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Three-element groupoids with minimal clones

B. CSÁKÁNY

To Professor Béla Szőkefalvi-Nagy on his seventieth birthday

A set of finitary operations on a set M is called a *clone* on M if it is closed under composition and contains all projections. The clones on a finite set M form an atomic lattice whose atoms are called *minimal clones*. The set of all term functions (polynomials in the terminology of [5]) of any algebra $\langle M; F\rangle$ is a clone on M. In this paper we give a complete list of those essentially distinct three-element algebras with one essentially binary operation whose clones of term functions are minimal.

The lattice of all clones on a finite set M is also coatomic, and the coatoms are called maximal clones. The knowledge of all maximal clones on M provides a method for deciding whether an algebra $\langle M; F\rangle$ is primal. The maximal clones on a two-element set, on a three-element set, and on any finite set have been determined in [9], [6], and [11], respectively. By the Galois connection between operations and relations on a finite set (see [4], [1]), the knowledge of all minimal clones on M enables us to decide whether a set of (finitary) relations on M generates all relations on M (in the sense of [1]). The minimal clones on a two-element set are determined in [9]; however, for sets consisting of more than two elements the problem of listing the minimal clones ([10], Problem 12) is open.

Our result may be considered as a first step towards the solution of this problem. Indeed, the complete description of the maximal clones on a three-element set suggests how the maximal clones on a finite set can behave in general, and the same may be expected for minimal clones. On the other hand, it is known ([10], p. 115) that any minimal clone on a three-element set is generated by an essentially at most ternary operation. The unary case is trivial, and here we settle the binary case.

Throughout this paper, $\mathbf{n}$ denotes the set $\{0, 1, \ldots, n-1\}$. For the sake of simplicity, we consider operations on the base set $\mathbf{3}$ only and, for brevity, we call them *functions*. The symbol $[f]$ stands for the clone generated by the function f (i.e.,

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consisting of all term functions of $\langle 3; f\rangle$). Instead of $g\in[f]$ we write also $f\to g$; we say in this case that f *produces* g. Projections will often be referred to as *trivial functions*.

We start with two basic observations (see [10], Ch. 4.4):

(α) *A clone* C *is minimal iff it contains a non-trivial function, and* $f\to g$ *for any non-trivial* $f, g\in C$.

(β) *An essentially at least binary function* f *generating a minimal clone is idempotent* (i.e., $f(x, \dots, x)=x$ holds identically).

By (β), we have to consider idempotent functions only. Such a functions has a Cayley table of form

$$(*)\qquad \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & n_5 & n_4 \\ 1 & n_3 & 1 & n_2 \\ 2 & n_1 & n_0 & 2 \end{array}$$

where $n_i\in\mathbf{3}$ ($i=0, \dots, 5$). The function defined by ($*$) will be denoted by the integer $\sum_{i=0}^{5} 3^i n_i$. Thus, the functions we study will be numbered by 0, 1, ..., 728. E.g., 44 is the first binary projection (i.e., the function $f(x, y)=x$), and 424 is the second one. We shall denote our functions multiplicatively, with a subscript indicating the number of the considered function; e.g., we shall write $((xy)x)_{728}$ instead of $(x\,728\,y)728\,x$. For $f, g\in\mathbf{729}$, $f\cong g$ or $f\cong_a g$ means that $\langle 3; f\rangle$ and $\langle 3; g\rangle$ are

0 (3)	17	48	94 (6)	130 (6)
1	21	49	95	132
3	22	50	96	135 (6)
4	23	52 (6)	97	136
5	24	57 (6)	104 (3)	138
6	25	58	105	139
7	26 (6)	59	106	140 (6)
8	30	67	108	141
10 (6)	31	68 (6)	109 (6)	142
11 (6)	32	76 (6)	110	144
12	33 (6)	84 (6)	111 (6)	150 (4)
13	34	85	113	156 (6)
14	35	86	126	178 (2)
15	39 (6)	87	127	624 (1)
16	44 (2)	88	129	

Table 1

isomorphic or anti-isomorphic, respectively. The functions f and g are said to be *essentially distinct* if neither $f \cong g$ nor $f \cong_a g$ holds. In other words, the permutations of **3** and the dualization of functions generate a 12 element permutation group A on **729**, and f is essentially distinct from g iff they belong to distinct orbits of A.

