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On the Riesz summability of eigenfunction expansions

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Dedicated to Professor Béla Szőkefalvi-Nagy on the occasion of his 70th birthday

Let Ω be an arbitrary bounded domain in $\mathbf{R}^N$ $(N\geqq 3)$ having C^∞-smooth boundary, and q an arbitrary non-negative function from the class $L_2(\Omega)$. Consider the Schrödinger operator

$$L = L(x, D) = -\Delta + q(x)\cdot .$$

Denote $\hat{L}$ an arbitrary positive selfadjoint extension of the operator L from the domain $C_0^\infty(\Omega)$ with discrete spectrum. According to a theorem of K. O. FRIEDRICHS [1, 2] there exists such a selfadjoint extension. Let $0<\lambda_1\leqq\lambda_2\leqq\dots$ denote the sequence of the eigenvalues of the operator $\hat{L}$ and let $\{u_n\}_1^\infty$ be the complete orthonormal system of the corresponding eigenfunctions in $L_2(\Omega)$. For any $s\geqq 0$ and $f\in L_2(\Omega)$, consider the s-th Riesz means of the spectral expansion of f:

$$E_\lambda^s f(x) = \sum_{\lambda_n<\lambda}\left(1-\frac{\lambda_n}{\lambda}\right)^s (f, u_n)\, u_n(x).$$

It is assumed in this work that the potential q is spherically symmetric. Namely, let $a\in C^\infty(0, \infty)$ be a non-negative function satisfying

$$\text{(1)}\qquad t^k|a^{(k)}(t)| \leqq C_\tau t^{\tau-1} \quad (t>0;\ k=0, 1 \dots, [N/2])$$

for some $\tau>0$. If $N=3$, then it is assumed that $\tau>1/2$. In particular, we have

$$\text{(2)}\qquad a(t) \leqq C_\tau t^{\tau-1} \quad (t>0).$$

The constant C_τ depends only on τ. Now assume that the potential q has the form

$$q(x) = \frac{a(|x-x_0|)}{|x-x_0|}, \quad x\in\Omega,$$

where $x_0\in\Omega$ is an arbitrary but fixed point.

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Denote by $\mathring{L}_p^l(\Omega)$ the set of those elements of $L_p^l(\mathbf{R}^N)$ for which $\operatorname{supp} f \subset \bar{\Omega}$. It is well known that $C_0^\infty(\Omega)$ is dense in $\mathring{L}_p^l(\Omega)$ with respect to the norm of $L_p^l(\mathbf{R}^N)$ (cf. [14], 4.3.2/1(b)).

We shall prove the following theorems.

Theorem 1. *Let* $p \geqq 1, s \geqq 0, l > 0, l+s \geqq (N-1)/2, pl > N$. *Then for any* $f \in \mathring{L}_p^l(\Omega)$,

(3) $$\lim_{\lambda\to\infty} E_\lambda^s f(x) = f(x), \quad x \in \Omega.$$

Theorem 2. *Let* $s \geqq 0, l \geqq 0, l+s \geqq (N-1)/2$. *Then for any* $f \in \mathring{L}_2^l(\Omega)$,

(4) $$\lim_{\lambda\to\infty} E_\lambda^s f(x) = 0, \quad x \in \Omega \setminus \operatorname{supp} f.$$

Remark. For $q \equiv 0$, Theorem 1 was proved in [10] and it was extended in [5], for an arbitrary elliptic operator with smooth coefficients. Earlier, the case of $q \equiv 0, s = 0$, when l is an integer was settled by V. A. Il'in [8]. In [10] it is proved that if $l+s < (N-1)/2, p = \infty$ and $q \equiv 0$, then (3) does not hold for any $f \in \mathring{L}_p^l(\Omega)$.

For the proof of the Theorems we have to estimate the eigenfunctions u_n. We use the method of V. A. Il'in [9] and to this we need a mean value formula for the functions u_n. Thereafter, the theorems follow by applying Hörmander's Tauber type theorem [7]. To this it is necessary to estimate the Fourier coefficients of functions from Liouville classes using interpolation theorems and to estimate the resolvent of the operator $\hat{L}$ outside of an angular domain, which contains the spectrum $\{\lambda_n\}$.

1. The mean value formula and its application

Define

$$W(t, r) = \frac{1}{4}\pi^{1-N/2}\Gamma\left(\frac{N}{2}\right)[J_{N/2-1}(t)Y_{N/2-1}(r) - Y_{N/2-1}(t)J_{N/2-1}(r)],$$

$$\omega(t) = \min(1, t^{(1-N)/2}),$$

where J_ν and Y_ν denote the ν-th Bessel and Neumann function [6], respectively.

Lemma 1.1. *The estimate*

(1.1) $$r^{1-N/2}\int_0^r \omega(t\mu)|W(t\mu, r\mu)|t^{N/2-1}a(t)\,dt \leqq C_1\mu^{-\tau}\omega(\mu r)$$

holds for every $r > 0$ *and* $\mu > 0$.

