Acta Sci. Math., 45 (1983), 67—74

Quasisimilarity and properties of the commutant of C_{11} contractions

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Dedicated to Professor Béla Szőkefalvi-Nagy on his 70th birthday

An operator T acting on the complex Hilbert space $\mathfrak{H}$ is said to have property (Q) if $T|\ker X$ and $(T^*|\ker X^*)^*$ are quasisimilar for every X in the commutant $\{T\}'$ of T. This property was introduced by UCHIYAMA [11] in connection with a conjecture of SZ.-NAGY and FOIAŞ [8].

We say that T has property (P) if $\ker X^* = \{0\}$ for every operator X in $\{T\}'$ such that $\ker X = \{0\}$.

In this note we prove that a weak C_{11} contraction has property (Q) whenever it has property (P). None of the assumptions of this result can be omitted. Indeed, there are weak C_{11} contractions (even unitary operators) that do not have property (P) and we will show that there are C_{11} contractions having property (P) but not property (Q). Since (P) is a quasisimilarity invariant in C_{11} (cf. [4]) and, as we shall see, for unitary operators (P) and (Q) are equivalent, we obtain in particular that the property of being a weak contraction and property (Q) are not quasisimilarity invariants in C_{11}.

These examples show that the results of [2] concerning weak C_0 contractions and [1] concerning C_0 contractions with property (Q) cannot be extended to the class of C_{11} contractions.

It is easy to see that our Theorem 2.7 extends (via [4]) the result of WU [12] concerning completely nonunitary C_{11} contractions with finite defect indices.

We note that every C_{11} contraction with property (P) is the direct sum of a singular unitary operator and an operator on a separable space. (Cf. [4, Corollary 5].)

Received December 7, 1981.

1. The residual part of a contraction

Let T be a contraction acting on the Hilbert space $\mathfrak{H}$ and let U_+ acting on $\mathfrak{K}_+$ be the minimal isometric dilation of T, that is U_+ is an isometry, $T^*=U_+^*|\mathfrak{H}$ and $\mathfrak{K}_+=\bigvee_{n\geq 0} U_+^n\mathfrak{H}$. Let $\mathfrak{K}_+=\mathfrak{M}\oplus\mathfrak{R}$ be the Wold decomposition of $\mathfrak{K}_+$ with respect to U_+, with $\mathfrak{R}=\bigcap_{n\geq 0} U_+^n\mathfrak{K}_+$.

Definition 1.1. The unitary operator $R_T=U_+|\mathfrak{R}$ is called the *residual part* of T. (Cf. [9, ch. II. 2].)

It is obvious that $R_{V\oplus T}=V\oplus R_T$ whenever V is a unitary operator.

Sz.-Nagy and Foiaş proved the following (cf. [10, Theorem 1.3]):

Proposition 1.2. *If the contractions T and T' are similar, then R_T and $R_{T'}$ are unitary equivalent.*

Let us recall that a contraction T acting on $\mathfrak{H}$ is said to be of class C_{11} if $\lim_{n\to\infty}\|T^nh\|=0$ or $\lim_{n\to\infty}\|T^{*n}h\|=0$ implies $h=0$. The following result is proved in [9, Proposition II. 3.5].

Proposition 1.3. *Any C_{11} contraction T is quasisimilar to R_T.*

It follows by [9, Proposition II. 3.4] that in the class C_{11} R_T is a quasisimilarity invariant and even a quasiaffine invariant. Therefore R_T is the unique unitary operator (up to unitary equivalence), quasisimilar to the operator T of class C_{11}.

We do not know whether R_T is in general a quasisimilarity invariant. It is easy to see that R_T is not a quasiaffine invariant; indeed, if S denotes the unilateral shift on H^2, we have $S\prec S^*$ [7] and $R_S\not\cong R_{S^*}$.

The following result follows from [9, Chapter VII, §1].

Lemma 1.4. *If T is a completely nonunitary contraction on $\mathfrak{H}$ and $\mathfrak{H}'$ is an invariant subspace for T, then $R_T\cong R_{T'}\oplus R_{T''}$, where $T'=T|\mathfrak{H}'$ and $T''=(T^*|\mathfrak{H}\ominus\mathfrak{H}')^*$.*

The following two results will help us extend this lemma to arbitrary contractions. The first of them is proved in [5, Lemma 2], while the proof of the second one is essentially the same as that in [5, Lemma 1].

