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Attractors of systems close to autonomous ones

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1. Introduction. Most of the papers on stability theory of non-autonomous systems of differential equations start with the non generic assumption that $x=0$ is a solution of the system $\dot{x}=f(t, x)$, i.e. $f(t, 0)\equiv 0$. (The reason of this is that this state of affairs is achieved in case, originally, another system is given with a known solution and the system for the variation of the solutions is formed.) Now, clearly, in the generic case the solution x of the equation $f(t, x)=0$ will depend on t and will *not* be a constant. The method of averaging helps us to get rid of the above mentioned assumption by substituting the original non-autonomous system by the averaged autonomous one (see, e.g., [4], [5]). The cases successfully dealt with by the method of averaging are the most important ones, still, these are special cases in which it is assumed that the original system is periodic, almost periodic, asymptotically almost periodic, etc. These assumptions make it possible to say something about the stable solution of the original system that emanates from the stable equilibrium of the averaged autonomous one.

In this paper these assumptions will be dismissed apart from the assumption that the system is close in a certain simple sense to an autonomous one. An asymptotically stable equilibrium of the latter system gives rise to an attractor of the original non-autonomous system. This attractor is, in general, not the integral curve of a single solution but an invariant set which is the thinner the closer the two systems are to each other. The existence of this attractor is ensured by theorems due to YOSHIZAWA [7], [8].

We are giving an explicit upper estimate of this attractor and a lower estimate to its region of attractivity. We omit here the proof of the estimates since it is similar to the proof of the theorem of paper [1]. As an example, the results are applied to van der Pol's equation under bounded perturbation in case time tends to minus infinity.

The problem treated here is connected with the problem of structurally stable ("rough"=грубые in the Soviet literature) properties of systems (see., e.g.,

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Sections 18 and 19 in [2]). However, in our case the majorant function is a constant and so the perturbation need not disappear at the equilibrium point of the unperturbed system. What could be considered here as a "structurally stable property" is the perseverance of an attractor whose character may change. The problem is also connected to the concept of practical stability (see [3] Section 25) and the result can be considered as a method to estimate the "region of practical stability".

2. The attractor and the region of attractivity of a non-autonomous system close to an autonomous one. Assume that $\Omega\subset R^n$ is an open set containing the origin,

$$f\in C^0[R^+\times\Omega, R^n],\quad f_x'\in C^0[R^+\times\Omega, R^{n^2}],\quad g\in C^2[\Omega, R^n],$$

and for any compact $Q\subset\Omega$, $|f_x'|$ is bounded over $R^+\times Q$ where $R^+=[0, \infty)$ and $x=\mathrm{col}\,(x_1, \ldots, x_n)\in R^n$. Consider the systems of differential equations

$$\dot{x} = f(t, x) \tag{1}$$

and

$$\dot{x} = g(x) \tag{2}$$

where dot denotes differentiation with respect to $t\in R^+$. Assume further that there exists an $\eta>0$ such that

$$|f(t, x)-g(x)| < \eta, \quad (t, x) \in R^+\times\Omega. \tag{3}$$

Without loss of generality let $g(0)=0$, and assume that the real parts of all the eigenvalues of the matrix $g'(0)$ are negative.

Under these conditions, as it is well known, one can find a positive definite quadratic form $w(x)=x^{\mathrm{T}}Wx$ where x^{T} denotes the transpose of the column vector x, such that the derivative of w with respect to system

$$\dot{y} = g'(0)y \tag{4}$$

is negative definite. Moreover, there exist constants $\varrho_1>0$, $\varrho_2>0$ such that

$$U_{\varrho_1} = \{x\in R^n\colon |x| < \varrho_1\} \subset \Omega, \quad \text{and} \quad \dot{w}_{(2)}(x) \leqq -\varrho_2 w(x) \quad \text{for} \quad x\in U_{\varrho_1}. \tag{5}$$

Let us denote the eigenvalues of the positive definite matrix W by λ_i, $i=1, 2, \ldots, n$, and let $0<\lambda_1\leqq\lambda_2\leqq\ldots\leqq\lambda_n$. Clearly,

$$\lambda_1|x|^2 \leqq w(x) \leqq \lambda_n|x|^2, \quad x\in R^n. \tag{6}$$

Finally, let us introduce the notations

$$A_\eta = \{x\in R^n\colon w(x) \leqq 4\lambda_n^2\eta^2/\varrho_2^2\lambda_1\}, \tag{7}$$

$$B = \{x\in R^n\colon w(x) < \lambda_1\varrho_1^2\}. \tag{8}$$

We are now in the position to state the following

Theorem. *Under the conditions imposed upon systems* (1), (2) *and* (4), *if*

(9) $$0 < \eta < \lambda_1 \varrho_1 \varrho_2 / 2\lambda_n,$$

then the set $R^+ \times A_\eta$ *is a uniform asymptotically stable invariant set of system* (1) *and its region of attractivity contains the set* $R^+ \times B$.

