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Analytic generators for one-parameter cosine families

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Dedicated to B. Szőkefalvi-Nagy on the occasion of his seventieth birthday

One parameter cosine families of linear operators have been recently used in several papers on operator algebras ([6], [7], [13]). Some technical results of these papers suggested us to develop here a general theory of the analitic generator of one-parameter cosine families similarly to that presented in [3] for one-parameter groups. It is proved, that a one-parameter cosine family of 0 exponential type is uniquely determined by its analytic generator and explicit formulas are given.

We remark that the theory developed here can be used to give intrinsic characterizations for the analytic generators of one-parameter groups of automorphisms of operator algebras; this is due to the fact that while the analytic generator of such a goup frequently has "bad" spectral properties [4], the analytic generator of its "cosine part" has always a "thin" spectrum.

1. Analytic extensions of cosine families

Let us first specify the frame in which cosine families are to be considered.

We call a *dual pair of Banach spaces* any pair $(\mathbf{X}, \mathscr{F})$ of complex Banach spaces, together with a bilinear functional

$$\mathbf{X}\times\mathscr{F}\ni(x, \varphi)\to\langle x, \varphi\rangle\in\mathbf{C},$$

such that

(i) $\|x\| = \sup_{\|\varphi\|\leq 1} |\langle x, \varphi\rangle|$ for any $x\in\mathbf{X}$;

(ii) $\|\varphi\| = \sup_{\|x\|\leq 1} |\langle x, \varphi\rangle|$ for any $\varphi\in\mathscr{F}$;

(iii) the convex hull of any relatively $\sigma(\mathbf{X}, \mathscr{F})$-compact subset of $\mathbf{X}$ is relatively $\sigma(\mathbf{X}, \mathscr{F})$-compact;

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(iv) the convex hull of any relatively $\sigma(\mathscr{F}, \mathbf{X})$-compact subset of $\mathscr{F}$ is relatively $\sigma(\mathscr{F}, \mathbf{X})$-compact.

If $(\mathbf{X}, \mathscr{F})$ is a dual pair of Banach spaces, then $(\mathscr{F}, \mathbf{X})$ endowed with the same bilinear pairing, is also a dual pair of Banach spaces. We note that if $\mathbf{X}$ is a complex Banach space and $\mathbf{X}^*$ its dual, then the pairs $(\mathbf{X}, \mathbf{X}^*)$ and $(\mathbf{X}^*, \mathbf{X})$, endowed with the natural pairing between $\mathbf{X}$ and $\mathbf{X}^*$, are dual pairs of Banach spaces. We recall that if $(\mathbf{X}, \mathscr{F})$ is a dual pair of Banach spaces, then the uniform boundedness principle holds in $\mathbf{X}$ with respect to $\sigma(\mathbf{X}, \mathscr{F})$ ([8], Th. 2.8.6); in particular, the analyticity of $\mathbf{X}$-valued mappings of complex variable does not depend on the topology considered on $\mathbf{X}$ ([8], Th. 3.10.1). On the other hand quite general $\mathbf{X}$-valued mappings, defined on a locally compact space endowed with a Radon measure, are $\sigma(\mathbf{X}, \mathscr{F})$-integrable ([2], Prop. 1.2; [3], Prop. 1.4.).

If $(\mathbf{X}, \mathscr{F})$ is a dual pair of Banach spaces and T is a $\sigma(\mathbf{X}, \mathscr{F})$-densely defined linear operator in $\mathbf{X}$, then one can define the adjoint $T^{\mathscr{F}}$ of T in $\mathscr{F}$ by

$$(\varphi, \psi) \in \operatorname{graph}(T^{\mathscr{F}}) \Leftrightarrow \langle x, \psi \rangle = \langle T(x), \varphi \rangle \quad \text{for all} \quad x \in \mathscr{D}_T.$$

$T^{\mathscr{F}}$ is always $\sigma(\mathscr{F}, \mathbf{X})$-closed. If moreover T is $\sigma(\mathbf{X}, \mathscr{F})$-closed, then $T^{\mathscr{F}}$ will be $\sigma(\mathscr{F}, \mathbf{X})$-densely defined and $(T^{\mathscr{F}})^{\mathbf{X}} = T$ holds ([11], IV. 7.1). Denote by $\mathscr{B}_{\mathscr{F}}(\mathbf{X})$ the Banach algebra of all $\sigma(\mathscr{F}, \mathbf{X})$-continuous linear operators on $\mathbf{X}$. For $T \in \mathscr{B}_{\mathscr{F}}(\mathbf{X})$ we have

$$T^{\mathscr{F}}(\varphi) = \varphi \circ T, \quad \varphi \in \mathscr{F} \quad \text{and} \quad T^{\mathscr{F}} \in \mathscr{B}_{\mathbf{X}}(\mathscr{F}).$$

If $(\mathbf{X}, \mathscr{F})$ is a dual pair of Banach spaces and T is a $\sigma(\mathbf{X}, \mathscr{F})$-closed linear operator in $\mathbf{X}$; then the *resolvent set of* T is

$$\varrho(T) = \{\lambda \in \mathbf{C};\ \lambda - T \quad \text{is injective and} \quad (\lambda - T)^{-1} \in \mathscr{B}_{\mathscr{F}}(\mathbf{X})\},$$

and *the spectrum* of T is $\sigma(T) = \mathbf{C} \setminus \varrho(T)$. The standard power series argument shows that $\varrho(T)$ is open in $\mathbf{C}$, thus $\sigma(T)$ is closed. If T is also $\sigma(\mathbf{X}, \mathscr{F})$-densely defined, then $\sigma(T) = \sigma(T^{\mathscr{F}})$. We note that if $\mathscr{F} = \mathbf{X}^*$ or $\mathbf{X} = \mathscr{F}^*$, then, by the closed graph theorem, the Banach—Smulian theorem on the weak continuity of linear functionals, and the Alaoglu theorem, we have

$$\varrho(T) = \{\lambda \in \mathbf{C};\ \lambda - T \text{ is bijective}\}.$$

Let $(\mathbf{X}, \mathscr{F})$ be a dual pair of Banach spaces; a *one-parameter cosine family* C in $\mathscr{B}_{\mathscr{F}}(\mathbf{X})$ is a mapping $C : \mathbf{R} \to \mathscr{B}_{\mathscr{F}}(\mathbf{X})$ such that

$$C_0 = I_{\mathbf{X}}, \text{ where } I_{\mathbf{X}} \text{ is the identity map of } \mathbf{X};$$

$$C_{s+t} + C_{s-t} = 2C_s C_t \text{ for all } s, t \in \mathbf{R}.$$

It follows directly from this definition that

$$C_t = C_{-t}, \quad t \in \mathbf{R} \quad \text{and} \quad C_t C_s = C_s C_t, \quad s, t \in \mathbf{R}.$$

C is called *$\sigma(\mathbf{X}, \mathscr{F})$-continuous* if for each $x \in \mathbf{X}$ the mapping $\mathbf{R} \ni t \to C_t(x) \in \mathbf{X}$ is $\sigma(\mathbf{X}, \mathscr{F})$-continuous. In this case one can define the *dual cosine family* $C^{\mathscr{F}}$ in $\mathscr{B}_{\mathbf{X}}(\mathscr{F})$ by $C_t^{\mathscr{F}} = (C_t)^{\mathscr{F}}$, $t \in \mathbf{R}$, and $C^{\mathscr{F}}$ is $\sigma(\mathscr{F}, \mathbf{X})$-continuous. We note that a quite complete infinitesimal generator theory for strongly continuous one-parameter cosine families is done in [12].