Lemma 1.5. *Any absolutely continuous unitary operator is similar to a completely nonunitary contraction.*

Lemma 1.6. *Let U be a singular unitary operator and let T be a completely nonunitary contraction. Every invariant subspace $\mathfrak{M}$ of $U\oplus T$ has the form $\mathfrak{N}\oplus\mathfrak{P}$, where $\mathfrak{N}$ is invariant for U and $\mathfrak{P}$ is invariant for T.*

Theorem 1.7. *Let T be any contraction acting on $\mathfrak{H}$, $\mathfrak{H}'$ an invariant subspace for T. Then we have $R_T \cong R_{T'} \oplus R_{T''}$, where $T' = T|\mathfrak{H}'$ and $T'' = (T^*|\mathfrak{H} \ominus \mathfrak{H}')^*$.*

Proof. Let T_1 be another contraction acting on $\mathfrak{H}_1$, and $X: \mathfrak{H} \to \mathfrak{H}_1$ an invertible operator such that $T_1 X = XT$; set $\mathfrak{H}_1' = X\mathfrak{H}'$. Then T' and T'' are similar to $T_1' = T_1|\mathfrak{H}_1'$ and $T_1'' = (T_1^*|\mathfrak{H}_1 \ominus \mathfrak{H}_1')^*$, respectively. This shows by Proposition 1.2 that in proving the theorem we may replace T by a similar operator. It follows then from Lemma 1.5 that we may assume $T = U \oplus T_1$, where U is a singular unitary operator and T_1 is completely nonunitary. (Cf. also [9, Theorem I. 3.2].) Now Lemma 1.6 shows that we can further reduce the proof to the cases where T is a singular unitary or completely nonunitary. If T is completely nonunitary the proposition follows by Lemma 1.4. In turn, if T is a singular unitary operator, then every invariant subspace of T reduces T (cf. [6, Proposition 1.11]) and so the statement becomes obvious. The proof is complete.

2. C_{11} contractions with property (P)

The following result was proved in [4].

Proposition 2.1. *A contraction T of class C_{11} has property (P) if and only if R_T has property (P).*

Now, unitary operators having property (P) are easily characterized in terms of properties of their commutant.

Lemma 2.2. *A unitary operator T has property (P) if and only if the commutant $\{T\}'$ is a finite von Neumann algebra.*

Proof. Assume first that $\{T\}'$ is not finite. Then there exists a nonunitary isometry U in $\{T\}'$; in particular U is one-to-one but $\ker U^* \neq \{0\}$ so that T does not have property (P).

Conversely, if T does not have property (P), there exists X in $\{T\}'$ such that $\ker X = \{0\}$ and $\ker X^* \neq \{0\}$. If $X = UP$ is the polar decomposition of X, we have $U \in \{T\}'$ (cf. the proof of [9, Proposition II. 3.4]), $\ker U = \ker X = \{0\}$ and $\ker U^* = \ker X^* \neq \{0\}$ so that $\{T\}'$ is not finite. The lemma is proved.

It follows from the results of [3] that unitary operators having property (P) also have the following "cancellation" property: if $T \oplus U$ is unitarily equivalent to $T \oplus V$ for some unitary operators T, U and V, and $T \oplus U$ has property (P), then U and V are unitarily equivalent.

Proposition 2.3. *Let T be a C_{11} contraction having property (P). For every X in $\{T\}'$ the operators $R_{T|\ker X}$ and $R_{(T^*|\ker X^*)^*}$ are unitarily equivalent.*

Proof. By Theorem 1.7 we have $R_T \cong R_{T|\ker X} \oplus R_{(T^*|(\ker X)^\perp)^*} \cong R_{T|(\operatorname{ran} X)^-} \oplus \oplus R_{(T^*|\ker X^*)^*}$. The operators $(T^*|(\ker X)^\perp)^*$ and $T|(\operatorname{ran} X)^-$ are of class C_{11} (cf. [5, Lemma 5]) and they are quasisimilar (cf., e.g., [12, Corollary 3.4]), so that $R_{(T^*|(\ker X)^\perp)^*}$ and $R_{T|(\operatorname{ran} X)^-}$ are unitarily equivalent. The proposition now follows from the cancellation property described above.

