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# A combinatorial proof of a theorem of P. Lévy on the local time

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*Dedicated to Professor B. Szőkefalvi-Nagy on the occasion of his 70th birthday*

## 1. Introduction

Let $\{W(t);\ t\geqq 0\}$ be a Wiener process and introduce the notations

$$M(t)=\sup_{0\leqq s\leqq t} W(s),\quad Y(t)=M(t)-W(t)$$

and for any Borel set $A$ let

$$H(A,t)=\lambda\{s\colon\ s\leqq t,\quad W(s)\in A\}$$

be the occupation time of $W$ where $\lambda$ is the Lebesgue measure. It is well-known that $H(A,t)$ (for any fixed $t$) is a random measure absolutely continuous with respect to $\lambda$ with probability 1. The Radon—Nikodym derivative of $H$ is called the local time of $W$ and it will be denoted by $\eta$ i.e. $\eta(x,t)$ is defined by

$$H(A,t)=\int_A \eta(x,t)\,dx.$$

Finally let $\eta(0,t)=\eta(t)$.

A celebrated result of P. Lévy reads as follows (see for example KNIGHT [7], Theorem 5.3.7).

Theorem A. *We have*

$$\{Y(t), M(t);\ t\geqq 0\}\overset{\mathcal{D}}{=}\{|W(t)|, \eta(t);\ t\geqq 0\}$$

*i.e. the finite dimensional distributions of the vector valued process* $\{(Y(t), M(t));\ t\geqq 0\}$ *are equal to the corresponding distributions of* $\{(|W(t)|, \eta(t));\ t\geqq 0\}$.

A natural question arises: what is the analogue of Theorem A in the case of random walk. In order to formulate our problem precisely introduce the following notations.

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Let $X_1, X_2, \ldots$ be a sequence of i.i.d.r.v.'s with $\mathbf{P}(X_1=1)=\mathbf{P}(X_1=-1)=1/2$ and let

$$S(0)=0, \quad S(n)=X_1+X_2+\ldots+X_n \quad (n=1,2,\ldots);$$
$$m(n)=\max_{1\le k\le n} S(k), \quad y(n)=m(n)-S(n),$$
$$\xi(0,n)=\xi(n)=\mathcal{N}\{k: k\le n, S(k)=0\}$$

where $\mathcal{N}\{\cdot\}$ is the cardinality of the set in brackets. Now our question is: does Theorem A remain true if we replace $W(t)$, $Y(t)$, $M(t)$; $\eta(t)$ by $S(n)$; $y(n)$, $m(n)$, $\xi(n)$ respectively (and $n$ runs over the integers). The answer of this question is negative. This fact can be seen from the following well-known

Theorem B.

$$\mathbf{P}\{\xi(2n)=k\}=\frac{1}{2^{2n-k}}\binom{2n-k}{n} \quad (k=0,1,2,\ldots,n)$$

*and*

$$\mathbf{P}\{m(n)=k\}=\frac{1}{2^n}\binom{n}{\left[\frac{n-k}{2}\right]} \quad (k=0,1,2,\ldots,n).$$

In spite of this disappointing fact we prove that Theorem A is "nearly true" for random walks. In fact we have

Theorem 1. *Let $X_1, X_2, \ldots$ be a sequence of i.i.d.r.v.'s with $\mathbf{P}(X_1=1)=\mathbf{P}(X_1=-1)=1/2$ defined on a probability space $\{\Omega, S, \mathbf{P}\}$. Then one can define a sequence $Z_1, Z_2, \ldots$ of i.i.d.r.v.'s on the same probability space $\{\Omega, S, \mathbf{P}\}$ such that $\mathbf{P}(Z_1=1)=\mathbf{P}(Z_1=-1)=1/2$ and*

$$n^{-1/4-\varepsilon} d\big((\hat{y}(n), \hat{m}(n)), (|S(n)|, \xi(n))\big) \to 0 \quad a.s.$$

*for any $\varepsilon>0$ where*

$$S(0)=T(0)=0, \quad S(n)=X_1+X_2+\ldots+X_n, \quad T(n)=Z_1+Z_2+\ldots+Z_n \quad (n=1,2,\ldots),$$
$$\xi(n)=\mathcal{N}\{k: k\le n, \quad S(k)=0\}, \quad \hat{m}(n)=\max_{1\le k\le n} T(k), \quad \hat{y}(n)=\hat{m}(n)-T(n)$$

*and $d$ is the Euclidean distance of the vectors i.e.*

$$d\big((a_1,a_2),(b_1,b_2)\big)=\big((b_2-a_2)^2+(b_1-a_1)^2\big)^{1/2}.$$

The proof of this Theorem is very elementary and will be presented in Section 2. In Section 3 we show that Theorem A can be obtained as a simple consequence of Theorem 1.