Let C be a $\sigma(\mathbf{X}, \mathscr{F})$-continuous one-parameter cosine-family in $\mathscr{B}_{\mathscr{F}}(\mathbf{X})$. For $z \in \mathbf{C}$, denote by $D_z = \{\zeta \in \mathbf{C};\ \mathrm{Im}\,\zeta \cdot \mathrm{Im}\,z \leqq 0,\ |\mathrm{Im}\,\zeta| \leqq |\mathrm{Im}\,z|\}$. Suppose that for some $x \in \mathbf{X}$, the mapping $\mathbf{R} \ni t \to C_t(x) \in \mathbf{X}$ has a $\sigma(\mathbf{X}, \mathscr{F})$-continuous extension on D_z which is analytic on its interior; such an extension will be called *$\sigma(\mathbf{X}, \mathscr{F})$-regular*. By the symmetry principle ([1], Ch. V, 1.6) it follows that this extension is uniquely determined. Thus we can define a linear operator C_z in $\mathbf{X}$ by

$$(x, y) \in \mathrm{graph}\, C_z \Leftrightarrow \mathbf{R} \ni t \to C_t(x) \in \mathbf{X} \text{ has a } \sigma(\mathbf{X}, \mathscr{F})\text{-regular extension on } D_z \text{ whose value at } z \text{ is } y.$$

C_z is called the *analytic extension* of C at z.

Lemma 1.1. *Let $(\mathbf{X}, \mathscr{F})$ be a dual pair of Banach spaces and C a $\sigma(\mathbf{X}, \mathscr{F})$-continuous one-parameter cosine family in $\mathscr{B}_{\mathscr{F}}(\mathbf{X})$. Then*

$$C_z = C_{-z}, \quad z \in \mathbf{C},$$

$$C_{s+z} + C_{s-z} = 2C_s C_z \subset 2C_z C_s, \quad s \in \mathbf{R}, \quad z \in \mathbf{C}.$$

Proof. Let $z \in \mathbf{C}$. For each $x \in \mathscr{D}_{C_z}$ the mapping $D_{-z} \ni \zeta \to C_{-\zeta}(x) \in \mathbf{X}$ is $\sigma(\mathbf{X}, \mathscr{F})$-regular and extends $\mathbf{R} \ni t \to C_{-t}(x) = C_t(x) \in \mathbf{X}$, hence $x \in \mathscr{D}_{D_{-z}}$, $C_{-z}(x) = C_z(x)$. Thus $C_z \subset C_{-z}$ and changing z with $-z$, one gets also the converse inclusion.

Let further $s \in \mathbf{R}$ and $z \in \mathbf{C}$. For each $x \in \mathscr{D}_{C_z} = \mathscr{D}_{C_{s+z}} = \mathscr{D}_{C_{s-z}}$ the mappings

$$D_z \ni \zeta \to C_{s+\zeta}(x) + C_{s-\zeta}(x) \in \mathbf{X}, \quad D_z \ni \zeta \to 2C_s C_\zeta(x) \in \mathbf{X}$$

are $\sigma(\mathbf{X}, \mathscr{F})$-regular extensions of

$$\mathbf{R} \ni t \to C_{s+t}(x) + C_{s-t}(x) = 2C_s C_t(x) = 2C_t C_s(x) \in \mathbf{X};$$

thus

$$C_s(x) \in \mathscr{D}_{C_z}, \quad C_{s+z}(x) + C_{s-z}(x) = 2C_s C_z(x) = 2C_z C_s(x).$$

Therefore $C_{s+z} + C_{s-z} = 2C_s C_z \subset 2C_z C_s$.

According to Lemma 1.1 and to the symmetry principle, for each $z \in \mathbf{C}$, it holds

$$(x, y) \in \mathrm{graph}\, C_z \Leftrightarrow \mathbf{R} \ni t \to C_t(x) \in \mathbf{X} \text{ has a } \sigma(\mathbf{X}, \mathscr{F})\text{-regular extension on the strip } \{\zeta \in \mathbf{C};\ |\mathrm{Im}\,\zeta| \leqq |\mathrm{Im}\,z|\} \text{ whose value in } z \text{ is } y.$$

In particular, if $z \in \mathbf{C}$, $\mathrm{Im}\,z \neq 0$ and $x \in \mathscr{D}_{C_z}$, then by [8], Th. 3.10.1, we have that $\mathbf{R} \ni t \to C_t(x) \in \mathbf{X}$ is norm-continuous.

Lemma 1.2. *Let* $(\mathbf{X}, \mathscr{F})$ *be a dual pair of Banach spaces and* C *a* $\sigma(\mathbf{X}, \mathscr{F})$-*continuous one-parameter cosine family in* $\mathscr{B}_{\mathscr{F}}(\mathbf{X})$. *Then we have:*

(i) *for each* $z \in \mathbf{C}$ *and* $x \in \mathscr{D}_{C_z}$

$$\|C_z(x)\| \leqq \sup_{\substack{|\mathrm{Re}\,\zeta| \leqq 1 \\ \mathrm{Im}\,\zeta = \mathrm{Im}\, z}} \|C_\zeta(x)\| \cdot \sum_{\substack{k \in \mathbf{Z} \\ |k| \leqq |\mathrm{Re}\, z|}} \|C_k\|;$$

(ii) *for each* $\varepsilon > 0$ *and* $x \in \mathscr{D}_{C_{-\varepsilon i}} = \mathscr{D}_{C_{\varepsilon i}}$

$$\overline{\lim_{\delta \to +\infty}} \frac{1}{\delta} \ln \sup_{\substack{|\mathrm{Re}\, z| \leqq \delta \\ |\mathrm{Im}\, z| \leqq \varepsilon}} \|C_z(x)\| \leqq \overline{\lim_{t \to +\infty}} \frac{1}{t} \ln \|C_t\| = \overline{\lim_{\mathbf{Z} \ni k \to +\infty}} \frac{1}{k} \ln \|C_k\| \leqq \ln(1 + 2\|C_1\|).$$

Proof. (i) Let $\alpha \in \mathbf{R}$ and $x \in \mathscr{D}_{C_{\alpha i}}$, and denote for convenience

$$c = \sup_{|s| \leqq 1} \|C_{s+\alpha i}(x)\|.$$

We prove by induction, that for $n \geqq 1$,

$$\|C_{t+\alpha i}(x)\| \leqq c \sum_{\substack{k \in \mathbf{Z} \\ |k| \leqq n-1}} \|C_k\| \quad \text{for} \quad |t| \leqq n.$$

Indeed, the above statement holds obviously for $n=1$. Assuming that it holds for some $n \geqq 1$ and that $n < |t| \leqq n+1$, we successively get by Lemma 1.1

$$C_{t+\alpha i}(x) = C_{|t| + \mathrm{sign}(t)\alpha i}(x) = 2C_n C_{|t|-n+\mathrm{sign}(t)\alpha i}(x) - C_{n-(|t|-n)-\mathrm{sign}(t)\alpha i}(x) =$$
$$= 2C_n C_{\mathrm{sign}(t)(|t|-n)+\alpha i}(x) - C_{-\mathrm{sign}(t)(n-(|t|-n))+\alpha i}(x),$$
$$\|C_{t+\alpha i}(x)\| \leqq 2\|C_n\| \cdot c + c \cdot \sum_{|k| \leqq n-1} \|C_k\| = c \cdot \sum_{|k| \leqq n} \|C_k\|.$$

(ii) Again by induction, it is easy to verify that

$$\|C_k\| \leqq (1 + 2\|C_1\|)^k \quad \text{for} \quad k \geqq 1.$$

Now one can easily complete the proof.