An obvious consequence of Proposition 2.3 is the following.

Corollary 2.4. *Let the C_{11} contraction T be such that $T|\ker X$ and $T^*|\ker X^*$ are of class C_{11} for every X in $\{T\}'$. Then T has property (Q) if and only if it has property (P).*

The hypothesis of the preceding Corollary can be weakened; to do this we need some definitions from [5]. For a C_{11} contraction $\operatorname{lat}_1 T$ denotes the set of those invariant subspaces $\mathfrak{M}$ for T such that $T|\mathfrak{M}$ is of class C_{11}. For every invariant subspace $\mathfrak{M}$ for T there exists a largest subspace in $\operatorname{lat}_1 T$ contained in $\mathfrak{M}$, this subspace (the C_{11}-part of $\mathfrak{M}$) is denoted by $\mathfrak{M}^{(1)}$. For a subspace $\mathfrak{M}$ in $\operatorname{lat} T^*$ we set $\mathfrak{M}^{\perp_1} = (\mathfrak{M}^\perp)^{(1)}$.

Let us say that the C_{11} contraction T has property (R) if $\ker X \in \operatorname{lat}_1 T$ for every X in $\{T\}'$.

Proposition 2.5. *Let T be a C_{11} contraction having property (P). Then T has property (R) if and only if T^* has property (R).*

Proof. By [5, Lemma 5] a subspace $\mathfrak{M}$ is in $\operatorname{lat}_1 T^*$ if and only if it has the form $(\ker X)^\perp$ for some X in $\{T\}'$. It follows that T has property (R) if and only if $\mathfrak{M}^\perp \in \operatorname{lat}_1 T$ for every $\mathfrak{M}$ in $\operatorname{lat}_1 T^*$.

Let us assume that T has property (R) and $\mathfrak{M} \in \operatorname{lat}_1 T$; it follows from [5, Proposition 2] that $(\mathfrak{M}^{\perp_1})^{\perp_1} = \mathfrak{M}$. Now, $\mathfrak{M}^{\perp_1} \in \operatorname{lat}_1 T^*$ and T has property (R) so that $(\mathfrak{M}^{\perp_1})^\perp \in \operatorname{lat}_1 T$. Consequently $(\mathfrak{M}^{\perp_1})^\perp = (\mathfrak{M}^{\perp_1})^{\perp_1} = \mathfrak{M}$ and therefore $\mathfrak{M}^{\perp_1} = \mathfrak{M}^\perp$; that is $\mathfrak{M}^\perp \in \operatorname{lat}_1 T^*$. We proved that T^* has property (R).

By [4, Corollary 4] T has property (P) if and only if T^* has property (P). Thus the proof is completed by the same argument applied to T^* instead of T.

Now we can reformulate Corollary 2.4 as follows.

Theorem 2.6. *Let T be a C_{11} contraction having property (P). Then T has property (Q) if and only if $T|\ker X$ is of class C_{11} for every X in $\{T\}'$.*

Proof. The sufficiency obviously follows from Corollary 2.4 and Proposition 2.5. Conversely, if T has property (Q) and $X \in \{T\}'$, then $T|\ker X$ is of class $C_{1\cdot}$ and $(T^*|\ker X^*)^*$ is of class $C_{\cdot 1}$; it follows that both operators are of class C_{11} since they are quasisimilar. The theorem is proved.

Let us recall that a contraction T is said to be weak if $I-T^*T$ is a trace class operator and $\lambda I-T$ is invertible for some λ with $|\lambda|<1$.

Theorem 2.7. *A weak C_{11} contraction has property (P) if and only if it has property (Q).*

Proof. It is enough to prove that a weak C_{11} contraction T having property (P) also has property (Q). By virtue of Theorem 2.6 it suffices to show that, if T is a weak C_{11} contraction then $T|\ker X$ is of class C_{11} for every X in $\{T\}'$.