In Section 4 we show that replacing the number of roots of the random walk $S(n)$ of Theorem 1 by the number of crossing points of that walk we can obtain a much better rate than that of Theorem 1. In Section 5 as an application of Theorem A (or that of Theorem 1) we prove a Strassen-type law of iterated logarithm for local time.

## 2. Proof of Theorem 1

Using the notations of Theorem 1 we also introduce the following notations:

$$\begin{aligned} \varrho_1 &= \min\{i:\ i>0,\ S_i=0\},\\ \varrho_2 &= \min\{i:\ i>\varrho_1,\ S_i=0\},\\ &\ \vdots\\ \varrho_{l+1} &= \min\{i:\ i>\varrho_l,\ S_i=0\},\ \dots, \end{aligned}$$

$$Z_j = \begin{cases} -X_1 X_{j+1} & \text{if } 1 \leqq j \leqq \varrho_1-1,\\ -X_{\varrho_1+1}X_{j+2} & \text{if } \varrho_1 \leqq j \leqq \varrho_2-2,\\ \quad\vdots & \\ -X_{\varrho_l+1}X_{j+l+1} & \text{if } \varrho_l-(l-1) \leqq j \leqq \varrho_{l+1}-(l+1),\\ \quad\vdots & \end{cases}$$

The following lemma is immediately clear by the above definitions.

Lemma 1.

1°. $Z_1, Z_2, \dots$ *is a sequence of i.i.d.r.v.'s with* $\mathbf{P}(Z_1=+1)=\mathbf{P}(Z_1=-1)=1/2$.

2°.
$$T(k)-T(\varrho_l-l) = \sum_{j=\varrho_l-(l-1)}^{k} Z_j =$$
$$= -X_{\varrho_l+1}\sum_{j=\varrho_l-(l-1)}^{k} X_{j+l+1}\begin{cases} \leqq 0 & \text{if } \varrho_l-(l-1)\leqq k \leqq \varrho_{l+1}-l-3,\\ =0 & \text{if } k=\varrho_{l+1}-l-2,\\ =1 & \text{if } k=\varrho_{l+1}-(l+1). \end{cases}$$

3°. $\xi(\varrho_{l+1}) = l+1 = T(\varrho_{l+1}-(l+1)) = \hat{m}(\varrho_{l+1}-(l+1)) = \hat{m}(\varrho_{l+1}-\xi(\varrho_{l+1}))$
$(l=0,1,2,\dots)$.

4°. *For any* $\varrho_{l+1} \leqq n < \varrho_{l+2}$ *we have*

$$\varrho_{l+1}-\xi(\varrho_{l+1}) \leqq n-\xi(n) < \varrho_{l+2}-\xi(\varrho_{l+1}) = \varrho_{l+2}-\xi(\varrho_{l+2})+1,$$

*hence*

$$\xi(n) = \xi(\varrho_{l+1}) = \hat{m}(\varrho_{l+1}-\xi(\varrho_{l+1})) \leqq \hat{m}(n-\xi(n))$$

*and*

$$\xi(n) = \xi(\varrho_{l+2})-1 = \hat{m}(\varrho_{l+2}-\xi(\varrho_{l+2}))-1 \geqq \hat{m}(n-\xi(n))-1$$

*i.e.*

$$|\xi(n)-\hat{m}(n-\xi(n))| \leqq 1 \quad (n=1,2,\dots).$$

5°.
$$\hat{y}(k) = \begin{cases} |S(k+1)|-1 & \text{if } 1 \leqq k \leqq \varrho_1-2,\\ 0=|S(k)|-1 & \text{if } k=\varrho_1-1 \end{cases}$$

*and*

$$\hat{y}(k) = \begin{cases} |S(k+l)|-1 & \text{if } \varrho_{l-1}-(l-1) \leqq k \leqq \varrho_l-(l+1),\\ 0=|S(k+l-1)|-1 & \text{if } k=\varrho_l-l. \end{cases}$$

The following strong laws are known:

Lemma 2.