Proof. Using well known asymptotic formulas (cf. [6], 7.2.1 (2), (4); 7.13.1 (3), (4)) it follows

$$|W(t,r)| \leqq C_0 \frac{1}{\sqrt{tr}}, \quad \text{if} \quad t \geqq 1, \tag{1.2}$$

$$|W(t,r)| \leqq C_0 \frac{1}{t^{N/2-1}\sqrt{r}}, \quad \text{if} \quad 0 < t \leqq 1 \leqq r, \tag{1.3}$$

$$|W(t,r)| \leqq C_0 \left(\frac{r}{t}\right)^{N/2-1}, \quad \text{if} \quad 0 < t \leqq r \leqq 1. \tag{1.4}$$

If $r\mu \leqq 1$ then, by (1.4), it follows

$$I \leqq C_0 \int_0^r a(t)\,dt \leqq C_0 C_\tau \int_0^r t^{\tau-1}\,dt \leqq \frac{C_0 C_\tau}{\tau} r^\tau \leqq \frac{C_0 C_\tau}{\tau} \mu^{-\tau}\omega(r\mu),$$

where I denotes the left hand side of (1.1). If $r\mu > 1$, then we use the decomposition $I = \int_0^{1/\mu} + \int_{1/\mu}^r = I_1 + I_2$ and apply to the estimation of I_1 and I_2, (1.3) and (1.2), respectively. It follows

$$I_1 \leqq C_0 r^{1-N/2} \int_0^{1/\mu} (t\mu)^{1-N/2} \frac{1}{\sqrt{r\mu}} t^{N/2-1} a(t)\,dt = C_0 (r\mu)^{(1-N)/2} \int_0^{1/\mu} a(t)\,dt \leqq$$

$$\leqq \frac{C_0 C_\tau}{\tau} \mu^{-\tau}\omega(r\mu),$$

$$I_2 \leqq C_0 r^{1-N/2} \int_{1/\mu}^r (t\mu)^{1-N/2} \frac{1}{\sqrt{t\mu}} \frac{1}{\sqrt{\mu r}} t^{N/2-1} a(t)\,dt =$$

$$= C_0 (r\mu)^{(1-N)/2} (1/\mu) \int_{1/\mu}^r \frac{a(t)}{t}\,dt \leqq \frac{C_0 C_\tau}{\tau} \omega(r\mu) \int_{1/\mu}^r t^{-2+\tau}\,dt \leqq \frac{C_0 C_\tau}{1-\tau} \mu^{-\tau}\omega(r\mu).$$

Lemma 1.2. *For every $r > 0$, $\mu_n > C_2$,*

$$\int_\theta u_n(x_0 + r\theta)\,d\theta = u_n(x_0)[C_N (r\mu_n)^{1-N/2} J_{N/2-1}(r\mu_n) + \alpha(r, \mu_n)], \tag{1.5}$$

$$|\alpha(r, \mu_n)| \leqq \text{const}\, \mu_n^{-\tau}\omega(r\mu_n), \tag{1.6}$$

where $\mu_n = \sqrt{\lambda_n}$, $C_N = 2^{N/2-1}\Gamma(N/2)$. Here $\int_\theta f(x_0 + r\theta)d\theta$ denotes the integration with respect to the normalized Lebesgue measure over the sphere of radius r and centred in x_0.

Proof. Recall the mean value formula of E. C. TITCHMARSH (cf. [13]; p. 232) stating

$$\int_{\theta} u_n(x_0+r\theta)\,d\theta = u_n(x_0)C_N(r\mu_n)^{1-N/2}J_{N/2-1}(r\mu_n)+$$

$$+\int_0^r\Big(\int_{\theta} q(x_0+t\theta)u_n(x_0+t\theta)\,d\theta\Big)(t/r)^{N/2-1}tW(t\mu_n, r\mu_n)\,dt.$$

In the case of our spherically symmetrical potential we get

$$\int_{\theta} u_n(x_0+r\theta)\,d\theta = u_n(x_0)C_N(r\mu_n)^{1-N/2}J_{N/2-1}(r\mu_n)+$$

$$+r^{1-N/2}\int_0^r\Big(\int_{\theta} u_n(x_0+t\theta)\,d\theta\Big)W(t\mu_n, r\mu_n)t^{N/2-1}a(t)\,dt,$$

i.e., the function $v^*(r, \mu_n)=\int_{\theta} u_n(x_0+r\theta)d\theta$ satisfies the integral equation

$$v^*(r, \mu_n) = u_n(x_0)C_N(r\mu_n)^{1-N/2}J_{N/2-1}(r\mu_n)+$$

$$+r^{1-N/2}\int_0^r v^*(t, \mu_n)W(t\mu_n, r\mu_n)t^{N/2-1}a(t)\,dt$$

of Volterra type (cf. [2]). Define

$$v_0(r, \mu_n) = u_n(x_0)C_N(r\mu)^{1-N/2}J_{N/2-1}(r\mu),$$

$$v_k(r, \mu) = r^{1-N/2}\int_0^r v_{k-1}(t, \mu)W(t\mu, r\mu)t^{N/2-1}a(t)\,dt.$$

The estimates $|v_k(r; \mu)|\leqq \text{const}\,\omega(r\mu)[c_1/\mu^{\tau}]^k$ $(r>0,\ \mu>0)$ follow by induction on k. On the other hand, it is easy to see that

$$v^*(r, \mu_n) = u_n(x_0)v_0(r, \mu_n)+u_n(x_0)\sum_{k=1}^{\infty} v_k(r, \mu_n).$$

Hence (1.5) and (1.6) follow for the function

$$\alpha(r, \mu) \overset{\text{def}}{=} \sum_{k=1}^{\infty} v_k(r, \mu).$$

Lemma 1.3. *We have*

(1.7) $$\sum_{\substack{n \\ |\mu_n-\mu|\leqq 1}} |u_n(x_0)|^2 \leqq C_3\mu^{N-1} \quad (\mu \geqq 1).$$

The constant C_3 *does not depend on* μ.