We note that, if $f: \mathbf{R} \to \mathbf{C}$ is a Lebesgue-measurable function with

$$\int_{-\infty}^{+\infty} |f(t)| e^{\omega |t|} \, dt < +\infty$$

where $\omega > \overline{\lim_{t \to +\infty}} \frac{1}{t} \ln \|C_t\|$, then there exists

$$C_f = \int_{-\infty}^{+\infty} f(t) C_t \, dt \in \mathscr{B}_{\mathscr{F}}(\mathbf{X})$$

uniquely defined by

$$\left\langle \left(\int_{-\infty}^{+\infty} f(t) C_t \, dt \right)(x), \varphi \right\rangle = \int_{-\infty}^{+\infty} f(t) \langle C_t(x), \varphi \rangle \, dt, \quad x \in \mathbf{X}, \ \varphi \in \mathscr{F}$$

([2], Prop. 1.2; [3], Prop. 1.4). For each $s\in\mathbf{R}$; we have:

$$C_f C_s = C_s C_f = \int_{+\infty}^{-\infty} f(t) C_s C_t td = \int_{-\infty}^{+\infty} f(t)\frac{1}{2}(C_{s+t}+C_{s-t})\,dt =$$

$$= \int_{-\infty}^{+\infty} \frac{f(t-s)+f(-t+s)}{2} C_t\,dt.$$

Thus, if f is additionally even, then

$$C_f C_s = C_s C_f = \int_{-\infty}^{+\infty} f(t-s) C_t\,dt, \quad s\in\mathbf{R}.$$

Lemma 1.3. *Let* $(\mathbf{X}, \mathscr{F})$ *be a dual pair of Banach spaces and* C *a* $\sigma(\mathbf{X}, \mathscr{F})$-*continuous one-parameter cosine family in* $\mathscr{B}_{\mathscr{F}}(\mathbf{X})$. *Let us denote*

$$f_\delta(t) = \sqrt{\frac{\delta}{\pi}}\, e^{-\delta t^2}, \quad \delta > 0, \ t\in\mathbf{R}.$$

Then

$$C_{f_\delta}(\mathbf{X}) \subset \bigcap_{z\in\mathbf{C}} \mathscr{D}_{C_z}, \quad \delta > 0,$$

$$C_{f_\delta} C_z \subset C_z C_{f_\delta} = \int_{-\infty}^{+\infty} \sqrt{\frac{\delta}{\pi}}\, e^{-\delta(t-z)^2} C_t\,dt\in\mathscr{B}_{\mathscr{F}}(\mathbf{X}), \quad \delta > 0, \ z\in\mathbf{C},$$

$$\sigma(\mathbf{X}, \mathscr{F}) - \lim_{\delta\to+\infty} C_{f_\delta}(x) = x, \quad x\in\mathbf{X}.$$

Proof. Let $\delta>0$. Since

$$\int_{-\infty}^{+\infty} \sqrt{\frac{\delta}{\pi}}\, e^{-\delta(t-z)^2} |e^{\omega|t|}\,dt < +\infty, \quad z\in\mathbf{C}, \ \omega > 0,$$

the integral

$$\int_{-\infty}^{+\infty} \sqrt{\frac{\delta}{\pi}}\, e^{-\delta(t-z)^2} C_t\,dt\in\mathscr{B}_{\mathscr{F}}(\mathbf{X}), \quad z\in\mathbf{C}$$

exists. It is easy to see that the mapping

$$\mathbf{C}\ni z \to \int_{-\infty}^{+\infty} \sqrt{\frac{\delta}{\pi}}\, e^{-\delta(t-z)^2} C_t\,dt\in\mathscr{B}_{\mathscr{F}}(\mathbf{X})$$

is analytical and extends

$$\mathbf{R}\ni s \to C_f C_s = C_s C_f = \int_{-\infty}^{+\infty} \sqrt{\frac{\delta}{\pi}}\, e^{-\delta(t-s)^2} C_t\,dt\in\mathscr{B}_{\mathscr{F}}(\mathbf{X}).$$

It follows that

$$C_{f_\delta}(\mathbf{X}) \subset \bigcap_{z\in\mathbf{C}} \mathcal{D}_{C_z},$$

$$C_{f_\delta} C_z \subset C_z C_{f_\delta} = \int_{-\infty}^{+\infty} \sqrt{\frac{\delta}{\pi}}\, e^{-\delta(t-z)^2} C_t\, dt \in \mathcal{B}_{\mathcal{F}}(\mathbf{X}), \quad z\in\mathbf{C}.$$

Now let $x\in\mathbf{X}$ be arbitrary; for each $\varphi\in\mathcal{F}$ and $\delta, \varepsilon>0$ the following holds:

$$|\langle C_{f_\delta}(x)-x, \varphi\rangle| = \Big|\int_{-\infty}^{+\infty} \sqrt{\frac{\delta}{\pi}}\, e^{-\delta t^2} \langle C_t(x)-x, \varphi\rangle\, dt\Big| \leqq$$

$$\leqq \int_{|t|<\varepsilon} \sqrt{\frac{\delta}{\pi}}\, e^{-\delta t^2} |\langle C_t(x)-x, \varphi\rangle|\, dt + \int_{|t|\geqq\varepsilon} \sqrt{\frac{\delta}{\pi}}\, e^{-\delta t^2} |\langle C_t(x)-x, \varphi\rangle|\, dt \leqq$$

$$\leqq \sup_{|t|<\varepsilon} |\langle C_t(x)-x, \varphi\rangle| + \int_{|t|\geqq\varepsilon} \sqrt{\frac{\delta}{\pi}}\, e^{-\delta t^2} (\|C_t\|+1)\, dt\, \|x\|\, \|\varphi\|.$$

Hence

$$\overline{\lim_{\delta\to+\infty}} |\langle C_{f_\delta}(x)-x, \varphi\rangle| \leqq \inf_{\varepsilon\to 0} \sup_{|t|<\varepsilon} |\langle C_t(x)-x, \varphi\rangle| = 0.$$