The proof is similar to the proof of the Theorem in [1].

3. Van der Pol's equation under bounded perturbation. It is well known (see e.g. [6]) that for van der Pol's equation

(10) $$d^2u/d\tau^2 + m(u^2-1)\,du/d\tau + u = 0, \quad m > 0$$

the origin of the phase plane $(u, du/d\tau) = (0, 0)$ is an asymptotically stable equilibrium *in the past*, i.e., for $\tau \to -\infty$, whose region of attractivity is the open region inside the path of the single non-constant periodic solution. Substituting $t = -\tau$ equation (10) turns into

(11) $$\ddot{u} + m(1-u^2)\dot{u} + u = 0, \quad m > 0$$

where dot denotes differentiation with respect to t. For equation (11) the origin $(u, \dot{u}) = (0, 0)$ is asymptotically stable (in the future) with a bounded region of attractivity.

We are going to consider (11) under a bounded non-autonomous perturbation. First of all a Liapunov function will be constructed to the system

(12) $$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 - mx_2 + mx_1^2 x_2$$

which is equivalent to (11): $x_1 = u$. The linearized system is

(13) $$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 - mx_2.$$

The latter system is, clearly, asymptotically stable and, thus, it is easy to find a positive definite quadratic form whose derivative with respect to system (13) is negative definite. For instance, the quadratic form

(14) $$w(x) = \frac{m^2+2}{2m} x_1^2 + x_1 x_2 + \frac{1}{m} x_2^2$$

is positive definite and $\dot{w}_{(13)}(x) = -(x_1^2 + x_2^2)$. Moreover,

(15) $$\dot{w}_{(13)}(x) \leqq -\alpha w(x)$$

if $0 < \alpha \leqq m\left(1 - m/(m^2+4)^{1/2}\right)$. The derivative of w with respect to the system (12) is

$$\dot{w}_{(12)}(x) = -(x_1^2 + x_2^2) + x_1^2(mx_1x_2 + 2x_2^2).$$

Introducing the notation

(16) $$\varrho_2 = m\left(1-m/(m^2+4)^{1/2}\right)-\delta$$

where $0<\delta<m\left(1-m/(m^2+4)^{1/2}\right)$, we are going to determine $\varrho_1>0$ so that $\dot{w}_{(12)}(x)\leqq -\varrho_2 w(x)$ should hold for $|x|<\varrho_1$. In the expression

$$-\dot{w}_{(12)}(x)-\varrho_2 w(x) = -x_1^2(mx_1x_2+2x_2^2)+\delta w(x)-[\dot{w}_{(13)}(x)+m\left(1-m/(m^2+4)^{1/2}\right)w(x)]$$

the quadratic form in square brackets is negative semidefinite in view of (15). Thus, the whole expression is non-negative provided that

(17) $$\delta w(x)-x_1^2(mx_1x_2+2x_2^2) = \left(\delta\frac{m^2+2}{2m}-2x_2^2\right)x_1^2+(\delta-mx_1^2)\,x_1x_2+\frac{\delta}{m}x_2^2 \geqq 0.$$

In case $x_1x_2<0$, if

$$\delta\frac{m^2+2}{2m}-2x_2^2 \geqq 0 \text{ and } 2\left(\frac{\delta}{m}\left(\delta\frac{m^2+2}{2m}-2x_2^2\right)\right)^{1/2} \geqq \delta-mx_1^2 \geqq 0,$$

then

$$0 \leqq \left(\left(\delta\frac{m^2+2}{2m}-2x_2^2\right)^{1/2}x_1+\left(\frac{\delta}{m}\right)^{1/2}x_2\right)^2 \leqq$$

$$\leqq \left(\delta\frac{m^2+2}{2m}-2x_2^2\right)x_1^2+\frac{\delta}{m}x_2^2+(\delta-mx_1^2)x_1x_2.$$