Proof. We use the method of V. A. IL'IN [8, 9]. Consider the function

$$d(r,\mu)=\begin{cases}\mu^{N/2}\dfrac{J_{N/2-1}(r\mu)}{r^{N/2-1}} & \text{if} \quad R<r<2R\\ 0 & \text{if} \quad r\notin(R,2R),\end{cases}$$

where $0<2R<\operatorname{dist}(x_0,\partial\Omega)$, $r=|x-x_0|$, $\mu>0$. Calculate the Fourier coefficients of $d(r,\mu)$ with respect to the system $\{u_n\}$. Taking into consideration (1.5), we obtain

$$d_n=d_n(\mu)=\int_\Omega d(|x-x_0|,\mu)\,u_n(x)\,dx=$$

$$=\mu^{N/2}\int_R^{2R}\frac{J_{N/2-1}(r\mu)}{r^{N/2-1}}\Big(\int_\theta u_n(x_0+r\theta)\,d\theta\Big)r^{N-1}\,dr=$$

$$=\mu^{N/2}u_n(x_0)\Big[C_N\int_R^{2R}J_{N/2-1}(r\mu)\frac{J_{N/2-1}(r\mu_n)}{\mu_n^{N/2-1}}\,r\,dr+\int_R^{2R}J_{N/2-1}(r\mu)\alpha(r,\mu_n)r^{N/2}\,dr\Big].$$

It is proved in [8] that

$$\Big|\int_R^{2R}J_{N/2-1}(r\mu)J_{N/2-1}(r\mu_n)\,r\,dr\Big|\geqq c\cdot\frac{1}{\mu},$$

if $|\mu-\mu_n|\leqq 1$ and μ is large enough, where the constant c does not depend on μ. Hence

$$\Big|\mu^{N/2}u_n(x_0)\int_R^{2R}J_{N/2-1}(r\mu)\frac{J_{N/2-1}(r\mu_n)}{\mu_n^{N/2-1}}\,r\,dr\Big|\geqq c\cdot|u_n(x_0)|,$$

if $|\mu-\mu_n|\leqq 1$ and $\mu\geqq\mu_0$. On the other hand, using (1.6), we obtain

$$\Big|\int_R^{2R}J_{N/2-1}(r\mu)\alpha(r,\mu_n)r^{N/2}\,dr\Big|\leqq\text{const}\,\mu_n^{-\tau}\Big|\int_R^{2R}J_{N/2-1}(r\mu)\omega(r\mu_n)r^{N/2}\,dr\Big|\leqq$$

$$\leqq\text{const}\,\mu_n^{-\tau}\mu_n^{(1-N)/2}\Big|\int_R^{2R}J_{N/2-1}(r\mu)\sqrt{r}\,dr\Big|=O\Big(\mu_n^{-\tau}\mu_n^{(1-N)/2}\cdot\frac{1}{\sqrt{\mu}}\Big).$$

Summarising our estimates, we get

$$|d_n(\mu)|\geqq\text{const}\,|u_n(x_0)|\Big[1+O\Big(\frac{1}{\mu^\tau}\Big)\Big]\geqq\text{const}\,|u_n(x_0)|,$$

if $|\mu_n-\mu|\leqq 1$ and μ is large enough. Hence the desired estimate (1.7) follows by applying the Parseval equality and the relation

$$\int_\Omega|d(|x-x_0|,\mu)|^2\,dx=O(\mu^{N-1}).$$

2. Estimates for the Fourier coefficients of functions from Liouville spaces

Lemma 2.1. *Let k be a natural number and $\alpha=(\alpha_1, \dots, \alpha_N)$ a multiindex such that $0\leqq|\alpha|<k\leqq N/2$. Then for every $\varepsilon>0$, there exists a constant $C_4=C_4(\varepsilon)$ for which*

$$\||x-x_0|^{\varepsilon-(k-|\alpha|)}D^\alpha f(x)\|_{L_2(\Omega)} \leqq C_4\|f\|_{W_2^k(\Omega)} \tag{2.1}$$

holds for all $f\in W_2^k(\Omega)$.

Proof. The estimate follows immediately using classical imbedding theorems and the Hölder inequality.

Lemma 2.2. *For every natural number k, $0\leqq k\leqq N/2$, the estimate*

$$\|\hat{L}^{k/2}f\|_{L_2} \leqq C_5\|f\|_{W_2^k} \tag{2.2}$$

holds for every $f\in \mathring{W}_2^k(\Omega)$. The constant C_5 does not depend on f.

Proof. According to the spectral theorem, we have to prove the estimate

$$\sum_{n=1}^{\infty} \lambda_n^k|(f, u_n)|^2 \leqq \text{const}\,\|f\|_{W_2^k}. \tag{2.3}$$

By definition, $\mathring{W}_2^k(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the space $W_2^k(\Omega)$. Hence it is enough to prove (2.3) for functions from the class $C_0^\infty(\Omega)$. If $k=2m$, then we have

$$\sum_{n=1}^{\infty} \lambda_n^k|(f, u_n)|^2 = \|(\Delta-q)^m f\|_{L_2}^2$$

for every $f\in C_0^\infty(\Omega)$. In this case we use Lemma 2.1 and the following simple facts: for any natural number $1\leqq m\leqq N/2$, we can write

$$(\Delta-q)^m = \Delta^m + \sum_{|\alpha|\leqq 2m-2} C_{m,\alpha}(x)D^\alpha, \tag{2.4}$$

where the functions $C_{m,\alpha}(x)$ belong to $C^\infty(\Omega\setminus\{x_0\})$ and (for each multiindex β) we have

$$|D^\beta C_{m,\alpha}(x)| \leqq \text{const}\,|x-x_0|^{|\alpha|+\tau-2m-|\beta|}. \tag{2.5}$$