We can now give

Proposition 1.4. *Let $(\mathbf{X}, \mathcal{F})$ be a dual pair of Banach spaces, C a $\sigma(\mathbf{X}, \mathcal{F})$-continuous one-parameter cosine family in $\mathcal{B}_{\mathcal{F}}(\mathbf{X})$ and $z\in\mathbf{C}$. Then C_z is $\sigma(\mathbf{X}; \mathcal{F})$-densely defined and $\sigma(\mathbf{X}, \mathcal{F})$-preclosed. Moreover, we have*

$$\overline{C_z}^{\sigma(\mathbf{X}, \mathcal{F})} = \overline{C_z|\bigcap_{\zeta\in\mathbf{C}} \mathcal{D}_{C_\zeta}}^{\sigma(\mathbf{X}, \mathcal{F})} = (C_z^{\mathcal{F}})^{\mathbf{X}}.$$

Proof. By Lemma 1.3 it is clear that C_z and $C_z^{\mathcal{F}}$ are $\sigma(\mathbf{X}, \mathcal{F})$, resp. $\sigma(\mathcal{F}, \mathbf{X})$-densely defined. For each $x\in\mathcal{D}_{C_z}$ and $\varphi\in\mathcal{D}_{C_z^{\mathcal{F}}}$ the functions $\zeta\to\langle C_\zeta(x), \varphi\rangle$ and $\zeta\to\langle x, C_\zeta^{\mathcal{F}}(\varphi)\rangle$ defined on the strip $\{\zeta\in\mathbf{C};\ |\mathrm{Im}\,\zeta|\leqq|\mathrm{Im}\,z|\}$, are regular extensions of the function

$$\mathbf{R}\ni t \to \langle C_t(x), \varphi\rangle = \langle x, C_t^{\mathcal{F}}(\varphi)\rangle,$$

hence $\langle C_z(x), \varphi\rangle = \langle x, C_z^{\mathcal{F}}(\varphi)\rangle$. It follows $C_z \subset (C_z^{\mathcal{F}})^{\mathbf{X}}$. In particular, C_z is $\sigma(\mathbf{X}, \mathcal{F})$-preclosed.

To end the proof, we have only to prove that the domain of $(C_z^{\mathcal{F}})^{\mathbf{X}}$ is contained in the domain of $\overline{C_z|\bigcap_{\zeta\in\mathbf{C}} \mathcal{D}_{D_\zeta}}^{\sigma(\mathbf{X}, \mathcal{F})}$. Let x be in the domain of $(C_z^{\mathcal{F}})^{\mathbf{X}}$. By Lemma 1.3, for each $\delta>0$ we have $C_{f_\delta}^{\mathcal{F}} C_z^{\mathcal{F}} \subset C_z^{\mathcal{F}} C_{f_\delta}^{\mathcal{F}}$, thus

$$C_{f_\delta}(C_z^{\mathcal{F}})^{\mathbf{X}} = (C_z^{\mathcal{F}} D_{f_\delta}^{\mathcal{F}})^{\mathbf{X}} \subset (C_{f_\delta}^{\mathcal{F}} C_z^{\mathcal{F}})^{\mathbf{X}} = (C_z^{\mathcal{F}})^{\mathbf{X}} C_{f_\delta}.$$

Again by Lemma 1.3, it follows that

$$C_{f_\delta}(x) \in \bigcap_{\zeta \in \mathbf{C}} \mathscr{D}_{C_\zeta}, \quad \delta > 0,$$

$$\sigma(\mathbf{X}, \mathscr{F}) - \lim_{\delta \to +\infty} C_{f_\delta}(x) = x,$$

$$\sigma(\mathbf{X}, \mathscr{F}) - \lim_{\delta \to +\infty} C_z C_{f_\delta}(x) = \sigma(\mathbf{X}, \mathscr{F}) - \lim_{\delta \to +\infty} (C_z^{\mathscr{F}})^{\mathbf{X}} C_{f_\delta}(x) =$$

$$= \sigma(\mathbf{X}, \mathscr{F}) - \lim_{\delta \to +\infty} C_{f_\delta}(C_z^{\mathscr{F}})^{\mathbf{X}}(x) = (C_z^{\mathscr{F}})^{\mathbf{X}}(x).$$

Consequently x is in the domain of $\overline{C_z | \bigcap_{\zeta \in \mathbf{C}} \mathscr{D}_{C_\zeta}}^{\sigma(\mathbf{X}, \mathscr{F})}$.

In the sequel we shall denote $\bar{C}_z^{\sigma(\mathbf{X}, \mathscr{F})}$ and $\overline{C_z^{\mathscr{F}}}^{\sigma(\mathscr{F}, \mathbf{X})}$ simply by $\bar{C}_z$, respectively $\overline{C_z^{\mathscr{F}}}$. We call $\overline{C_i} = \overline{C_{-i}}$ *the analytic generator* of the cosine family C.

2. Spectral properties of the analytic generator of cosine families

Let $(\mathbf{X}, \mathscr{F})$ be a dual pair of Banach spaces and C a $\sigma(\mathbf{X}, \mathscr{F})$-continuous one-parameter cosine family in $\mathscr{B}_{\mathscr{F}}(\mathbf{X})$. We recall that by Lemma 1.2 (ii)

$$\lim_{t \to +\infty} \frac{1}{t} \ln \|C_t\| = \overline{\lim_{k \to +\infty}} \frac{1}{k} \ln \|C_k\| < +\infty.$$

On the other hand, if $\mathbf{X} \neq \{0\}$, then

$$\overline{\lim_{t \to +\infty}} \frac{1}{t} \ln \|C_t\| \geqq 0.$$

Indeed, we have for each $t \in \mathbf{R}$

$$1 = \|C_0\| = \|2C_tC_t - C_{2t}\| \leqq 2\|C_t\|^2 + \|C_{2t}\| \leqq 3 \max\{\|C_t\|^2, \|C_{2t}\|\}$$

so that

$$0 = \lim_{t \to +\infty} \frac{1}{2t} \ln 1 \leqq \overline{\lim_{t \to +\infty}} \max\left\{\frac{1}{t} \ln \|C_t\|, \frac{1}{2t} \ln \|C_{2t}\|\right\} = \lim_{t \to +\infty} \frac{1}{t} \ln \|C_t\|.$$

We say that C is of 0 *exponential type* if

$$\overline{\lim_{t \to +\infty}} \frac{1}{t} \ln \|C_t\| \leqq 0$$

that is, if $\mathbf{X} \neq \{0\}$,

$$\overline{\lim_{t \to \infty}} \frac{1}{t} \ln \|C_t\| = 0.$$

Let $\mu \in \mathbf{C} \setminus (-\infty, 0]$. We denote

$$\arg \mu = \theta \text{ where } \mu = |\mu| e^{i\theta}, \ |\theta| < \pi; \quad \ln \mu = \ln |\mu| + i\theta.$$

Then $\mathbf{C} \ni z \to \mu^z = e^{z \ln \mu} \in \mathbf{C}$ is an entire function. The next lemma is the main technical result of this paragraph.

