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A proof of the spectral theorem for J-positive operators

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Dedicated to Béla Sz.-Nagy on the occasion of his 70th birthday

A Krein space is a Hilbert space with the usual (positive definite) inner product (f, g) and a non-degenerate (in general, indefinite) J-inner product $[f, g]=(Jf, g)$, where J is a symmetry: $J^*=J$ and $J^2=I$. A Krein space with one (or both) of the eigenspaces of J having finite dimension is called a Pontrjagin space.

Let A be a bounded or unbounded linear operator in the Krein space $\mathfrak{H}$. If JA is selfadjoint (in the Hilbert space sense), then A is said to be J-selfadjoint. If JA is positive, that is, $[Af, f]=(JAf, f)\geqq 0$ for every f in the domain of A, then we say A is J-positive. Further, if there is a non-zero polynomial p such that $p(A)$ is J-positive we say A is J-positizable.

In 1963, M. G. KREĬN and H. LANGER [1] proved a spectral theorem for J-selfadjoint operators with real spectrum in a Pontrjagin space. The proof made use, among other things, of the J-positizability of these operators. LANGER [2] generalized the theorem to J-positizable J-selfadjoint operators with real spectrum in a Krein space (see also [3]—[5] for statement of the result). Proofs for the bounded J-positive case have also been given by M. G. KREĬN and JU. L. ŠMUL'JAN [6], T. ANDO [7], and for further generalizations by B. N. HARVEY [8] and P. JONAS [9]—[10].

In our opinion, the spectral theory based on these results has not gained the popularity it deserves. The situation can perhaps be improved by reducing the machinery required in the proofs. ANDO [7] has already made the decisive step in this direction.

Our proof below was inspired by a paper of C. S. WONG [11] and is hoped to be a further step in eliminating unnecessary tools. Restricted to the bounded J-positive case, it uses only the basic facts of Hilbert space spectral theory as treated by B. SZ.-NAGY [12] and the elements of Krein space theory [13]. In particular, neither an auxiliary space nor complex variables are needed.

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Theorem (Kreĭn, Langer). *Let A be a bounded J-positive operator on the Krein space $\mathfrak{H}$. Then to every real number $\lambda \neq 0$ there is one and only one J-selfadjoint projection E_λ on $\mathfrak{H}$ such that the function $\lambda \mapsto E_\lambda$ has the following properties:*

1. *If $\lambda \leqq \mu$, then $E_\lambda E_\mu = E_\mu E_\lambda = E_\lambda$.*
2. *If $\lambda < \mu < 0$, then $[E_\lambda f, f] \geqq [E_\mu f, f]$; if $0 < \lambda < \mu$, then $[E_\lambda f, f] \leqq [E_\mu f, f]$ for every $f \in \mathfrak{H}$.*
3. *If $\lambda < -\|A\|$, then $E_\lambda = O$; if $\lambda > \|A\|$, then $E_\lambda = I$.*
4. *If $\lambda \neq 0$, then the strong limit $E_{\lambda+0}$ exists and $E_{\lambda+0} = E_\lambda$.*
5. *If T is a bounded linear operator on $\mathfrak{H}$ such that $TA = AT$, then $TE_\lambda = E_\lambda T$ for every λ.*
6. *The spectrum $\sigma(A|E_\lambda \mathfrak{H})$ is contained in the interval $(-\infty, \lambda]$, while $\sigma(A|(I-E_\lambda)\mathfrak{H})$ is contained in $[\lambda, \infty)$.*

Moreover,

$$\int_{-\|A\|-0}^{\|A\|} \nu \, dE_\nu$$

is a strongly convergent improper integral with singular point 0, *and*

$$S := A - \int_{-\|A\|-0}^{\|A\|} \nu \, dE_\nu$$

is a bounded J-positive operator such that $S^2 = O$, $SE_\lambda = E_\lambda S = O$ if $\lambda < 0$, whereas $S(I-E_\lambda) = (I-E_\lambda)S = O$ if $\lambda > 0$.