These facts follow easily by induction on m. For the sake of simplicity, in this section we assume $x_0=0$. In the case $k=2m$, (2.3) follows immediately. If $k=2m+1$, then we have

$$\sum_{n=1}^{\infty} \lambda_n^k|(f, u_n)|^2 = \|\nabla(\Delta-q)^m f\|_{L_2}^2 - \|\sqrt{q}\,(\Delta-q)^m f\|_{L_2}^2. \tag{2.6}$$

For the estimation of the first term on the right hand side we use (2.4):

$$\nabla(\Delta-q)^m f = \nabla\Delta^m f + \sum_{|\alpha|\leqq 2m-2} [(\nabla C_{m,\alpha})D^\alpha f + C_{m,\alpha}\nabla D^\alpha f],$$

so it is enough to prove the estimate

$$\| |x|^{|\alpha|-k+\tau} D^\alpha f \|_{L_2} \leqq \text{const} \|f\|_{W_2^k},$$

for every natural number $k, 0=k\leqq N/2$, but this is the statement of Lemma 2.1. To estimate the second term on the right hand side of (2.6), we use (2.4). It follows:

$$\sqrt{q}(\Delta-q)^m f = \sqrt{q}\Delta^m f + \sum_{|\alpha|\leqq 2m-2} \sqrt{q}\, C_{m,\alpha} D^\alpha f.$$

Hence, taking into account the trivial estimates

$$|(\sqrt{q}\,\Delta^m f)(x)| \leqq \text{const}\, |x|^{-1+\tau/2} |\Delta^m f(x)| \leqq \text{const} \sum_{|\alpha|=2m} |x|^{\tau/2-(k-|\alpha|)} |D^\alpha f(x)|$$

and

$$|(\sqrt{q}\, C_{m,\alpha} D^\alpha f)(x)| \leqq \text{const}\, |x|^{-1+\tau/2} |x|^{|\alpha|+\tau-2m} |D^\alpha f(x)| \leqq \text{const}\, |x|^{3\tau/2-(k-|\alpha|)},$$

the desired result follows by applying Lemma 2.1.

Lemma 2.3. *For any real number* s, $0\leqq s\leqq N/2$, *we have*

$$\|\hat{L}^{s/2} f\|_{L_2} \leqq C_6 \|f\|_{L_2^s} \tag{2.7}$$

for every $f\in \mathring{L}_2^s(\Omega)$. *The constant* C_6 *does not depend on* f.

Proof. We use Lemma 2.2 and apply Theorem 4.3.2/2 of [14] for $\mathscr{H}=L_2(\Omega)$; $W=\mathring{W}_2^{[N/2]}(\Omega)$, $A=\hat{L}^{[N/2]/2}$. Using the notations of [14], we obtain $(L_2; \mathring{W}_2^l)_{\theta,2}=\mathring{L}_2^{\theta l}$ for $l=[N/2]$, $s=\theta l$. (Here $A^\theta=\hat{L}^{s/2}$, $\mathring{L}_2^{\theta l}=\mathring{L}_2^s$.) Hence (2.7) follows.

Lemma 2.4. *There exists* $\sigma>N/4$ *such that*

$$C_0^\infty(\Omega)\subset \text{dom}\,(\hat{L}^\sigma). \tag{2.8}$$

Proof. Let $m=[N/4]$. Applying (2.4) we have

$$\hat{L}^m f = \Delta^m f + \sum_{|\alpha|\leqq 2m-2} C_{m,\alpha} D^\alpha f, \quad f\in C_0^\infty(\Omega),$$

and

$$|\hat{L}^m f(x)| \leqq \text{const}\, |x|^{\tau-2m}, \quad |\nabla \hat{L}^m f(x)| \leqq \text{const}\, |x|^{\tau-2m-1}.$$

Hence, $\nabla \hat{L}^m f\in L_p$ if $(\tau-2m-1)p>-N$, i.e., $N/p>2m-\tau+1$. It follows $\hat{L}^m f\in W_p^1$ if $f\in C_0^\infty(\Omega)$. By the classical imbedding theorem $W_p^1 \hookrightarrow W_2^\delta$ if $\delta-N/2=1-N/p$ (cf. [12], Ch. 6). It follows: $\hat{L}^m f\in \mathring{L}_2^\delta$ for every $\delta<N/2-2m+\tau$. Thus, using Lemma 2.3, we get $\hat{L}^{m+\delta/2} f\in L_2$, i.e., $f\in \text{dom}\,(\hat{L}^{m+\delta/2})$ for every $f\in C_0^\infty(\Omega)$. Choose $\delta= N/2-2m+\tau-\varepsilon$, where $\varepsilon>0$ is small enough; then $m+\delta/2=m+N/4-m+ +\tau/2-\varepsilon/2=N/4+(\tau-\varepsilon)/2$, and if $\varepsilon<\tau$, then we have $\sigma \overset{\text{def}}{=} m+\delta/2>N/4$.