Lemma 2.1. *Let* $(\mathbf{X}, \mathscr{F})$ *be a dual pair of Banach spaces,* C *a* $\sigma(\mathbf{X}, \mathscr{F})$-*continuous one-parameter cosine family of* 0 *exponential type in* $\mathscr{B}_{\mathscr{F}}(\mathbf{X})$, $\mu \in \mathbf{C} \setminus (-\infty, 0]$ *and* $\lambda = (\mu^2+1)/2\mu$. *Then the function* $g_\lambda \colon \mathbf{R} \to \mathbf{C}$, *defined by*

$$g_\lambda(t) = \begin{cases} \dfrac{\mu}{\mu^2-1} \cdot \dfrac{\mu^{it}-\mu^{-it}}{\sin i\pi t} = \dfrac{\mu}{i(\mu^2-1)} \cdot \dfrac{\mu^{it}-\mu^{it}}{\operatorname{sh} \pi t} & \text{if } \mu \neq 1, \\ \dfrac{it}{\sin i\pi t} = \dfrac{t}{\operatorname{sh} \pi t} & \text{if } \mu = 1 \end{cases}$$

depends only on λ, *the integral*

$$C_{g_\lambda} = \int_{-\infty}^{+\infty} g_\lambda(t) C_t \, dt \in \mathscr{B}_{\mathscr{F}}(\mathbf{X})$$

exists and $C_{g_\lambda}(\lambda + C_{-i}) \subset (\lambda + C_{-i}) C_{g_\lambda}; I_{\mathbf{X}}$.

Proof. Since the roots of the equation $\lambda = (w^2+1)/2w$ are μ and μ^{-1} and

$$\frac{\mu}{\mu^2-1}(\mu^{it}-\mu^{-it}) = \frac{\mu^{-1}}{\mu^{-2}-1}(\mu^{-it}-\mu^{it}), \quad t \in \mathbf{R},$$

g_λ depends only on λ.

Choosing some ω with $0 < \omega < \pi - |\arg \mu|$, we have

$$\int_{-\infty}^{+\infty} |g_\lambda(t)| e^{\omega |t|} \, dt < +\infty \quad \text{and} \quad \omega > \overline{\lim_{t \to +\infty}} \frac{1}{t} \ln \|C_t\| = 0.$$

By our remarks after Lemma 1.2 it follows that $C_{g_\lambda} \in \mathscr{B}_{\mathscr{F}}(\mathbf{X})$ is well defined and $C_{g_\lambda} C_s = C_s C_{g_\lambda}$. Let $x \in C_{-i}$ be arbitrary. Since the mapping

$$\{\zeta \in \mathbf{C};\ |\operatorname{Im} \zeta| \leqq 1\} \ni \zeta \to C_{g_\lambda} C_\zeta(x) \in \mathbf{X}$$

is $\sigma(\mathbf{X}, \mathscr{F})$-regular and extends

$$\mathbf{R} \ni s \to C_{g_\lambda} C_s(x) = C_s C_{g_\lambda}(x) \in \mathbf{X}$$

we have $C_{g_\lambda}(x) \in \mathscr{D}_{C_{-i}}$ and $C_{-i} C_{g_\lambda}(x) = C_{g_\lambda} C_{-i}(x)$. Consequently $(\lambda + C_{-i}) C_{g_\lambda}(x) = C_{g_\lambda}(\lambda + C_{-i})(x)$.

Finally we show that $C_{g_\lambda} C_{-i}(x) = x - \lambda C_{g_\lambda}(x)$, that is $C_{g_\lambda}(\lambda + C_{-i})(x) = x$.

Let us first assume $\mu\neq 1$ and fix some. $0<\varepsilon<1$. Since by Lemma 1.2. (ii)

$$\overline{\lim_{\delta\to+\infty}}\frac{1}{\delta}\ln\sup_{\substack{|\mathrm{Re}\,\zeta|\leqq 1\\ |\mathrm{Im}\,\zeta|\leqq 1}}\|C_\zeta(x)\|\leqq 0,$$

using Lemma 1.1 and the Cauchy integral theorem, we get

$$C_{g_\lambda}C_{-i}(x)=\int_{-\infty}^{+\infty}\frac{\mu}{2(\mu^2-1)}\cdot\frac{\mu^{it}-\mu^{-it}}{\sin i\pi t}C_{t-i}(x)\,dt+$$

$$+\int_{-\infty}^{+\infty}\frac{\mu}{2(\mu^2-1)}\cdot\frac{\mu^{it}-\mu^{it}}{\sin i\pi i}C_{t+i}(x)\,dt=$$

$$=\int_{-\infty}^{+\infty}\frac{\mu}{2(\mu^2-1)}\cdot\frac{\mu^{i(t+i\varepsilon)}-\mu^{-i(t+i\varepsilon)}}{\sin i\pi(t+i\varepsilon)}C_{t-(1-\varepsilon)i}(x)\,dt+$$

$$+\int_{-\infty}^{+\infty}\frac{\mu}{2(\mu^2-1)}\cdot\frac{\mu^{i(t-i\varepsilon)}-\mu^{-i(t-i\varepsilon)}}{\sin i\pi(t-i\varepsilon)}C_{t+(1-\varepsilon)i}(x)\,dt.$$

Defining the curves Γ_- and Γ_+ by

$$\Gamma_-(t)=t-(1-\varepsilon)i,\quad \Gamma_-(t)=1+(1-\varepsilon)i,\quad t\in\mathbf{R},$$

we obtain

$$C_{g_\lambda}C_{-i}(x)=\int_{\Gamma_-}\frac{\mu}{2(\mu^2-1)}\cdot\frac{\mu^{iz-1}-\mu^{-iz+1}}{\sin(i\pi z-\pi)}C_z(x)\,dz+$$

$$+\int_{\Gamma_+}\frac{\mu}{2(\mu^2-1)}\cdot\frac{\mu^{iz+1}-\mu^{-iz-1}}{\sin(i\pi z+\pi)}C_z(x)\,dz=$$

$$=\int_{\Gamma_-}\frac{\mu}{2(\mu^2-1)}\frac{\mu^{-iz+1}-\mu^{iz-1}}{\sin i\pi z}C_z(x)\,dz+\int_{\Gamma_+}\frac{\mu}{2(\mu^2-1)}\cdot\frac{\mu^{-iz-1}-\mu^{iz+1}}{\sin i\pi z}C_z(x)\,dz.$$

Further, the residue theorem gives

$$\int_{\Gamma_-}\frac{\mu}{2(\mu^2-1)}\frac{\mu^{-iz+1}-\mu^{iz-1}}{\sin i\pi z}C_z(x)\,dz=x+\int_{\Gamma_+}\frac{\mu}{2(\mu^2-1)}\frac{\mu^{-iz+1}-\mu^{iz-1}}{\sin i\pi z}Cz(x)\,dz$$

so that, using again the Cauchy integral theorem, we conclude

$$C_{g_\lambda}C_{-i}(x)=x+\int_{\Gamma_+}\frac{\mu}{2(\mu^2-1)}\cdot\frac{\mu^{-iz+1}-\mu^{iz-1}+\mu^{-iz-1}-\mu^{iz+1}}{\sin i\pi z}C_z(x)\,dz=$$

$$=x+\int_{\Gamma_+}\frac{\mu}{2(\mu^2-1)}\cdot\frac{(\mu+\mu^{-1})(\mu^{-iz}-\mu^{iz})}{\sin i\pi z}C_z(x)\,dz=$$

$$=x-\lambda\int_{\Gamma_+}\frac{\mu}{\mu^2-1}\cdot\frac{\mu^{iz}-\mu^{-iz}}{\sin i\pi z}C_z(x)\,dz=x-\lambda C_{g_\lambda}(x).$$

For $\mu=1$ the proof is completely similar.