For our aim, it is sufficient to study one representative from each orbit as the property of generating minimal clone is preserved under isomorphism and anti-isomorphism. We represent each orbit by its least element. In Table 1 we list the full system of representatives; the number in parentheses is the number of functions in the represented orbit if it does not equal 12.

Now we are ready to formulate the result announced above.

Theorem. *Every three-element groupoid with essentially binary operation having a minimal clone of term functions is isomorphic or anti-isomorphic to exactly one of the following twelve groupoids:*

$$\langle \mathbf{3}; f\rangle \text{ with } f \in M_2 = \{0, 8, 10, 11, 16, 17, 26, 33, 35, 68, 178, 624\}.$$

Proof. A three-element groupoid with the properties in the Theorem is, by (β), idempotent, and hence is isomorphic or anti-isomorphic to exactly one $\langle \mathbf{3}; f\rangle$ where f is an entry of Table 1. Therefore it is sufficient to prove that the functions listed in the Theorem generate minimal clones on **3** while the remaining functions in Table 1 do not. The second job is mainly of computational character, and we shall do it at first. We apply the following simple fact:

(γ) *For any function f, if there exist a clone C and a non-trivial function $g \in C$ such that $f \notin C$ and $f \to g$, then $[f]$ is not minimal.*

Put $f=3$ and let C_{ij} be the clone of all functions preserving the set $\{i, j\} \subseteq \mathbf{3}$. Then $3 \notin C_{02}$ and $3 \to ((xy)y)_3 = 0 \in C_{02}$. Thus, by ($\gamma$), [3] is not minimal. The same consideration (with C_{02} and $(xy)y$) is applicable also for the functions 4 ($((xy)y)_4 = 1$) 5 (2), 12 (9), 13 (10), 14 (11), 21 (18), 22 (19), 23 (20), 57 (33), 58 (34), 59 (35), 67 (43), 76 (52), 84 (9), 85 (10), 86 (11), 87 (15), 88 (16), 104 (182), 105 (186), 106 (187), 108 (36), 109 (37), 110 (38), 126 (207), 127 (208), 129 (210), 132 (213), 135 (9), 136 (10), 138 (42), 139 (43), 141 (69), 142 (70), and 156 (213). Similarly, by the help of $x(xy)$ we can settle the functions 30 ($(x(xy))_{30} = 37 \in C_{02}$), 31 (8), 39 (37), 48 (47), 50 (53), 95 (17), 96 (17), 97 (16), 111 (37), and 140 (26), and by $(xy)x$ the function 150 (178). Further, C_{12} and $x(xy)$ take care of 15 (17), 24 (26), 34 (43), while C_{12} with $x(yx)$ and $(xy)y$ settles 32 (40) and 113 (41), respectively. Finally, taking C_{01} and $(xy)y$, we can cast off also 144 (90).

A binary function satisfying the identity $(xy)(uv) = (xu)(yv)$ is called *medial*. If f is medial and $f \to g$, then the function g is also medial (cf. Prop. III. 3.2 in [2]). Thus, if a non-medial g produces a non-trivial medial f then $[g]$ is not minimal by (α). This is the case for $g=49$ and $f=41=(x(yx))_{49}$ as $((12)(02))_{49} = 1 \neq 2 =$

$=((10)(22))_{49}$ while one can check the mediality of 41 immediately. Therefore, [49] is not minimal.

In order to show that $[f]$ is not minimal for $f=1, 6, 25$, we apply (α) as follows. Observe that $1\to 0$; namely, $((xy)x)_1=0$; on the other hand, the binary term functions of $\langle \mathbf{3}; 0\rangle$ are 44, 424, and $(xy)_0=0$, i.e., 0 does not produce 1. Further, $(x(xy))_6=8$, but not $8\to 6$, as the binary term functions of $\langle \mathbf{3}; 8\rangle$ are 44, 424, 8, and 180. Similarly, $(x(yx))_{25}=17$, but the binary term functions of $\langle \mathbf{3}; 17\rangle$ are only 44, 424, 17, 181.