3. Estimation of the Green function

Let $R(\lambda, \hat{L})$ denote the resolvent of the operator $\hat{L}$, i.e., $R(\lambda, \hat{L})=(\hat{L}-\lambda I)^{-1}$ and $G_\mu=R(\mu^2, \hat{L})=(\hat{L}-\mu^2 I)^{-1}$. Let $0<\varepsilon_0<\pi/2$ be an arbitrary small real number and define

$$Z_0 = \{z\in\mathbf{C}:\ \varepsilon_0 \leqq \arg z \leqq \pi-\varepsilon_0\}.$$

Set $\mu=\sqrt{\lambda}$ with $\operatorname{Im}\mu\geqq 0$, i.e., $0\leqq\arg\mu\leqq\pi$.

The aim of the present paragraph is to investigate (estimate) the Green function of the operator $\hat{L}-\lambda I$, i.e., the kernel function of the resolvent $(\hat{L}-\lambda I)^{-1}$. Using the method of E. E. Levi (cf. [7], [13]) first we construct a fundamental solution $E(x, y, \mu)$ of the operator $\hat{L}-\lambda I$, i.e., a function for which

$$(-\Delta+q(x)\cdot-\mu^2 I)E(x, y, \mu) = \delta(x-y) \quad (x, y\in\Omega).$$

In case of $q\equiv 0$, the fundamental solution $E_0(x, y, \mu)$ which decreases exponentially for $\operatorname{Im}\mu>0$ is the following:

$$E_0(x, y, \mu) = C_N(\mu/r)^{N/2-1}H^{(1)}_{N/2-1}(r\mu)$$

(cf. [13], (13.7.2)). Here $H^{(1)}_\nu(z)$ denotes the ν-th Hankel function of first order. Obviously, the exponentially decreasing fundamental solution E is the solution of the integral equation

$$E(x, y, \mu) = E_0(x, y, \mu)-\int_\Omega E_0(x, u, \mu)E(u, y, \mu)\,q(u)\,du.$$

Now define

$$E_k(x, y, \mu)\overset{\text{def}}{=} E_0(x, y, \mu)-\int_\Omega E_0(x, u, \mu)E_{k-1}(u, y, \mu)\,q(u)\,du,$$

$$F_0(x, y, \mu)\overset{\text{def}}{=} E_0(x, y, \mu), \quad F_k(x, y, \mu)\overset{\text{def}}{=} E_k(x, y, \mu)-E_{k-1}(x, y, \mu) \quad (k = 1, 2, \ldots),$$

and $k_0\overset{\text{def}}{=}[(N-2)/\tau]$. Obviously,

$$E(x, y, \mu) = \sum_{k=0}^{\infty} F_k(x, y, \mu) \tag{3.1}$$

if the series is uniformly convergent. Furthermore,

$$F_k(x, y, \mu) = -\int_\Omega E_0(x, u, \mu)F_{k-1}(u, y, \mu)\,q(u)\,du. \tag{3.2}$$

Our first aim is to prove that the series (3.1) has good convergence properties. To this we must estimate the functions F_k. If $p>N/(N-2+\tau)$, then $p'<N/(2-\tau)$ $(1/p+1/p'=1)$ and hence, taking into account that, according to our assumption on q,

$$q(x) = a(|x-x_0|)|x-x_0|^{-1} \leqq c_\tau|x-x_0|^{-2+\tau}\in L_{p'}(\Omega),$$

we obtain the following estimate for F_k:

(3.3) $$|F_k(x, y, \mu)|^p \leqq \|q\|_{L_{p'}}^p \int_\Omega |E_0(x, u, \mu)|^p |F_{k-1}(u, y, \mu)|^p \, du$$

for every $p > N/(N-2+\tau)$.

Lemma 3.1. *If $k \leqq k_0$, then for any $\delta \in (0, \tau)$,*

(3.4) $$|F_k(x, y, \mu)| \leqq C_7 |x-y|^{2-N+k\delta} e^{-\alpha|x-y||\mu|} \quad (x, y \in \Omega, \ \mu \in Z_0).$$

Here C_7 and α are positive constants not depending on x, y and μ.

Proof. For $k=0$ we have $F_0 = E_0$, and (3.4) follows from

(3.5) $$|E_0(x, y, \mu)| \leqq C_7 |\mu|^{-\varkappa} |x-y|^{2-\varkappa-N} e^{-\alpha|x-y||y|} \quad (x, y \in \Omega; \ 0 \leqq \varkappa \leqq 2).$$

Here C_7 and α are positive constants, $\mu \in Z_0$. This estimate is immediate from [6] (7.2.1 (2) and (5); 7.3.1 (1)). Suppose, (3.4) is fulfilled for $k-1$ in place of k. Using (3.3) and the fact that $|x-y| \leqq |x-u| + |u-y|$ implies $e^{-\alpha|x-u||\mu|} e^{-\alpha|u-y||\mu|} \leqq \leqq e^{-\alpha|x-y||\mu|}$, we obtain by the induction hypothesis that

$$|F_k(x, y, \mu)|^p \leqq \text{const}\, e^{-\alpha|x-y||\mu|} \int_\Omega |x-u|^{(2-N)p} |u-y|^{(2-N-(k-1)\delta)\delta} \, du \leqq$$

$$\leqq \text{const}\, e^{-\alpha|x-y||\mu|p} |x-y|^{(2-N-k\delta)p}.$$

Thus (3.4) follows by induction.

Lemma 3.2. *If $k > k_0$, then for every $\varkappa \in (0, \tau)$,*

(3.6) $$|F_k(x, y, \mu)| \leqq C_8 e^{-\alpha|x-y||\mu|} (C^* |\mu|^{-\varkappa})^{k-k_0-1} \quad (x, y \in \Omega; \ \mu \in Z_0).$$

The constants C_8 and C^* do not depend on x, y and μ.