Theorem 2.2. *Let* $(\mathbf{X}, \mathscr{F})$ *be a dual pair of Banach spaces and* C *a* $\sigma(\mathbf{X}, \mathscr{F})$-*continuous one-parameter cosine family of* 0 *exponential type in* $\mathscr{B}_{\mathscr{F}}(\mathbf{X})$. *Then* $\sigma(\overline{C}_i) \subset [1, +\infty)$. *Moreover, for each* $\lambda \in \mathbf{C} \setminus (-\infty, -1]$ *the roots of the equation* $\lambda = (\mu^2+1)/2\mu$ *belong to* $\mathbf{C} \setminus (-\infty, 0)$ *and*

$$(\lambda + \overline{C_{-i}})^{-1} = \begin{cases} \int_{-\infty}^{+\infty} \frac{\mu}{i(\mu^2-1)} \cdot \frac{\mu^{it} - \mu^{-it}}{\operatorname{sh} \pi t} C_t \, dt & \text{if } \lambda \neq 1 \\ \int_{-\infty}^{+\infty} \frac{t}{\operatorname{sh} \pi t} C_t \, dt & \text{if } \lambda = 1. \end{cases}$$

Proof. Let $\lambda \in \mathbf{C} \setminus (-\infty, -1]$ be arbitrary. If one of the roots of the equation $\lambda = (\mu^2+1)/2\mu$, say μ_1, belonged to $(-\infty, 0]$, then, taking in account that the other root is μ_1^{-1}, we would have

$$\lambda = \frac{\mu_1 + \mu_1^{-1}}{2} = -\frac{|\mu_1| + |\mu_1|^{-1}}{2} \leqq -1.$$

Now let $\mu \in \mathbf{C} \setminus (-\infty, 0]$ be arbitrary, with $\lambda = (\mu^2+1)/2\mu$, define $g_\lambda : \mathbf{R} \to \mathbf{C}$ as in Lemma 2.1. Then by this lemma $\overline{C_{g_\lambda}(\lambda + C_{-i})} \subset (\lambda + C_{-i}) C_{g_\lambda} = I_{\mathbf{X}}$ holds, and this implies that $C_{g_\lambda}(\lambda + \overline{C_{-i}}) \subset (\lambda + \overline{C_{-i}}) C_{g_\lambda}$. On the other hand, since $(\lambda + \overline{C_{-i}}) C_{g_\lambda} | \mathscr{D}_{C_{-i}} = C_{g_\lambda}(\lambda + C_{-i}) \subset I_{\mathbf{X}}$, and $\mathscr{D}_{C_{-i}}$ is $\sigma(\mathbf{X}, \mathscr{F})$-dense in $\mathbf{X}$, one gets easily that $(\lambda + \overline{C_{-i}}) C_{g_\lambda} = I_{\mathbf{X}}$. Consequently $\lambda + \overline{C_{-i}}$ is invertible and $(\lambda + \overline{C_{-i}})^{-1} = C_{g_\lambda} \in \mathscr{B}_{\mathscr{F}}(\mathbf{X})$.

A first consequence of Theorem 2.2 is the following unicity result:

Corollary 2.3. *Let* $(\mathbf{X}, \mathscr{F})$ *be a dual pair of Banach spaces and* C *and* D *two* $\sigma(\mathbf{X}, \mathscr{F})$-*continuous one-parameter cosine families of* 0 *exponential type in* $\mathscr{B}_{\mathscr{F}}(\mathbf{X})$. *If* $\overline{C_{-i}} \subset \overline{D_{-i}}$, *then* $C = D$.

Proof. By Theorem 2.2 we have for each $s \in \mathbf{R} - \{0\}$

$$\left(\frac{e^{2s}+1}{2e^s} + \overline{C_{-i}} \right)^{-1} = \int_{-\infty}^{+\infty} \frac{e^s}{i(e^{2s}-1)} \cdot \frac{e^{its} - e^{-its}}{\operatorname{sh} \pi t} C_t \, dt,$$

that is,

$$\frac{e^{2s}-1}{2e^s} \left(\frac{e^{2s}+1}{2e^s} + \overline{C_{-i}} \right)^{-1} = \int_{-\infty}^{+\infty} \frac{\sin ts}{\operatorname{sh} \pi t} C_t \, dt, \quad s \in \mathbf{R}.$$

Similarly,

$$\frac{e^{2s}-1}{2e^s} \left(\frac{e^{2s}+1}{2e^s} + \overline{D_{-i}} \right)^{-1} = \int_{-\infty}^{+\infty} \frac{\sin ts}{\operatorname{sh} \pi t} D_t \, dt, \quad s \in \mathbf{R}.$$

Since $\overline{C_{-i}} \subset \overline{D_{-i}}$ implies $\left(\frac{e^{2s}+1}{2e^s}+\overline{C_{-i}}\right)^{-1}=\left(\frac{e^{2s}+1}{2e^s}+\overline{D_{-i}}\right)^{-1}$ for every $s \in \mathbf{R}$, the above considerations yield

$$\int_{-\infty}^{+\infty} \frac{\sin ts}{\operatorname{sh} \pi t} C_t \, dt = \int_{-\infty}^{+\infty} \frac{\sin ts}{\operatorname{sh} \pi t} D_t \, dt, \quad s \in \mathbf{R}.$$

Using the inequality $|\sin \alpha - \sin \beta| \leqq |\alpha - \beta|$, $\alpha, \beta \in \mathbf{R}$, and the Lebesgue dominated convergence theorem, it is easy to see that one can differentiate with respect to s under the sign of integration and we get

$$\int_{-\infty}^{+\infty} \cos ts \frac{t}{\operatorname{sh} \pi t} C_t \, dt = \int_{-\infty}^{+\infty} \cos ts \frac{t}{\operatorname{sh} \pi t} D_t \, dt.$$

In other words, for each $x \in \mathbf{X}$ and $\varphi \in \mathscr{F}$, the integrable continuous even functions

$$\mathbf{R} \ni t \to \frac{t}{\operatorname{sh} \pi t} \langle C_t(x), \varphi \rangle \quad \text{and} \quad \mathbf{R} \ni t \to \frac{t}{\operatorname{sh} \pi t} \langle D_t(x), \varphi \rangle$$

have equal Fourier cosine transforms, so they coincide. Consequently $C_t = D_t$; $t \in \mathbf{R}$.