(2.1) $$\limsup_{n\to\infty} \frac{\xi(n)}{(2n\log\log n)^{1/2}} = 1 \quad a.s.$$

(cf. Kesten [6]).

(2.2) $$\limsup_{n\to\infty} \frac{\hat{m}(n)-\hat{m}(n-a_n)}{\left(a_n \log \frac{n}{a_n}\right)^{1/2}} \leqq 2^{1/2} \quad a.s.$$

(cf. Csörgő—Révész [4], Theorem 3.1.1) *where* $a_n = ((2+\varepsilon)n\,\text{loglog}\,n)^{1/2}$, $\varepsilon \geqq 0$.

(2.3) $$\limsup_{n\to\infty} \max_{0\leqq l\leqq b_n} \frac{|S(n+l)|-|S(n)|}{\left(2b_n \log \frac{n}{b_n}\right)^{1/2}} = 1 \quad a.s.$$

(cf. Csörgő—Révész [4], Theorem 3.1.1 and Remark 3.1.1) *where* $b_n = n^{1/2}\,\text{loglog}\,n$. *Consequently*

(2.4) $$\limsup_{n\to\infty} \frac{\hat{m}(n)-\hat{m}(n-\xi(n))}{(2n\,\text{loglog}\,n)^{1/4}(\log n)^{1/2}} \leqq 2^{1/2} \quad a.s.$$

*Further we have*

(2.5) $$\liminf_{n\to\infty} 4(n^{-2}\,\text{loglog}\,n)\varrho_n = 1 \quad a.s.$$

(cf. Mijnheer [9], p. 53 and Rényi [10] p. 236).

Remark. Theorem 3.1.1 of Csörgő—Révész [4] states that

$$\limsup_{n\to\infty} \sup_{0\leqq s\leqq a_n} \frac{T(n+s)-T(n)}{\left(a_n \log \frac{n}{a_n}\right)^{1/2}} = 2^{1/2} \quad \text{a.s.}$$

what easily implies (2.2). Applying Theorem A, Csáki—Csörgő—Földes—Révész [1] also proved that

$$\limsup_{n\to\infty} \frac{\hat{m}(n)-\hat{m}(n-a_n)}{\left(a_n \log \frac{n}{a_n}\right)^{1/2}} = 1 \quad \text{a.s.}$$

(2.4) of Lemma 2 and 4° of Lemma 1 together imply

Lemma 3. $$\limsup_{n\to\infty} \frac{|\xi(n)-\hat{m}(n)|}{(2n\,\text{loglog}\,n)^{1/4}(\log n)^{1/2}} \leqq \sqrt{2} \quad a.s.$$

For any positive integer $k$ let $l=l(k)$ be defined by $\varrho_{l-1}-(l-1)\leqq k<\varrho_l-l$. Then by (2.5) of Lemma 2 we have

Lemma 4. $$\lim_{n\to\infty}\frac{l(n)}{n^{1/2}\operatorname{loglog} n}=0\quad a.s.$$

5° of Lemma 1, (2.3) of Lemma 2 and Lemma 4 together imply

Lemma 5. $$\limsup_{n\to\infty}\frac{\big|\hat{y}(n)-|S(n)|\big|}{n^{1/4}((\log n)(\operatorname{loglog} n))^{1/2}}\leqq\sqrt{2}\quad a.s.$$

Lemmas 3 and 5 together prove Theorem 1.

## 3. Proof of Theorem A

The proof of Theorem A is based on Theorem 1 and the following invariance principle

Theorem C (Révész [11]). *Let $\{W(t);\ t\geqq 0\}$ be a Wiener process defined on a probability space $\{\Omega, S, \mathbf{P}\}$. Then on the same probability space $\Omega$ one can define a sequence $X_1, X_2, \dots$ of i.i.d.r.v.'s with $\mathbf{P}(X_1=+1)=\mathbf{P}(X_1=-1)=1/2$ such that*

$$\lim_{n\to\infty} n^{-1/4-\varepsilon}|\xi(n)-\eta(n)|=0\quad a.s.$$

*and*

$$\lim_{n\to\infty} n^{-1/4-\varepsilon}|S(n)-W(n)|=0\quad a.s.$$

*for any $\varepsilon>0$.*

In order to prove Theorem A it is enough to prove

Lemma 6.