Clearly, $[f]$ is not minimal if there exists a non-trivial g such that $f\to g$ and $[g]$ is not minimal. Hence it follows that $[f]$ is not minimal for $f=7, 94$, and 130. Indeed, $((xy)x)_7=6$, $((xy)y)_{90}=91\cong_a 13$, $((xy)y)_{130}=211\cong_a 49$; and we have already shown that [6], [13], and [49] are not minimal.

[44] is the clone of all projections. Thus it remains to show that [52] is not minimal. Put $q(x, y, z)=((xy)(zx))_{52}$. Then $52\to q$ and q is not trivial, as $q(1, 0, 0)=1$ and $q(1, 2, 0)=2$. However, $q\to 52$ is not valid, since $q(x, x, y)=$ $=q(x, y, x)=q(x, y, y)=x$ and hence $[q]$ contains no essentially binary function. Now, by (α), [52] is not minimal.

Next we prove that the functions in M_2 generate minimal clones. Their Cayley tables can be seen here:

$$
0:\ \begin{array}{|ccc}\hline 0&0&0\\ 0&1&0\\ 0&0&2\end{array}\quad
8:\ \begin{array}{|ccc}\hline 0&0&0\\ 0&1&0\\ 2&2&2\end{array}\quad
10:\ \begin{array}{|ccc}\hline 0&0&0\\ 0&1&1\\ 0&1&2\end{array}\quad
11:\ \begin{array}{|ccc}\hline 0&0&0\\ 0&1&1\\ 0&2&2\end{array}
$$

$$
16:\ \begin{array}{|ccc}\hline 0&0&0\\ 0&1&1\\ 2&1&2\end{array}\quad
17:\ \begin{array}{|ccc}\hline 0&0&0\\ 0&1&1\\ 2&2&2\end{array}\quad
26:\ \begin{array}{|ccc}\hline 0&0&0\\ 0&1&2\\ 2&2&2\end{array}\quad
33:\ \begin{array}{|ccc}\hline 0&0&0\\ 1&1&0\\ 2&0&2\end{array}
$$

$$
35:\ \begin{array}{|ccc}\hline 0&0&0\\ 1&1&0\\ 2&2&2\end{array}\quad
68:\ \begin{array}{|ccc}\hline 0&0&0\\ 2&1&1\\ 1&2&2\end{array}\quad
178:\ \begin{array}{|ccc}\hline 0&0&2\\ 0&1&1\\ 2&1&2\end{array}\quad
624:\ \begin{array}{|ccc}\hline 0&2&1\\ 2&1&0\\ 1&0&2\end{array}
$$

The functions 0 and 10 are semilattice operations, hence they generate minimal clones (cf. [10], 4.4.4). It was proved by Płonka that 624 ($=2x+2y \bmod 3$) generates a minimal clone (see [8]). Demetrovics, Hannák and Marčenkov proved that [178] is also minimal ([3]; for a proof, see [7]).

Now we can prove that for each $f\in M_2$ any non-trivial binary $g\in[f]$ produces f. We do this by establishing the following property of any function in M_2:

A. *It produces no essentially binary function except itself and its dual.*

Indeed, a trivial computation shows that $x(xy)$, $x(yx)$, $(xy)x$, $(xy)y$, $(xy)(yx)\in$ $\in\{x, y, xy, yx\}$ whenever multiplication means anyone of the functions in M_2.

Hence $g\in\{x, y, xy, yx\}$ follows by induction on the depth of the shortest f-term representing g. Thus $g\to f$ provided g is not a projection.

It remains to prove that, for each $f\in M_2$, a non-trivial $g\in[f]$ *of arbitrary arity* does produce f. In view of the preceding paragraph it is enough to show only that g produces a non-trivial binary function. For the cases $f=8, 11, 16, 17, 26$ (and also for the known cases $f=0, 10, 178$) this can be done by the following argument. The restriction of f to $\mathbf{2}=\{0, 1\}$ is the minimum function $\wedge$; hence any term function g of $\langle\mathbf{3}; f\rangle$ has the form $x_1\wedge\ldots\wedge x_k$ when restricted to $\mathbf{2}$. Identifying all variables but one of g we obtain a binary function which is not a projection because its restriction to $\mathbf{2}$ is the minimum function again.