Proof. The positive operator $B := JA$ satisfies

$$A = JB. \tag{1}$$

The operator

$$C := B^{1/2} J B^{1/2} \tag{2}$$

is selfadjoint and

$$CB^{1/2} = B^{1/2} A. \tag{3}$$

Since $\|C\| \leqq \|A\|$, the spectral decomposition of C can be written in the form

$$C = \int_{-\|A\|-0}^{\|A\|} \nu \, dF_\nu, \tag{4}$$

where $\{F_\lambda\}_{\lambda=-\infty}^{\infty}$ is the right-continuous spectral family of C. We set

$$C_\lambda := C|F_\lambda \mathfrak{H} \quad \text{for} \quad \lambda < 0, \tag{5}$$

$$C_\lambda := C|(I-F_\lambda)\mathfrak{H} \quad \text{for} \quad \lambda > 0 \tag{6}$$

and

$$E_\lambda := JB^{1/2} C_\lambda^{-1} F_\lambda B^{1/2} \quad \text{for} \quad \lambda < 0, \tag{7}$$

$$I - E_\lambda := JB^{1/2} C_\lambda^{-1} (I - F_\lambda) B^{1/2} \quad \text{for} \quad \lambda > 0. \tag{8}$$

Clearly, E_λ is a bounded operator on $\mathfrak{H}$ for every real number $\lambda \neq 0$. Further, if $\lambda < 0$ then $C_\lambda^{-1} F_\lambda$ is selfadjoint and therefore $E_\lambda^* = J E_\lambda J$, which is equivalent to E_λ being J-selfadjoint. If $\lambda > 0$, the J-selfadjointness of $I - E_\lambda$ and hence the same property of E_λ follow similarly. Thus

$$(9) \qquad E_\lambda^* = J E_\lambda J \quad \text{for every} \quad \lambda \neq 0,$$

a relation needed later on.

Let $\lambda \leqq \mu < 0$. Then from (7), (2), and the relation

$$(10) \qquad C C_\lambda^{-1} F_\lambda = F_\lambda \quad (\lambda < 0)$$

(see (5)) we obtain $E_\lambda E_\mu = E_\lambda$. Similarly, if $0 < \lambda \leqq \mu$ then (8), (2), and the relation

$$(11) \qquad C C_\lambda^{-1}(I - F_\lambda) = I - F_\lambda \quad (\lambda > 0)$$

(see (6)) yield $(I - E_\lambda)(I - E_\mu) = I - E_\mu$, that is, $E_\lambda E_\mu = E_\lambda$. Finally, in the case $\lambda < 0 < \mu$ from (7), (8), (2) and (11) we get $E_\lambda(I - E_\mu) = O$ and therefore $E_\lambda E_\mu = E_\lambda$ again. The relation $E_\mu E_\lambda = E_\lambda$ follows by taking adjoints and applying (9). Thus Property 1 is valid. Choosing $\lambda = \mu$ we see that E_λ is a projection.

Let us prove Property 2. If $\lambda < \mu < 0$, then by (7), (5), and (4)

$$[E_\mu f, f] - [E_\lambda f, f] = (C_\mu^{-1} F_\mu B^{1/2} f, B^{1/2} f) - (C_\lambda^{-1} F_\lambda B^{1/2} f, B^{1/2} f) =$$
$$= \int_\lambda^\mu \frac{1}{\nu} d(F_\nu B^{1/2} f, B^{1/2} f) \leqq 0$$

for every $f \in \mathfrak{H}$. On the other hand, if $0 < \lambda < \mu$ then by (8), (6), and (4)

$$[E_\mu f, f] - [E_\lambda f, f] = [(I - E_\lambda) f, f] - [(I - E_\mu) f, f] =$$
$$= \big(C_\lambda^{-1}(I - F_\lambda) B^{1/2} f, B^{1/2} f\big) - \big(C_\mu^{-1}(I - F_\mu) B^{1/2} f, B^{1/2} f\big) =$$
$$= \int_\lambda^\mu \frac{1}{\nu} d(F_\nu B^{1/2} f, B^{1/2} f) \geqq 0,$$

as required.