Proof. Let $k = k_0 + 1$. Choose $\delta \in (0, \tau)$ such that $N-2 < (k_0+1)\delta$. According to the definition of k_0, this is possible. Then choose $p\ (> N/(N-2+\tau))$ so that $1/p = (N-2)/N + \delta/N$, apply (3.3) and (3.5) for $\varkappa = 0$ and (3.4) for $k = k_0$, respectively. Using the notation $(N-2-k_0\delta)p = \varepsilon$ it follows that

$$|F_k(x, y, \mu)|^p \leqq \text{const}\, e^{-\alpha|x-y||\mu|p} \int_\Omega |x-u|^{p\delta-N} |u-y|^{-\varepsilon} \, du.$$

According to the definition of k_0, we have $k_0\tau \leqq N-2 < (k_0+1)\tau$ and hence $k_0\delta < N-2$, i.e., $\varepsilon > 0$. Furthermore, according to the choice of δ we have $\varepsilon = (N-2-k_0\delta)p = p\delta((N-2)/\delta - k_0) < p\delta(k_0+1-k_0) = p\delta$. A result of Titchmarsh's book ([13], 22.1) states that, in this case,

$$\int_\Omega |x-u|^{p\delta-N} |u-y|^{-\varepsilon} \, du < \infty.$$

Thus (3.6) is proved for $k = k_0 + 1$. Then (3.6) follows by induction on k.

It follows from (3.6) that the series in (3.1) converges for every $x, y \in \Omega$, if μ is large enough, and hence shifting the spectrum of the operator $\hat{L}$ we obtain

Lemma 3.3. *For any* $x, y \in \Omega$ *and* $\mu \in Z_0$, *we have the estimate*

$$|E(x, y, \mu)| \leqq C_9 |x-y|^{2-N} e^{-\alpha |x-y||\mu|}. \tag{3.7}$$

Let $E^t(x, y, \mu) \overset{\text{def}}{=} E(y, x, \mu)$ (the formal adjoint of E). A standard calculation shows that for any $f \in W_p^2(\Omega)$ with $p > N/2$ and $\operatorname{supp} f \subset \Omega$, the equality

$$\int_\Omega E^t(x, y, \mu)[L(y, D) - \mu^2] f(y)\, dy = f(x), \quad x \in \Omega$$

holds. Let Ω_0 and Ω_1 be two domains in $\mathbf{R}^N$ for which $x_0 \in \Omega_0$, $\bar{\Omega}_0 \subset \Omega_1$ and $\bar{\Omega}_1 \subset \Omega$. Let $\eta \in C_0^\infty(\Omega)$ be such that $\eta(x) = 1$ if $x \in \Omega_1$. Define

$$H(x, y, \mu) \overset{\text{def}}{=} E^t(x, y, \mu) \cdot \eta(y)$$

and

$$K(x, y, \mu) \overset{\text{def}}{=} 2(\nabla_y \eta(y)) \nabla_y E^t(x, y, \mu) + (\Delta_y \eta(y)) E^t(x, y, \mu).$$

Obviously,

$$(L(y, D) - \mu^2) H(x, y, \mu) = \eta(y)(L(y, D) - \mu^2) E^t(x, y, \mu) -$$

$$- 2(\nabla \eta) \nabla_y E^t(x, y, \mu) - (\Delta \eta) E^t(x, y, \mu).$$

Furthermore, $K(x, y, \mu) = 0$ if $y \in \Omega$ and hence, using (3.7), we get

$$|K(x, y, \mu)| \leqq \operatorname{const} e^{-c|\mu|} \quad (x \in \Omega_0,\ y \in \Omega,\ \mu \in Z_0). \tag{3.8}$$

It is easy to verify that for every $f \in W_p^{2, \text{loc}}(\Omega)$ $(p > N/2)$ the equality

$$[\hat{H}(L - \mu^2 I) f](x) = [f - \hat{K} f](x), \quad x \in \Omega_0 \tag{3.9}$$

holds, where the operation $\hat{}$ is defined by

$$(\hat{\varphi} f)(x) = \int_\Omega \varphi(x, y) f(y)\, dy.$$

On the other hand, for any $f \in C_0^\infty(\Omega)$ we have $(\hat{L} - \lambda I)^{-1} f \in W_p^{2, \text{loc}}(\Omega)$ if $p > N/2$ and $q \in L_p(\Omega)$. Now apply (3.9) for $(\hat{L} - \lambda I)^{-1} f$ $(f \in C_0^\infty(\Omega))$ instead of f. It follows

$$\hat{H} f(x) = \hat{G}_\mu f(x) - \hat{K} \hat{G}_\mu f(x) \quad (f \in C_0^\infty(\Omega),\ x \in \Omega_0)$$

and hence, by continuous extension we get

Lemma 3.4. *For every* $f \in L_2(\Omega)$,

$$\hat{G}_\mu f(x) - \hat{H} f(x) = \hat{K} \hat{G}_\mu f(x), \quad x \in \Omega_0. \tag{3.10}$$

Using the equality (3.10) and the estimates (3.7) and (3.11) we obtain by an easy calculation