Our final task is to prove that 33, 35, and 68 generate minimal clones, too. First we check that each of them enjoys also the property

B. *It turns into a projection when restricted to a suitable two-element set.*

The two-element set in B is $\{0, 1\}$ for 33 and 35, and it is $\{1, 2\}$ for 68.

A function is called a *semi-projection* if it is not a projection and it turns into the same projection when any two of its variables are identified.

Lemma. *An idempotent function with properties* A *and* B *generates a minimal clone provided it produces no ternary semi-projection.*

Indeed, suppose that f has properties A and B, but $[f]$ is not minimal. Then there exists a non-trivial function $g\in[f]$ which does not produce f. The idempotence of f and A imply that g is at least ternary. We show that g produces an essentially ternary function. A well-known theorem of Świerczkowski ([12]; see also [5], p. 206) asserts that an at least three-element algebra with independent base set has only trivial operations. Hence it follows that any non-trivial function on $\mathbf{3}$ produces an at most ternary non-trivial function (cf. also [9], 4.4.7). In particular, g produces such a function h, which is, again by the idempotence of f and property A, essentially ternary. Let us identify two variables of h; then, by A, we always obtain a projection. Assume that two different identifications of variables furnish different projections; then the same is valid for the restriction of h to the two-element subset of $\mathbf{3}$ in property B. But this is impossible, as for a function composed from projections any identification of two variables gives the same projection. Hence h turns into the same projection under identification of any two of its variables, and, as it is not a projection, it has to be a semi-projection. We proved that f produces a semi-projection, which was needed.

In virtue of the lemma, it is enough to prove that none of 33, 35, and 68 does produce a ternary semi-projection. In these proofs, the actually considered function will be denoted as multiplication (no subscript will be used); we write pqr instead of $(pq)r$; finally, we write $f(x_1, \dots, x_k)=x_i\dots$ to indicate that x_i is the first-from-left entry in the term f. In this case $f(x_1, \dots, x_k)$ can be uniquely written in the form $x_i \cdot f_1(x_1, \dots, x_k)\dots f_n(x_1, \dots, x_k)$. We shall use the Cayley tables of the studied functions without further reference.

Case of 33. We need the following identities of $\langle \mathbf{3}; 33\rangle$:

$$xx = x(xy) = x(yx) = x, \quad (xy)x = (xy)(yx) = (xy)y = xy. \tag{1}$$

Suppose that $f(x, y, z)$ is a ternary 33-term of minimal length among those which are semi-projections: let $f(x, x, y)=f(x, y, y)=f(x, y, x)=x$ and $f(a, b, c)\neq a$ for suitable $a, b, c\in\mathbf{3}$. First suppose $a=0$. Let $f(x, y, z)=f_1(x, y, z)\cdot f_2(x, y, z)$; then by (1), $f_1(x, x, y)=f_1(x, y, y)=f_1(x, y, x)=x$, and, by the minimality of f, identically $f_1(x, y, z)=x$, i.e. $f(x, y, z)=x\cdot f_2(x, y, z)$, whence $f(a, b, c)= 0\cdot f_2(a, b, c)=0=a$, a contradiction. Therefore, $a\neq 0$.

Let, e.g., $a=1, b=2, c=0$. Now f is a 33-term with

$$f(x, y, y) = y, \quad f(1, 2, 0) \neq 1. \tag{2}$$

We shall be ready if we prove that for any 33-term g satisfying the requirements in (2) there exists a shorter 33-term also satisfying (2). Observe that $g(x, y, z)= =x\dots$, otherwise (1) implies $g(x, y, y)=y$ or $g(x, y, y)=yx$. Thus $g(x, y, z)= =x\cdot d_1(x, y, z)\dots d_n(x, y, z)$, and $g(x, y, y)=x\cdot d_1(x, y, y)\dots d_n(x, y, y)$. Hence, by (1), $d_i(x, y, y)\neq y$ for every i. On the other hand, $g(1, 2, 0)=1\cdot d_1(1, 2, 0)\dots d_n(1, 2, 0)\neq 1$, showing that $d_j(1, 2, 0)=2$ for at least one j. Now, $d_j(x, y, z)=y\dots=y\cdot h_1(x, y, z)\dots$ $\dots h_m(x, y, z)$. Using (1), we infer the existence of a k with $h_k(x, y, y)=x$. As $2\cdot h_1(1, 2, 0)\dots h_m(1, 2, 0)=d_j(1, 2, 0)=2$, we have $h_i(1, 2, 0)\neq 1$ for every i. In particular, $h_k(1, 2, 0)\neq 1$, i.e. h_k is the 33-term we required. For $\langle a, b, c\rangle=\langle 1,0,2\rangle$, the same argument works. As (12) is an automorphism of $\langle \mathbf{3}; 33\rangle$, we do not have to deal with the case $a=2$ separately.