Property 3 is a simple consequence of (7), (8) and (4).

To prove Property 4, first let $\lambda < \mu < 0$. Then

$$\|E_\mu f - E_\lambda f\|^2 = \|J B^{1/2} (C_\mu^{-1} F_\mu - C_\lambda^{-1} F_\lambda) B^{1/2} f\|^2 \leqq$$
$$\leqq \|B\| \int_\lambda^\mu \frac{1}{\nu^2} d(F_\nu B^{1/2} f, B^{1/2} f) \leqq \|B\| \frac{1}{\mu^2} \|(F_\mu - F_\lambda) B^{1/2} f\|^2,$$

and the last member tends to 0 as $\mu \to \lambda + 0$. Therefore $E_{\lambda+0} = E_\lambda$ if $\lambda < 0$. A similar reasoning applies in the case $\lambda > 0$.

Next assume that T is a bounded linear operator which commutes with A, i.e., $TA = AT$. To prove $TE_\lambda = E_\lambda T$ consider the case $\lambda < 0$ first.

By (7), (5), and (4)

$$E_\lambda = JB^{1/2} \int_{-\|A\|-0}^{\lambda} \frac{1}{\nu} dF_\nu \cdot B^{1/2} \quad (\lambda < 0).$$

Choose a sequence of polynomials $\{p_n\}_1^\infty$ which is bounded on $[-\|A\|, \|A\|]$ and satisfies the relation

$$\lim_{n\to\infty} p_n(\nu) = \begin{cases} 1/\nu & \text{if } -\|A\| \leqq \nu \leqq \lambda, \\ 0 & \text{if } \lambda < \nu \leqq \|A\|. \end{cases}$$

Then $JB^{1/2}p_n(C)B^{1/2} \to E_\lambda$ strongly. Hence it is sufficient to prove that T commutes with $JB^{1/2}C^m B^{1/2}$ for $m=0, 1, 2, \ldots$. But

$$JB^{1/2}C^m B^{1/2} = A^{m+1} \quad (m = 0, 1, 2, \ldots),$$

as one can verify by induction with the help of (1) and (3). This completes the proof of Property 5 for $\lambda<0$.

If $\lambda>0$, we start from the relation

$$I - E_\lambda = JB^{1/2} \int_{\lambda}^{\|A\|} \frac{1}{\nu} dF_\nu \cdot B^{1/2} \quad (\lambda > 0)$$

obtainable from (8), (6), and (4), and conclude as above that T commutes with $I-E_\lambda$.

Just as in Hilbert space, from the consequence $AE_\lambda=E_\lambda A$ of Property 5 it follows that the subspaces $E_\lambda\mathfrak{H}$ and $(I-E_\lambda)\mathfrak{H}$ are invariant under A.

As to Property 6, we first note that the relations (1), (7)—(8), (2) and (10)—(11) imply

(12) $$AE_\lambda = JB^{1/2}F_\lambda B^{1/2} \quad (\lambda \neq 0),$$

(13) $$A(I-E_\lambda) = JB^{1/2}(I-F_\lambda)B^{1/2} \quad (\lambda \neq 0).$$

Since $\sigma(T_1T_2)\subset\{0\}\cup\sigma(T_2T_1)$ for any pair T_1, T_2 of bounded linear operators (see [14], Problem 61), from (12) and (2) we obtain $\sigma(AE_\lambda)\subset\{0\}\cup\sigma(CF_\lambda)$. Therefore $\sigma(A|E_\lambda\mathfrak{H})\subset\{0\}\cup\sigma(C|F_\lambda\mathfrak{H})$ and, in view of (4),

$$\sigma(A|E_\lambda\mathfrak{H}) \subset \{0\}\cup(-\infty, \lambda].$$

We have to prove that if $\lambda<0$ then 0 does not belong to $\sigma(A|E_\lambda\mathfrak{H})$.