$$A=\{Y(t_1), Y(t_2), \dots, Y(t_n), M(t_1), M(t_2), \dots, M(t_n)\}\overset{\mathcal{D}}{=}$$
$$\overset{\mathcal{D}}{=}\{|W(t_1)|, |W(t_2)|, \dots, |W(t_n)|, \eta(t_1), \eta(t_2), \dots, \eta(t_n)\}=B$$

*provided that $0<t_1<t_2<\dots<t_n\leqq 1$.*

Applying the well-known formula $\left\{\frac{W(ct)}{\sqrt{c}};\ t\geqq 0\right\}\overset{\mathcal{D}}{=}\{W(t);\ t\geqq 0\}$ (for any $c>0$) one gets

Lemma 7. *For any $T>0$ we have*

$$A\overset{\mathcal{D}}{=}\left\{\frac{Y(t_1T)}{T^{1/2}}, \frac{Y(t_2T)}{T^{1/2}}, \dots, \frac{Y(t_nT)}{T^{1/2}}, \frac{M(t_1T)}{T^{1/2}}, \frac{M(t_2T)}{T^{1/2}}, \dots, \frac{M(t_nT)}{T^{1/2}}\right\}$$

*and*

$$B \stackrel{\mathcal{D}}{=} \left\{ \frac{|W(t_2T)|}{\sqrt{T}}, \frac{|W(t_2T)|}{\sqrt{T}}, \dots, \frac{|W(t_nT)|}{\sqrt{T}}, \frac{\eta(t_1T)}{\sqrt{T}}, \frac{\eta(t_2T)}{\sqrt{T}}, \dots, \frac{\eta(t_nT)}{\sqrt{T}} \right\}.$$

By Theorem C we have

Lemma 8. *One can define a random walk* $S(1), S(2), \dots$ *on the probability space of* $W$ *such that*

$$\frac{|W(t_iT)|}{\sqrt{T}} = \frac{|S(t_iT)|}{\sqrt{T}} + o(T^{-1/4+\varepsilon}) \quad (i = 1, 2, \dots, n)$$

*and*

$$\frac{\eta(t_iT)}{\sqrt{T}} = \frac{\xi(t_iT)}{\sqrt{T}} + o(T^{-1/4+\varepsilon}) \quad (i = 1, 2, \dots, n).$$

Applying Theorem 1 we have

Lemma 9. *Given the random walk of Lemma* 8 *one can define another random walk* $T(1), T(2), \dots$ *such that*

$$\frac{|S(t_iT)|}{\sqrt{T}} = \frac{\hat{y}(t_iT)}{\sqrt{T}} + o(T^{-1/4+\varepsilon}) \quad (i = 1, 2, \dots, n)$$

*and*

$$\frac{\xi(t_iT)}{\sqrt{T}} = \frac{\hat{m}(t_iT)}{\sqrt{T}} + o(T^{-1/4+\varepsilon}) \quad (i = 1, 2, \dots, n).$$

Applying again Theorem C we get

Lemma 10. *Given the random walk* $T(1), T(2), \dots$ *of Lemma* 9 *one can define a Wiener process* $\{\overline{W}(t);\ t > 0\}$ *such that*

$$\frac{\hat{y}(t_iT)}{\sqrt{T}} = \frac{\overline{Y}(t_iT)}{\sqrt{T}} + o(T^{-1/4+\varepsilon}) \quad (i = 1, 2, \dots, n)$$

*and*

$$\frac{\hat{m}(t_iT)}{\sqrt{T}} = \frac{\overline{M}(t_iT)}{\sqrt{T}} + o(T^{-1/4+\varepsilon}) \quad (i = 1, 2, \dots, n)$$

*where*

$$\overline{M}(t) = \sup_{0 \le s \le t} \overline{W}(s) \quad \text{and} \quad \overline{Y}(t) = \overline{M}(t) - \overline{W}(t).$$

Lemmas 8 and 10 together imply

Lemma 11.