It is clear that $I-(T|\ker X)^*(T|\ker X)=P_{\ker X}(I-T^*T)|\ker X$ is a trace class operator. By [9, Theorem VIII. 2.1] T is invertible. Since X commutes with T^{-1}, we have that $T^{-1}(\ker X)\subset\ker X$, and so $T|\ker X$ is also invertible. Therefore $T|\ker X$ is a weak contraction of class $C_{1\cdot}$, and so by [9, Theorem VIII. 2.1] it is of class C_{11}. The theorem follows.

Corollary 2.8. *A unitary operator has property (P) if and only if it has property (Q).*

3. Examples

It is known [9, Ch. VI. 4.2] that there exist C_{11} contractions whose spectrum coincides with the closed unit disk. The following result shows that there are C_{11} contractions having property (P) whose spectrum covers the unit disk.

Proposition 3.1. *Let U be an absolutely continuous unitary operator. There exists a C_{11} contraction T such that $\sigma(T)=\{\lambda\colon |\lambda|\leqq 1\}$ and R_T is unitarily equivalent to U.*

Proof. It suffices to prove the proposition in the case U is the operator of multiplication by e^{it} on $L^2(\sigma)$, where $\sigma\subset[0, 2\pi]$ has positive Lebesgue measure. Choose pairwise disjoint subsets σ_n of σ of positive measure such that $\bigcup\limits_{n\geqq 0}\sigma_n=\sigma$ and choose a sequence $\{\varepsilon_n\}_{n\geqq 0}$ of positive numbers less than 1. For each n there exists an outer function ϑ_n (uniquely determined up to a constant factor of modulus one) such that $|\vartheta_n(e^{it})|=1$ if $t\notin\sigma_n$ and $|\vartheta_n(e^{it})|=\varepsilon_n$ if $t\in\sigma_n$. It is clear by [4, Corollary 1] that the functional model T corresponding with the characteristic function $\theta(\lambda)=\operatorname{diag}(\vartheta_0(\lambda), \vartheta_1(\lambda), \dots)$ satisfies the condition $R_T\cong U$.

If the numbers ε_n satisfy the relation $\lim\limits_{n\to\infty}|\sigma_n|\log\varepsilon_n=-\infty$ (where $|\sigma_n|$ denotes the Lebesgue measure of σ_n) we have $\lim\limits_{n\to\infty}\vartheta_n(\lambda)=0$ for every λ, $|\lambda|<1$, and by [9, Theorem VI. 4.1] this implies that $\sigma(T)\supset\{\lambda\colon |\lambda|<1\}$. The proposition follows.

It is obvious that the operator T constructed in the preceding proof is not a weak contraction; in particular, a C_{11} contraction with a cyclic vector is not necessarily a weak contraction. Let us also note that if T is of class C_{11} then T and T^* cannot have eigenvalues of absolute value less than 1. Thus, if T is a C_{11} contraction and $\lambda\in\sigma(T)$, $|\lambda|<1$, then $\lambda I-T$ is one-to-one and has non-closed, dense range.

In the sequel we will identify a vector f of the Hilbert space $\mathfrak{H}$ with the operator $\mathbf{C}\to\mathfrak{H}$ defined by $\mathbf{C}\ni\lambda\mapsto\lambda f\in\mathfrak{H}$; the adjoint f^* is then defined by $f^*(g)=(g,f)$ for $g\in\mathfrak{H}$.

Lemma 3.2. *Let S be an injective contraction acting on $\mathfrak{H}$ such that $S\mathfrak{H}\neq\mathfrak{H}$. There exists a vector $f\in\mathfrak{H}$ such that the operator $(S,f)\colon\mathfrak{H}\oplus\mathbf{C}\to\mathfrak{H}$ defined by $(S,f)(h\oplus\lambda)=Sh+\lambda f$ is an injective contraction.*