Let $\lambda<0$ and assume that $AE_\lambda f_n\to 0$ $(n\to\infty)$ for some sequence $\{f_n\}_1^\infty\subset\mathfrak{H}$. Then also $B^{1/2}AE_\lambda f_n\to 0$ or, by (12) and (2), $CF_\lambda B^{1/2}f_n\to 0$. Since, according to (4) and the assumption $\lambda<0$, the value 0 is regular for $C|F_\lambda\mathfrak{H}$, it follows that $F_\lambda B^{1/2}f_n\to 0$. Applying the operator $JB^{1/2}C_\lambda^{-1}$ and using (7) we obtain $E_\lambda f_n\to 0$ $(n\to\infty)$.

Thus 0 belongs to neither the continuous nor the point spectrum of $A|E_\lambda\mathfrak{H}$. But $A|E_\lambda\mathfrak{H}$, being a selfadjoint operator on the "negative Hilbert space" $E_\lambda\mathfrak{H}$

(cf. Properties 2—3 as well as [13], Theorems II. 3.10 and V. 3.5), has no residual spectrum. This proves one half of Property 6. The proof of the other half is similar.

Assume that $\lambda\mapsto E'_\lambda$ (λ real, $\lambda\neq 0$, E'_λ a J-selfadjoint projection on $\mathfrak{H}$) is also a function with the properties 1—6. Let $\lambda<\mu<0$. Obviously

$$E'_\lambda = E'_\lambda E_\mu + E'_\lambda(I-E_\mu).$$

By Property 5, E'_λ and E_μ commute with A and with each other. In particular, $E'_\lambda(I-E_\mu)$ is a J-selfadjoint projection which commutes with A.

Similarly to the case of a selfadjoint projection, if E is a J-selfadjoint projection and $AE=EA$ then

$$\sigma(A)=\sigma(A|E\mathfrak{H})\cup\sigma\big(A|(I-E)\mathfrak{H}\big)\supset\sigma(A|E\mathfrak{H}).$$

Indeed, the A-invariant subspaces $E\mathfrak{H}$ and $(I-E)\mathfrak{H}$ are orthogonal with respect to the J-inner product; therefore [13], Theorem V. 3.5, implies that they are orthogonal also in a Hilbert space with norm equivalent to the original one.

Applying this fact to the Krein spaces (cf. [13], Theorem V. 3.4) $E'_\lambda\mathfrak{H}$, $(I-E_\mu)\mathfrak{H}$, and using Property 6 we obtain

$$\sigma\big(A|E'_\lambda(I-E_\mu)\mathfrak{H}\big)\subset\sigma(A|E'_\lambda\mathfrak{H})\cap\sigma\big(A|(I-E_\mu)\mathfrak{H}\big)\subset(-\infty,\lambda]\cap[\mu,\infty)=\emptyset.$$

But the spectrum of a selfadjoint operator on the "negative Hilbert space" $E'_\lambda(I-E_\mu)\mathfrak{H}\subset E'_\lambda\mathfrak{H}$ can be empty only if the space is zero. Thus $E'_\lambda(I-E_\mu)=O$,

$$E'_\lambda = E'_\lambda E_\mu. \tag{14}$$

(14) remains valid if $0<\lambda<\mu$, the only difference in the proof being that $E'_\lambda(I-E_\mu)\mathfrak{H}\subset(I-E_\mu)\mathfrak{H}$ now are ordinary Hilbert spaces. Letting $\mu\to\lambda+0$, from (14) and Property 4 we conclude that $E'_\lambda=E'_\lambda E_\lambda$. Similarly, $E_\lambda=E_\lambda E'_\lambda$. Therefore, in view of Property 5, $E'_\lambda=E_\lambda$.