$$\left( \frac{|S(t_1T)|}{\sqrt{T}}, \frac{|S(t_2T)|}{\sqrt{T}}, \dots, \frac{|S(t_nT)|}{\sqrt{T}}, \frac{\xi(t_1T)}{\sqrt{T}}, \frac{\xi(t_2T)}{\sqrt{T}}, \dots, \frac{\xi(t_nT)}{\sqrt{T}} \right) =$$

$$= b(T) \stackrel{\mathcal{D}}{\Rightarrow} B \quad \text{as} \quad T \to \infty$$

*and*

$$\left(\frac{\hat{y}(t_1T)}{\sqrt{T}}, \frac{\hat{y}(t_2T)}{\sqrt{T}}, \ldots, \frac{\hat{y}(t_nT)}{\sqrt{T}}, \frac{\hat{m}(t_1T)}{\sqrt{T}}, \frac{\hat{m}(t_2T)}{\sqrt{T}}, \ldots, \frac{\hat{m}(t_nT)}{\sqrt{T}}\right) =$$
$$= a(T) \overset{\mathcal{D}}{\Rightarrow} A \quad \text{as} \quad T \to \infty.$$

By Lemma 9 the limit distributions of $a(T)$ and $b(T)$ cannot be different. Hence we have Lemma 6, and hence Theorem A.

## 4. Roots and crossings

Theorem 1 says that the vector $(|S(n)|, \xi(n))$ can be approximated by the vector $(\hat{y}(n), \hat{m}(n))$ in order $n^{1/4+\varepsilon}$ while Theorem C says that the vector $(\eta(n), W(n))$ can be approximated by the vector $(\xi(n), S(n))$ in the same order $n^{1/4+\varepsilon}$. It is natural to ask whether this order is the best possible or not. Unfortunately we do not know the answer of this question. However we can show that considering the number of crossings $\theta(n)$ instead of the number of roots $\xi(n)$ better rates can be achieved in Theorems 1 and C.

Let

(4.1) $$\theta(n) = \mathcal{N}\{k:\ k \leqq n,\ S(k-1)S(k+1) < 0\}$$

be the number of crossings. Then we have

Theorem 2. *Let* $X_1, X_2, \ldots$ *be a sequence of i.i.d.r.v.'s with* $\mathbf{P}(X_1=1)= =\mathbf{P}(X_1=-1)=1/2$ *defined on a probability space* $\{\Omega, S, \mathbf{P}\}$. *Then one can define a sequence* $Z_1, Z_2, \ldots$ *of i.i.d.r.v.'s on the same probability space* $\{\Omega, S, \mathbf{P}\}$ *such that* $\mathbf{P}(Z_1=1)=\mathbf{P}(Z_1=-1)=1/2$ *and*

(4.2) $$|\hat{m}(n) - 2\theta(n)| \leqq 1,$$

(4.3) $$\big|\hat{y}(n) - |S(n)|\big| \leqq 2$$

*for* $n = 1, 2, \ldots$; *where*

$$S(0) = T(0) = 0, \quad S(n) = X_1 + X_2 + \ldots + X_n, \quad T(n) = Z_1 + Z_2 + \ldots + Z_n$$
$$(n = 1, 2, \ldots),$$
$$\hat{m}(n) = \max_{0 \leqq k \leqq n} T(k), \quad \hat{y}(n) = \hat{m}(n) - T(n).$$

Proof. Let

$$\tau_1 = \min\{i:\ i > 0,\ S(i-1)S(i+1) < 0\},$$
$$\tau_2 = \min\{i:\ i > \tau_1,\ S(i-1)S(i+1) < 0\},$$
$$\vdots$$
$$\tau_{l+1} = \min\{i:\ i > \tau_l,\ S(i-1)S(i+1) < 0\}, \ldots$$

and

$$Z_j = \begin{cases} -X_1X_{j+1} & \text{if } 1 \leqq j \leqq \tau_1, \\ X_1X_{j+1} & \text{if } \tau_1+1 \leqq j \leqq \tau_2, \\ \vdots & \\ (-1)^{l+1}X_1X_{j+1} & \text{if } \tau_l+1 \leqq j \leqq \tau_{l+1} \\ \vdots & \end{cases}.$$

This transformation was given in CSÁKI and VINCZE [3]. The following lemma is clearly true.

Lemma 12.

1°. $Z_1, Z_2, \ldots$ *is a sequence of i.i.d.r.v.'s with* $\mathbf{P}(Z_1=+1)=\mathbf{P}(Z_1=-1)=1/2$.

