, \quad k \in [k_1, k_2].$$

It is obvious that $k_1 = k_1(N)$ as $N = N_j \to \infty$. Furthermore, in view of (2.3) and (2.4), $\frac{k_2}{k_1} \geqq 1+\varepsilon_1$, $\varepsilon_1 > 0$ for every large N. Now we may apply the Lemma with

$k_1=N$, $v=(1+\varepsilon_1)$, $v_1=1+\frac{\varepsilon_1}{2}$. Putting $N=N_j$ we deduce that $f(n)\equiv 0 \pmod 1$ for $n<\frac{\varepsilon_1}{2}N_j$, i.e., for every n.

This completes the proof.

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