By Corollary 2.3, $\overline{C_{-i}}$ determines C uniquely. A second consequence of Theorem 2.2 is an invariance result:

Corollary 2.4. *Let* $(\mathbf{X}, \mathscr{F})$ *be a dual pair of Banach spaces,* C *a* $\sigma(\mathbf{X}, \mathscr{F})$*-continuous one-parameter cosine family of* 0 *exponential type in* $\mathscr{B}_{\mathscr{F}}(\mathbf{X})$ *and* $\mathbf{Y}$ *a* $\sigma(\mathbf{X}, \mathscr{F})$*-closed linear subspace of* $\mathbf{X}$. *If there exists some* $\lambda_0 \in \mathbf{C} \setminus (-\infty, -1]$ *with* $(\lambda_0 + \overline{C_{-i}})^{-1} \mathbf{Y} \subset \mathbf{Y}$ *then* $C_t \mathbf{Y} \subset \mathbf{Y}$, $t \in \mathbf{R}$.

Proof. If $\lambda \in \mathbf{C}$ and $|\lambda - \lambda_0| < \|(\lambda_0 + \overline{C_{-i}})^{-1}\|^{-1}$, then $(\lambda + \overline{C_{-i}})^{-1}$ exists and

$$(\lambda + \overline{C_{-i}})^{-1} = \sum_{k=0}^{\infty} (\lambda - \lambda_0)^k ((\lambda_0 + \overline{C_{-i}})^{-k-1} \in \mathscr{B}_{\mathscr{F}}(\mathbf{X}),$$

where the series converges in the operator norm. Thus, for such λ, we have $(\lambda + \overline{C_{-i}})^{-1} \mathbf{Y} \subset \mathbf{Y}$. Let $y \in \mathbf{Y}$ be arbitrary and $\varphi \in \mathscr{F}$ such that $\langle z, \varphi \rangle = 0$; $z \in \mathbf{Y}$. By the first part of the proof, the analytic function

$$\mathbf{C} \setminus (-\infty, -1] \ni \lambda \to \langle (\lambda + \overline{C_{-i}})^{-1}(y), \varphi \rangle$$

vanishes on some neighbourhood of λ_0, hence it vanishes identically. Using

Theorem 2.2 similarly as in the proof of Corollary 2.3, we get successively

$$\int_{-\infty}^{+\infty} \frac{\sin ts}{\operatorname{sh} t} \langle C_t(y), \varphi \rangle dt = 0, \quad s \in \mathbf{R},$$

$$\int_{-\infty}^{+\infty} \cos ts \frac{t}{\operatorname{sh} \pi t} \langle C_t(y), \varphi \rangle dt = 0, \quad s \in \mathbf{R},$$

$$\langle C_t(y), \varphi \rangle = 0, \quad t \in \mathbf{R}.$$

By the Hahn—Banach theorem we conclude that $C_t \mathbf{Y} \subset \mathbf{Y}$, $t \in \mathbf{R}$.

Corollary 2.5. *Let* $(\mathbf{X}, \mathscr{F})$ *be a dual pair of Banach spaces,* C *a* $\sigma(\mathbf{X}, \mathscr{F})$*-continuous one-parameter cosine family of* 0 *exponential type in* $\mathscr{B}_{\mathscr{F}}(\mathbf{X})$ *and* $\mathbf{Y}$ *and* $\mathbf{Z}$ *two* $\sigma(\mathbf{X}, \mathscr{F})$*-closed linear subspaces of* $\mathbf{X}$. *If* $(\overline{C_{-i}})^{-1}\mathbf{Y} \subset \mathbf{Z}$, $(\overline{C_{-i}})^{-1}\mathbf{Y} \subset \mathbf{Z}$, *then* $\mathbf{Y} = \mathbf{Z}$.

Proof. Let $x \in \mathscr{D}_{C_{-2i}}$. By Lemma 1.1 $2C_s C_{-i}(x) = C_{s+i}(x) + C_{s-i}(x)$, $s \in \mathbf{R}$, so that the mapping

$$\{\zeta \in \mathbf{C};\ |\operatorname{Im} \zeta| \leqq 1\} \ni \zeta \to C_{\zeta+i}(x) + C_{\zeta-i}(x) \in \mathbf{X},$$

which is $\sigma(\mathbf{X}, \mathscr{F})$-regular, extends

$$\mathbf{R} \ni s \to 2C_s C_{-i}(x) \in \mathbf{X}.$$

Thus $C_{-i}(x) \in \mathscr{D}_{C_{-i}}$, $2C_{-i}C_{-i}(x) = x + C_{-2i}(x)$, that is $I_{\mathbf{X}} + C_{-2i} \subset 2(C_{-i})^2 \subset 2(\overline{C_{-i}})^2$. But $(\overline{C_{-i}})^2$ is $\sigma(\mathbf{X}, \mathscr{F})$-closed, hence $I_{\mathbf{X}} + \overline{C_{-2i}} \subset 2(\overline{C_{-i}})^2$ and $(I_{\mathbf{X}} + \overline{C_{-2i}})^{-1} = 2^{-1}(\overline{C_{-i}})^{-2}$. From the last equality, we get

$$(I_{\mathbf{X}} + \overline{C_{-2i}})^{-1}\mathbf{Y} = 2^{-1}(\overline{C_{-i}})^{-2}\mathbf{Y} \subset 2^{-1}(\overline{C_{-i}})^{-1}\mathbf{Z} \subset \mathbf{Y};$$

$$(I_{\mathbf{X}} + \overline{C_{-2i}})^{-1}\mathbf{Z} = 2^{-1}(\overline{C_{-i}})^{-2}\mathbf{Z} \subset 2^{-1}(\overline{C_{-i}})^{-1}\mathbf{Y} \subset \mathbf{Z}.$$

Since $\overline{C_{-2i}}$ is the analytic generator of the cosine family

$$\mathbf{R} \ni t \to C_{2t} \in \mathscr{B}_{\mathscr{F}}(\mathbf{X}),$$

by Corollary 2.4 it follows that $C_t \mathbf{Y} \subset \mathbf{Y}$, $C_t \mathbf{Z} \subset \mathbf{Z}$, $t \in \mathbf{R}$. In particular, by Lemma 1.3 we have

$$\overline{\mathscr{D}_{C_{-i}} \cap \mathbf{Y}}^{\sigma(\mathbf{X}, \mathscr{F})} = \mathbf{Y}, \quad \overline{\mathscr{D}_{C_{-i}} \cap \mathbf{Z}}^{\sigma(\mathbf{X}, \mathscr{F})} = \mathbf{Z}.$$

Using now the invariance of $\mathbf{Y}$ under the action of C and the Hahn—Banach theorem, we deduce successively that $C_{-i}(y) \in \mathbf{Y}$ and $y = (\overline{C_{-i}})^{-1} C_{-i}(y) \in (\overline{C_{-i}})^{-1}\mathbf{Y} \subset \mathbf{Z}$ holds for each $y \in \mathscr{D}_{C_{-i}} \cap \mathbf{Y}$. Thus

$$\mathbf{Y} = \overline{\mathscr{D}_{C_{-i}} \cap \mathbf{Y}}^{\sigma(\mathbf{X}, \mathscr{F})} \subset \mathbf{Z}.$$

One obtains similarly also the inclusion $\mathbf{Z} \subset \mathbf{Y}$.