The existence of the strong integral

$$\int_{-\|A\|-0}^{-\varepsilon} \nu\, dE_\nu,$$

where $\varepsilon>0$, follows by reading the next relations from the right to the left (see (7), (5), and (4)):

$$\int_{-\|A\|-0}^{-\varepsilon} \nu\, dE_\nu = JB^{1/2}C_{-\varepsilon}^{-1}\int_{-\|A\|-0}^{-\varepsilon} \nu\, dF_\nu\cdot B^{1/2} = JB^{1/2}C_{-\varepsilon}^{-1}C_{-\varepsilon}F_{-\varepsilon}B^{1/2} = JB^{1/2}F_{-\varepsilon}B^{1/2}.$$

Similarly, from (8), (6), (4), and (1)

$$\int_{\varepsilon}^{\|A\|} \nu\, dE_\nu = -\int_{\varepsilon}^{\|A\|} \nu\, d(I-E_\nu) = -JB^{1/2}C_\varepsilon^{-1}\int_{\varepsilon}^{\|A\|} \nu\, d(I-F_\nu)\cdot B^{1/2} =$$

$$= JB^{1/2}C_\varepsilon^{-1}\int_{\varepsilon}^{\|A\|} \nu\, dF_\nu\cdot B^{1/2} = JB^{1/2}C_\varepsilon^{-1}C_\varepsilon(I-F_\varepsilon)B^{1/2} =$$

$$= JB^{1/2}(I-F_\varepsilon)B^{1/2} = A-JB^{1/2}F_\varepsilon B^{1/2}.$$

Thus

(15) $$\int_{-\|A\|-0}^{\|A\|} \nu \, dE_\nu = A - JB^{1/2}(F_0 - F_{-0})B^{1/2}$$

as a strong improper integral.

The operator $S := JB^{1/2}(F_0 - F_{-0})B^{1/2}$ appearing in (15) is obviously bounded and J-positive. Further $S^2 = O$, since according to (2) and (4)

$$B^{1/2}JB^{1/2}(F_0 - F_{-0}) = C(F_0 - F_{-0}) = O.$$

By the same reason, $SE_\lambda = E_\lambda S = O$ if $\lambda < 0$, and $S(I - E_\lambda) = (I - E_\lambda)S = O$ if $\lambda > 0$.

The proof is complete.

References

[1] M. G. Kreĭn and H. Langer, On the spectral function of a self-adjoint operator in a space with indefinite metric, *Dokl. Akad. Nauk SSSR,* **152** (1963), 39—42 (Russian).

[2] H. Langer, *Spektraltheorie linearer Operatoren in J-Räumen und einige Anwendungen auf die Schar* $L(\lambda) = \lambda^2 I + \lambda B + C$, Habilitationsschrift, Technische Universität Dresden, 1965.

[3] H. Langer, Über stark gedämpfte Scharen im Hilbertraum, *J. Math. Mech.,* **17** (1968), 685—706.

[4] H. Langer, Invariante Teilräume definisierbarer J-selbstadjungierter Operatoren, *Ann. Acad. Sci. Fenn. Ser. AI,* **475** (1971).

[5] P. Jonas and H. Langer, Compact perturbations of definitizable operators, *J. Operator Theory,* **2** (1979), 63—77.

[6] M. G. Kreĭn and Ju. L. Šmul'jan, J-polar representations of plus-operators, *Mat. Issled.,* **1** (1966), 172—210 (Russian).

[7] T. Ando, *Linear operators on Krein spaces,* Lecture notes, Hokkaido University (Sapporo, 1979).

[8] B. N. Harvey, Spectral operators with critical points, *Amer. J. Math.,* **96** (1974), 41—61.

[9] P. Jonas, Zur Existenz von Eigenspektralfunktionen mit Singularitäten, *Math. Nachr.,* **88** (1979), 345—361.

[10] P. Jonas, On the functional calculus and the spectral function for definitizable operators in Krein space, *Beiträge Anal.,* **16** (1981), 121—135.

[11] C. S. Wong, Characterizations of products of symmetric matrices, *Lin. Alg. Appl.,* **42** (1982), 243—251.

[12] B. Sz.-Nagy, *Spektraldarstellung linearer Transformationen des Hilbertschen Raumes,* Springer-Verlag (Berlin, 1942).

[13] J. Bognár, *Indefinite inner product spaces,* Springer-Verlag (Berlin—Heidelberg—New York, 1974).

[14] P. R. Halmos, *A Hilbert space problem book,* van Nostrand (Princeton, New Jersey, 1967).

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