Lemma 3.5. *Let* $0 \leqq l \leqq (N-1)/2$ *and* $0 \leqq \varepsilon < 1/2$. *Then for any* $f \in \mathring{L}_2^l(\Omega)$ *for which* $f(x)=0$ *whenever* $|x-x_0| \leqq r$ *we have*

$$(3.11)\qquad |\hat{G}_\mu f(x_0)| \leqq \text{const}\, \frac{r^{l-N/2}}{|\mu|^2} e^{-\alpha r|\mu|} (r|\mu|)^\varepsilon \|f\|_{L_2^l} \quad (\mu \in Z_0).$$

Lemma 3.6. *Let* $\sigma > N/4$. *Then for any* $f \in L_2(\Omega)$ *and on each compact set* $K \subset \Omega$ *we have the estimate*

$$(3.12)\qquad \|\hat{L}^{-\sigma} f\|_{L_\infty(K)} \leqq C_{10}(K) \|f\|_{L_2(\Omega)}.$$

Proof. First remark that the following fact is easily proved by induction on k: Let $N/4 = m+\delta$, where m is a natural number and $0 \leqq \delta < 1$ (i.e. $\delta = 0$, 1/4, 2/4, 3/4). Then for every $f \in L_2(\Omega)$ and $0 \leqq k \leqq m$ (k is a natural number) we have

$$(3.13)\qquad \|\hat{L}^{-k} f\|_{L_{p_k}(\Omega')} \leqq \text{const}\, \|f\|_{L_2(\Omega)} \quad (\Omega' \subset\subset \Omega,\ 1/p_k = 1/2 - 2k/N).$$

Thereafter, we prove (3.12) for some $\delta < \varepsilon < 1$ if $\sigma = m + \varepsilon$. Define

$$\hat{H}_\varepsilon \overset{\text{def}}{=} \frac{\sin \pi\varepsilon}{\pi} \int_0^\infty t^{-\varepsilon} \hat{H}_t \, dt;$$

then, using

$$\hat{L}^{-\sigma} = \hat{L}^{-m-\varepsilon} = \frac{\sin \pi\varepsilon}{\pi} \int_0^\infty t^{-\varepsilon} (\hat{L} + tI)^{-1} \hat{L}^{-m} \, dt$$

([14], 1.15.1 (1)) we obtain $\hat{L}^{-\sigma} = \hat{H}_\varepsilon \hat{L}^{-m} + \hat{K}_\varepsilon \hat{L}^{-m}$. Obviously,

$$\|\hat{K}_\varepsilon \hat{L}^{-m} f\|_{L_\infty(K)} \leqq \text{const}\, \|f\|_{L_2(\Omega)}$$

and

$$|H(x, y, \lambda)| \leqq \text{const}\, |x-y|^{2-N} e^{-\alpha|x-y|\,|\sqrt{\lambda}|},$$

hence $|H_\varepsilon(x, y)| \leqq \text{const}\, |x-y|^{2\varepsilon - N}$. By Hölder's inequality

$$\|\hat{H}_\varepsilon g\|_{L_\infty(K)} \leqq \text{const}\, \|g\|_{L_p(\Omega')} \quad (K \subset \Omega' \subset\subset \Omega,\ g = \hat{L}^{-m} f)$$

if $1/p < 2\varepsilon/N$, i.e., $\varepsilon > N/2p$; on the other hand, by (3.13),

$$\|\hat{L}^{-m} f\|_{L_p(\Omega')} \leqq \text{const}\, \|f\|_{L_2(\Omega)}, \quad 1/p = 1/2 - 2m/N.$$

But $\varepsilon > \delta = N/4 - m = (N/2)(1/2 - 2m/N) = (N/2)(1/p)$, i.e., $\varepsilon > N/2p$.

Corollary 1. *For every* $\sigma > N/4$ *the estimate*

$$(3.14)\qquad \sum_{n=1}^\infty |u_n(x)|^2 \lambda_n^{-2\sigma} \leqq C_{11}(K), \quad x \in K$$

holds uniformly on every compact subset $K \subset \Omega$.

Proof. By the spectral theorem, for any $f \in L_2(\Omega)$ we have

$$\hat{L}^{-\sigma} f \overset{L_2}{=} \sum_{n=1}^{\infty} (f, u_n) u_n(x) \lambda_n^{-\sigma},$$

and hence, using Lemma 3.6, the estimate

$$\left| \sum_{n=1}^{\infty} (f, u_n) u_n(x) \lambda_n^{-\sigma} \right| \leqq \text{const} \left(\sum_{n=1}^{\infty} |(f, u_n)|^2 \right)^{1/2}$$

follows, which implies the statement.

Corollary 2. *For any* $\sigma > N/4$ *and* $f \in \text{dom}(\hat{L}^\sigma)$, *the series*

$$\sum_{n=1}^{\infty} (f, u_n) u_n(x) \tag{3.15}$$

converges absolutely and uniformly on every compact set $K \subset \Omega$.

Proof. Using (3.14) the statement of Corollary 2 follows from the estimates

$$\sum_{k=n}^{n+p} |(f, u_k) u_k(x)| \leqq \left(\sum_{k=n}^{n+p} |(f, u_k)|^2 \lambda_k^{2\sigma} \right)^{1/2} \left(\sum_{k=n}^{n+p} |u_k(x)|^2 \lambda_k^{-2\sigma} \right)^{1/2} \leqq$$

$$\leqq \text{const} \left(\sum_{k=n}^{n+p} |(f, u_k)|^2 \lambda_k^{2\sigma} \right)^{1/2} \to 0 \quad (n, p \to \infty).$$

Corollary 3. *For every* $f \in C_0^\infty(\Omega)$ *the spectral expansion* $E_\lambda f(x)$ *tends to* $f(x)$ *as* $\lambda \to \infty$, *uniformly on every compact set* $K \subset \Omega$.