3. Connections with one-parameter groups of operators

Let $(\mathbf{X}, \mathscr{F})$ be a dual pair of Banach spaces. We recall that the *analytic extension* U_z of a $\sigma(\mathbf{X}, \mathscr{F})$-*continuous one-parameter group*

$$U\colon \mathbf{R}\ni t \to U_t \in \mathscr{B}_{\mathscr{F}}(\mathbf{X})$$

at $z\in\mathbf{C}$ is defined by

$$(x, y)\in \text{graph } U_z \Leftrightarrow \mathbf{R}\ni t\to U_t x\in\mathbf{X} \text{ has a } \sigma(\mathbf{X}, \mathscr{F})\text{-regular extension on the strip } D_z \text{ whose value at } z \text{ is } y$$

and U_z is a $\sigma(\mathbf{X}, \mathscr{F})$-closed and $\sigma(\mathbf{X}, \mathscr{F})$-densely defined linear operator in $\mathbf{X}$ ([3], Section 2). U_{-i} is called the *analytic generator* of U and provided that U is *of* 0 *exponential type,* that is,

$$\lim_{|t|\to\infty} \frac{1}{|t|} \|\ln U_t\| \leqq 0,$$

it uniquely determines U ([3], Section 4).

Proposition 3.1. *Let* $(\mathbf{X}, \mathscr{F})$ *be a dual pair of Banach spaces and* U *a* $\sigma(\mathbf{X}, \mathscr{F})$-*continuous one-parameter group in* $\mathscr{B}_{\mathscr{F}}(\mathbf{X})$. *Then the formula*

$$C_t = \frac{1}{2}(U_t + U_{-t})$$

defines a $\sigma(\mathbf{X}, \mathscr{F})$-*continuous one-parameter cosine family in* $\mathscr{B}_{\mathscr{F}}(\mathbf{X})$ *and*

$$\overline{C_z}^{\sigma(\mathbf{X}, \mathscr{F})} = \frac{1}{2}\overline{(U_z + U_{-z})}^{\sigma(\mathbf{X}, \mathscr{F})}, \quad z\in\mathbf{C}.$$

Proof. It is easy to verify that C is a $\sigma(\mathbf{X}, \mathscr{F})$ one-parameter cosine family in $\mathscr{B}_{\mathscr{F}}(\mathbf{X})$.

From the definition of the analytic extensions of U, respectively C, it follows immediately that $(1/2)(U_z + U_{-z})\subset C_z$. Thus, it remains to prove only the inclusion

$$C_z \subset \frac{1}{2}\overline{(U_z + U_{-z})}^{\sigma(\mathbf{X}, \mathscr{F})}.$$

Let $x\in\mathscr{D}_{C_z}$ be arbitrary and $f_\delta = \sqrt{\frac{\delta}{\pi}}\, e^{-\delta t^2}$, $\delta>0$, $t\in\mathbf{R}$; then, by Lemma 1.3, we have

$$C_{f_\delta} C_z(x) = \int_{-\infty}^{+\infty} \sqrt{\frac{\delta}{\pi}}\, e^{-\delta(t-z)^2} C_t(x)\, dt =$$

$$= \frac{1}{2}\left(\int_{-\infty}^{+\infty} \sqrt{\frac{\delta}{\pi}}\, e^{-\delta(t-z)^2} U_t(x)\, dt + \int_{-\infty}^{+\infty} \sqrt{\frac{\delta}{\pi}}\, e^{-\delta(t+z)^2} U_t(x)\, dt \right).$$

Since

$$\mathbf{C}\ni\zeta\to\int_{-\infty}^{+\infty}\sqrt{\frac{\delta}{\pi}}\,e^{-\delta(t-\zeta)^2}U_t(x)\,dt\in\mathbf{X}$$

is an entire extension of

$$\mathbf{R}\ni s\to U_sU_{f_\delta}(x)\in\mathbf{X}$$

where

$$U_{f_\delta}=\int_{-\infty}^{+\infty}\sqrt{\frac{\delta}{\pi}}\,U_t\,dt\in\mathscr{B}_{\mathscr{F}}(\mathbf{X}),$$

it follows that

$$C_{f_\delta}C_z(x)=\frac{1}{2}(U_z+U_{-z})_{f_\delta}(x),\quad \delta>0.$$

Finally, since

$$\sigma(\mathbf{X},\mathscr{F})-\lim_{\delta\to+\infty}U_{f_\delta}(x)=x,\quad \sigma(\mathbf{X},\mathscr{F})-\lim_{\delta\to+\infty}C_{f_\delta}C_z(x)=C_z(x),$$

we conclude that x belongs to the domain of $(1/2)\overline{(U_z+U_{-z})}^{\sigma(\mathscr{F},\mathbf{X})}$ and that $(1/2)\overline{(U_z+U_{-z})}^{\sigma(\mathbf{X},\mathscr{F})}(x)=C_z(x)$.

In particular, if U is of 0 exponential type, then by Corollary 2.3, $\overline{U_{-i}+U_i}$ uniquely determines the "cosine part" $t\to U_t+U_{-t}$ of V. The "cosine part" of U has the advantage that the spectrum of its analytic generator is always included in $[1,+\infty)$, while the spectrum of U_{-i} is quite frequently$=\mathbf{C}$ (see [4]); this motivates the interest of cosine families in handling one-parameter groups of operators.

Concerning applications, we restrict ourselves to a proof of the following result (see [9] and [5], Th. 4.1):

Theorem 3.2. *Let* $\mathbf{H}$ *be a complex Hilbert space and* C *a weakly continuous one-parameter cosine family of* 0 *exponential type of self-adjoint linear operators on* $\mathbf{H}$. *Then there exists an injective, positive, self-adjoint operator* B *in* $\mathbf{H}$ *such that*

$$C_t=\frac{1}{2}(B^{it}+B^{-it})=\cos(t\ln B),\quad t\in\mathbf{R}.$$

Proof. By Theorem 2.2, $\sigma(\overline{C_{-i}})\subset[1,+\infty)$ and

$$(\overline{C_{-i}})^{-1}=\int_{-\infty}^{+\infty}\frac{1}{e^{\frac{\pi}{2}t}+e^{-\frac{\pi}{2}t}}\,C_t\,dt;$$

thus $\overline{C_{-i}}$ is self-adjoint and $\overline{C_{-i}}\geq I_{\mathbf{H}}$. It follows that $B=\overline{C_{-i}}+((\overline{C_{-i}})^2-I_{\mathbf{H}})^{1/2}$ is an injective, positive, self-adjoint linear operator in $\mathbf{H}$ and

$$B^{-1}=\overline{\overline{C_{-i}}-((\overline{C_{-i}})^2-I_{\mathbf{H}})^{1/2}}\in\mathscr{B}(\mathbf{H})$$

(see, for example, [10], Section 128).

Now, the formula $U_t=B^{it}$, $t\in\mathbf{R}$ defines a strongly continuous one-parameter group of unitaries on $\mathbf{H}$ and $U_{-i}=B$ ([3], Th. 6.1). By Proposition 3.1 the cosine families C and $\mathbf{R}\ni t\to(1/2)(U_t+U_{-t})$ have equal analytic generators, so by Corollary 2.3

$$C_t=\frac{1}{2}(U_t+U_{-t}),\quad t\in\mathbf{R}.$$

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