Case of 35. The two-variable identities of $\langle \mathbf{3}; 35\rangle$ are

$$xx = x(xy) = x, \quad x(yx) = (xy)x = (xy)y = (xy)(yx) = xy. \tag{3}$$

As in the preceding case, we obtain that if f is a ternary 35-term which is a semi-projection of minimal length, and $f(x, y, y)=x, f(a, b, c)\neq a$, then $f(x, y, z)= =x\cdot f_2(x, y, z)$, and $a=1$. Furthermore, we have $f_2=f_{21}\cdot f_{22}$ and $f_{21}(a, b, c)=2$. Let, e.g., $\langle a, b, c\rangle=\langle 1, 2, 0\rangle$. Then $f_{21}(x, y, z)=y\cdot g_1(x, y, z)\dots g_n(x, y, z)$. From $f_{21}(2, 0, 0)=0\cdot g_1(2, 0, 0)\dots g_n(2, 0, 0)=0$ it follows $f_{21}(x, y, y)=y$ or $f_{21}(x, y, y)=yx$. Thus, $f_2(x, y, y)$ equals $y\cdot f_{22}(x, y, y)$ or $y\cdot x\cdot f_{22}(x, y, y)$, hence $f_2(x, y, y)=y$ or $f_2(x, y, y)=yx$ by (3). In both cases, $f(x, y, y)=xy$, a contradiction. If $\langle a, b, c\rangle=\langle 1, 0, 2\rangle$, then $f_{21}(x, y, z)=z\dots$ follows, and we can proceed similarly.

Case of 68. Again we need the two-variable identities of $\langle \mathbf{3}; 68\rangle$:

$$xx = x(xy) = (xy)y = (xy)(yx) = x, \quad x(yx) = (xy)x = xy. \tag{4}$$

Let f be a ternary 68-term which is a semi-projection; let $f(x, y, y)=x$ and $f(a, b, c)\neq a$. From $f(1, 0, 0)=1$ it follows $f(x, y, z)=x\cdot f_1(x, y, z)\dots f_n(x, y, z)$. This implies $a\neq 0$; so first suppose, e.g., $\langle a, b, c\rangle=\langle 1, 0, 2\rangle$. Now $(1\neq)f(1, 0, 2)= =1\dots=2$. At the same time, $f(1, 0, 2)=1\cdot f_1(1, 0, 2)\dots f_n(1, 0, 2)$. Hence there exists an odd number of f_i's such that $f_i(1, 0, 2)=0$. The last equality means $f_i(x, y, z)=y\dots$, therefore there is an odd number of f_i's whose first letter is y.

On the other hand, $f(x, y, z)=x\cdot f_1(x, y, x)\dots f_n(x, y, x)$. The identities (4) show that there exists an even number of f_j's with $f_j(x, y, x)=y$ or $f_j(x, y, x)=yx$. Observe that $f_j(x, y, z)=y\dots$ implies $f_j(x, y, x)=y$ or $f_j(x, y, x)=yx$, and $f_j(x, y, z)=x\dots$ or $f_j(x, y, z)=z\dots$ implies $f(x, y, x)=x$ or $f(x, y, x)=xy$ by (4). Hence we have an even number of f_j's with first letter y. This contradiction refutes $\langle a, b, c\rangle=\langle 1, 0, 2\rangle$. Assuming $\langle a, b, c\rangle=\langle 1, 2, 0\rangle$, we obtain a similar contradiction for the number of f_i's with first letter z. As (12) is an automorphism of $\langle \mathbf{3}; 68\rangle$, we have also settled the case $a=2$. Theorem is proved.

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JÓZSEF ATTILA UNIVERSITY
BOLYAI INSTITUTE
ARADI VÉRTANÚK TERE 1
6720 SZEGED, HUNGARY