In case $x_1x_2\geqq 0$, (17) holds provided that

$$\delta\frac{m^2+2}{2m}-2x_2^2 \geqq 0 \quad\text{and}\quad \delta-mx_1^2 \geqq 0.$$

A simple calculation yields that in both cases (17) holds if

$$x_1^2+x_1^2 \leqq \frac{\delta}{m}\min\left(1, \frac{m^2+4}{8}\right).$$

We can summarize the result in the following way. Let us define the function

$$r^2(m) = \begin{cases} (m^2+4)/8m & \text{if } \ 0<m\leqq 2 \\ 1/m & \text{if } \ 2<m. \end{cases}$$

If $|x|<r(m)\delta^{1/2}$ then $\dot{w}_{(12)}(x)\leqq -\varrho_2 w(x)$ where ϱ_2 is given by (16). Thus, our Theorem can be applied to the equation

$$\ddot{u}+m(1-u^2)\dot{u}+u = F(t, u, \dot{u}),$$

if $F, F'_u, F'_{\dot{u}}$ are continuous functions and $|F(t, u, \dot{u})|<\lambda_1\varrho_1\varrho_2/2\lambda_2$ for $t\in R^+$, $(u^2+\dot{u}^2)^{1/2}<\varrho_1$ where $\varrho_1=r(m)\delta^{1/2}$, ϱ_2 is given by (16) and $0<\lambda_1<\lambda_2$ are the easily computable eigenvalues of the quadratic form (14). Instead of giving the details in general, we are presenting a numerical example setting $m=0.20$.

Consider the equation

$$\ddot{u}+0.20(1-u^2)\dot{u}+u = F(t, u, \dot{u}). \tag{18}$$

The quadratic form (14) is now

$$w(x) = 5.1x_1^2+x_1x_2+5.0x_2^2.$$

$\varrho_2=0.18-\delta$, $r^2(0.20)=2.5$, $\varrho_1=1.6\delta^{1/2}$ and the eigenvalues of the quadratic form are $\lambda_1=4.6$, $\lambda_2=5.6$. It is assumed that F satisfies $|F(t, x_1, x_2)|<\eta$ for $t\in R^+$, $|x|<1.6\delta^{1/2}$. The value of η will be specified later. The projections of the attractor and its region of attractivity to the x-plane are, by (7) and (8),

$$A = \{x\in R^2\colon w(x) \leqq \eta^2 27/(0.18-\delta)^2\}, \quad B = \{x\in R^2\colon w(x) < 11\delta\},$$

respectively. According to (16), δ can be chosen arbitrarily between 0 and 0.18. We want to minimize the attractive set A_η and maximize its region of attractivity B at the same time. A way of doing this is to maximize the ratio

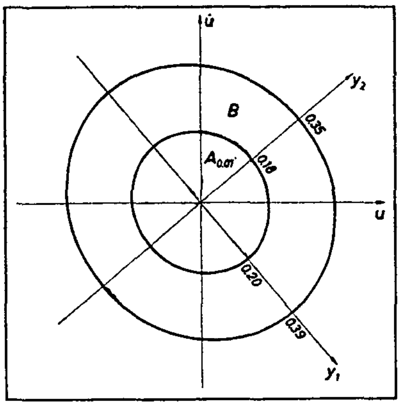
$$\frac{11\delta}{27/(0.18-\delta)^2} = 0.41\,\delta(0.18-\delta)^2.$$

One easily gets that the maximum of this ratio in the interval [0, 0.18] is achieved at $\delta=0.060$.

Substituting this value of δ into the formulae we get

$$A_\eta = \{x\in R^2\colon\ w(x) \leqq \eta^2 1900\}, \quad B = \{x\in R^2\colon\ w(x) < 0.66\}.$$

Thus, if $\eta<(0.66/1900)^{1/2}=0.019$ and $|F(t, x_1, x_2)|<\eta$ for $t\in R^+$, $|x|<0.39$, then $R^+\times A_\eta$ is a uniform asymptotically stable invariant set of the equation (18) and the set $R^+\times B$ is contained in its region of attractivity. The Figure below shows the projection of these sets into the $(u, \dot{u})=(x_1, x_2)$ plane in case $\eta=0.01$.



Figure

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