Proof. This follows immediately from Corollary 2, using Lemma 2.4.

4. Localization and convergence of the Riesz means

In this section we prove Theorem 1 only, because the proof of Theorem 2 goes on by the same argument.

Lemma 4.1. *Suppose* $0 \leqq l \leqq [N/2]$, $h > 0$, $t > 0$. *Then we have*

$$|\varphi[(t+h)^2] - \varphi(t^2)| \leqq C_{12} \|f\|_{L_2^l} (1 + \sqrt{h})(t+h)^{(N-1)/2-l} \tag{4.1}$$

where

$$\varphi(t) \overset{\text{def}}{=} E_t f(x_0) = \sum_{\lambda_n < t} (f, u_n) u_n(x_0), \quad f \in \mathring{L}_2^l(\Omega).$$

Proof. (4.1) follows from Lemma 1.3 immediately.

Using the method of [4] and the estimate (4.1), the following statement follows by applying Hörmander's Tauber type theorem (cf. [7]).

Lemma 4.2. *Suppose* $0 \leqq l < N/2$, $f \in \mathring{L}_2^l(\Omega)$ *and* $f(x)=0$ *if* $|x-x_0| \leqq r$. *Then we have the following estimate for every* $s \geqq 0$:

$$|E_\lambda^s f(x_0)| \leqq \text{const}\, \|f\|_{L_2^l} \lambda^{(1/2)(N/2-l)} (1+r\sqrt{\lambda})^{-1/2-s}. \tag{4.2}$$

Lemma 4.3. *Let* $s \geqq 0$, $l > 0$, $p > 1$ *and*

$$s+l = (N-1)/2, \quad 0 < l - N/p < 1. \tag{4.3}$$

Then for every $f \in \mathring{L}_p^l(\Omega)$,

$$|E_\lambda^s f(x_0)| \leqq \text{const}\, \|f\|_{L_p^l}. \tag{4.4}$$

Proof. 1°. First suppose that $f \in \mathring{L}_p^l(\Omega)$ is such that $f(x_0)=0$. Let $0 \leqq \varphi \in C_0^\infty(\Omega)$ be a function for which $\operatorname{supp} \varphi \subset (1/4, 1)$ and $\varphi(t/2)+\varphi(t)=1$ $(1/2<t<1)$. Taking into consideration that Ω is bounded, there exists a natural number k^* such that for any $f \in \mathring{L}_p^l(\Omega)$ and $x \in \Omega$ we have

$$f(x) = f(x) \sum_{k=-k^*}^{\infty} \varphi(2^k r), \quad r = |x-x_0|.$$

Denote

$$f_k(x) = f(x)\varphi(2^k r) \in \mathring{L}_p^l(\Omega) \quad (k \geqq -k^*).$$

Obviously $f_k(x)=0$ if $|x-x_0| \leqq c2^{-k}$, and by Lemma (4.2) it follows

$$|E_\lambda^s f_k(x_0)| \leqq \text{const}\, \|f_k\|_{L_p^l} \lambda^{(1/2)(N/2-l)} \cdot (1+2^{-k}\sqrt{\lambda})^{-1/2-s}.$$

Hence, using (4.3) and the estimate $\|f_k\|_{L_2^l} \leqq \text{const}\, 2^{-kN(1/2-1/p)} \|f\|_{L_p^l}$ (cf. [4], Lemma 1.1) we obtain

$$|E_\lambda^s f_k(x_0)| \leqq \text{const}\, 2^{-k(N/2-N/p-1/2-s)} (\sqrt{\lambda})^{(N-1)/2-l-s} \|f\|_{L_p^l} \leqq \text{const}\, 2^{-k(l-N/p)} \|f\|_{L_p^l}.$$

Consequently

$$|E_\lambda^s f(x_0)| \leqq \text{const}\, \|f\|_{L_p^l} \cdot \sum_{k=-k^*}^{\infty} 2^{-k(l-N/p)} \leqq \text{const}\, \|f\|_{L_p^l}.$$

2°. Now suppose $f(x_0) \neq 0$. Let $g \in C_0^\infty(\Omega)$ such that $g(x_0)=1$. Denote $f_1(x)=f(x)-f(x_0)g(x)$. According to 1° we have

$$|E_\lambda^s f_1(x_0)| \leqq \text{const}\, \|f\|_{L_p^l}. \tag{4.5}$$

By Corollary 3 of Lemma 3.6, the expansion $E_\lambda^s g(x_0)$ is bounded, and hence $|E_\lambda^s f(x_0) g(x)| \leqq \text{const}\, |f(x_0)|$. Using the imbedding theorem $L_p^l \hookrightarrow L_\infty$ if $l-N/p>0$ we get

$$\left|E_\lambda^s (f(x_0)g(x))\right| \leqq \text{const}\, \|f\|_{L_p^l}. \tag{4.6}$$

From (4.5) and (4.6) we obtain (4.4). Thus Lemma 4.3 is proved.

Proof of Theorem 1. Theorem 1 follows from Lemma 4.3, using the facts that, according to Corollary 3 of Lemma 3.6, it is true for every $f\in C_0^\infty(\Omega)$, and the set $C_0^\infty(\Omega)$ is dense in $\mathring{L}_p^l$.

Proof of Theorem 2. This theorem follows from Lemma 4.2 by the same argument as that of the proof of Theorem 1.

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