Proof. It is clear that (S,f) is injective if and only if $f\notin S\mathfrak{H}$. Let us set $f=u-SS^*u$, where $u\notin S\mathfrak{H}$ and $\|u\|^2\leqq 1/2$. Then clearly $f\notin S\mathfrak{H}$ and

$$\|u\|^2+\|f\|^2\leqq\|u\|^2+\|u\|^2\leqq 1. \tag{3.1}$$

We only have to prove that (S,f) is a contraction. Indeed, let $h\oplus\lambda\in\mathfrak{H}\oplus\mathbf{C}$; we have (using the notation $D=(I-S^*S)^{1/2}$)

$$\|Sh+\lambda f\|^2\leqq\|Sh\|^2+2|\lambda|\,|(Sh,f)|+|\lambda|^2\|f\|^2=$$

$$=\|Sh\|^2+2|\lambda|\,\big|((I-SS^*)Sh,u)\big|+|\lambda|^2\|f\|^2=\|Sh\|^2+2|\lambda|\,|(SDDh,u)|+|\lambda|^2\|f\|^2\leqq$$

$$\leqq\|Sh\|^2+2|\lambda|\,\|u\|\,\|Dh\|+|\lambda|^2\|f\|^2.$$

Using the inequality $2ab\leqq a^2+b^2$ in the middle term we get

$$\|Sh+\lambda f\|^2\leqq\|Sh\|^2+\|Dh\|^2+|\lambda|^2\|u\|^2+|\lambda|^2\|f\|^2=$$

$$=\|h\|^2+|\lambda|^2(\|u\|^2+\|f\|^2)\leqq\|h\|^2+|\lambda|^2$$

by (3.1). The lemma follows.

Theorem 3.3. *There exist C_{11} contractions having property (P) but not property (Q).*

Proof. Let T' and T'' be two noninvertible C_{11} contractions acting on $\mathfrak{H}'$ and $\mathfrak{H}''$, respectively. By Lemma 3.2 we can choose vectors $f\in\mathfrak{H}'$ and $g\in\mathfrak{H}''$ such that (T',f) and (T''^*,g) are injective contractions. It is then easy to see that the operator T defined on $\mathfrak{H}'\oplus\mathbf{C}\oplus\mathfrak{H}''$ by the matrix

$$\begin{bmatrix} T' & f & 0\\ 0 & 0 & g^*\\ 0 & 0 & T''\end{bmatrix}$$

is a C_{11} contraction. Let us note that the invariant subspace $\mathfrak{H}'\oplus\mathbf{C}$ for T is not in $\mathrm{lat}_1 T$, while its orthocomplement $\mathfrak{H}''$ obviously belongs to $\mathrm{lat}_1 T^*$; by the proof of Proposition 2.5 and by Theorem 2.6 we infer that T does not have property (Q).

By Theorem 1.7 we have $R_T\cong R_{T'}\oplus R_0\oplus R_{T''}\cong R_{T'}\oplus R_{T''}$, so that T has property (P) whenever T' and T'' have property (P) (cf. Proposition 2.1 and [4, Lemma 5]). The theorem follows by Proposition 3.1.

Remark 3.4. Proposition 3.1 shows in fact that the operator T in the preceding proof can be chosen so that R_T is unitarily equivalent to a given absolutely continuous unitary operator with property (P). In particular R_T could be chosen so that all its invariant subspaces are reducing (a reductive operator). This shows that the property "$\mathrm{lat}_1 T=\mathrm{lat}\, T$", generalizing reductivity, is not a quasisimilarity invariant in the class of C_{11} contractions or even in the class of C_{11} contractions having property (P).

Remark 3.5. Let us choose $T'=T''$ in the proof of Theorem 3.3; in this case we can produce an operator X in $\{T\}'$ for which $T|\ker X$ and $(T^*|\ker X^*)^*$ are not quasisimilar. Such an operator is defined by the matrix

$$\begin{bmatrix} 0 & 0 & I \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where I denotes the identity operator on $\mathfrak{H}'=\mathfrak{H}''$.

Remark 3.6. Finally we note that we have got by Theorems 3.3, 2.6 and by the proof of Proposition 2.5 that the C_{11}-orthogonal complement $\mathfrak{L}^{\perp_1}$ of a subspace $\mathfrak{L}\in\mathrm{lat}_1 T$, where T is a C_{11} contraction with property (P), does not generally coincide with the orthogonal complement $\mathfrak{L}^{\perp}$ of $\mathfrak{L}$.

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