2° $$T(k)-T(\tau_l) = \sum_{j=\tau_l+1}^{k} Z_j = (-1)^{l+1}X_1 \sum_{j=\tau_l+1}^{k} X_{j+1} =$$

$$= (-1)^{l+1}X_1\big(S(k+1)-S(\tau_l+1)\big) \begin{cases} \leqq 1 & \text{if } \tau_l+1 \leqq k \leqq \tau_{l+1}-2, \\ = 1 & \text{if } k = \tau_{l+1}-1, \\ = 2 & \text{if } k = \tau_{l+1}. \end{cases}$$

3°. $2\theta(\tau_l)=2l=T(\tau_l)=\hat{m}(\tau_l)$, $l=1,2,\ldots$.

4°. *For any* $\tau_l \leqq n < \tau_{l+1}$ *we have* $\theta(n)=l$, $2l \leqq \hat{m}(n) \leqq 2l+1$, *consequently* $0 \leqq \hat{m}(n)-2\theta(n) \leqq 1$.

5°. $$T(k) = \begin{cases} 2l+1-|S(k+1)| & \text{if } \tau_l+1 \leqq k \leqq \tau_{l+1}-1, \\ 2l+2-|S(k)| & \text{if } k = \tau_{l+1}, \end{cases}$$

*therefore*

$$\hat{y}(k) = \hat{m}(k)-T(k) \leqq |S(k+1)| \leqq |S(k)|+1$$

*and*

$$\hat{y}(k) = \hat{m}(k)-T(k) \geqq |S(k+1)|-1 \geqq |S(k)|-2.$$

This proves Theorem 2.

Corollary. *On a rich enough probability space* $\{\Omega, S, \mathbf{P}\}$ *one can define a Wiener process* $\{W(t); t \geqq 0\}$ *and a sequence* $X_1, X_2, \ldots$ *of i.i.d.r.v.'s with* $\mathbf{P}(X_1=+1)=\mathbf{P}(X_1=-1)=1/2$ *such that*

(4.4) $$\big||S(n)|-|W(n)|\big| = O(\log n) \quad a.s.$$

*and*

(4.5) $$|2\theta(n)-\eta(n)| = O(\log n) \quad a.s.$$

*where* $S(n)=X_1+X_2+\ldots+X_n$, $\theta(n)$ *is defined by* (4.1) *and* $\eta(\cdot)$ *is the local time at zero of* $W(\cdot)$.

Proof. Let us start with the random walk $X_1, X_2, \ldots$ . Then construct a random walk $Z_1, Z_2, \ldots$ according to Theorem 2. Then by the theorem of KOMLÓS, MAJOR and TUSNÁDY [8] one can construct a Wiener process $W_1(t)$ such that

$$\sup_{k\leqq n}|T(k)-W_1(k)| = O(\log n) \quad \text{a.s.} \tag{4.6}$$

where $T(k)=Z_1+\ldots+Z_k$. Put $M_1(t)=\max_{0\leqq s\leqq t} W_1(s)$. Then according to Lévy's theorem (Theorem A), $|W(t)|=M_1(t)-W_1(t)$ is the absolute value of a Wiener process whose local time $\eta(t)=M_1(t)$. Now

$$|\eta(n)-2\theta(n)| \leqq |M_1(n)-\hat{m}(n)|+|\hat{m}(n)-2\theta(n)|,$$

where $\hat{m}(n)=\max_{1\leqq k\leqq n} T(k)$. (4.5) follows from (4.2) and (4.6). Furthermore

$$\big||S(n)|-|W(n)|\big| \leqq \big||S(n)|-\hat{y}(n)\big|+|\hat{m}(n)-M_1(n)|+|T(n)-W_1(n)|,$$

where $\hat{y}(n)=\hat{m}(n)-T(n)$. (4.4) follows from (4.3) and (4.6).

## 5. A Strassen-type law of iterated logarithm

Let $\{W(t);\ t\geqq 0\}$ be a Wiener process and let

$$w_T(x) = w(x) = b_T^{-1}W(xT) \quad (0\leqq x\leqq 1)$$

where

$$b_T = (2T\log\log T)^{1/2} \quad (T > e).$$

Further let $\mathcal{S}\subset C(0,1)$ be the set of absolutely continuous functions (with respect to the Lebesgue measure) for which

$$f(0) = 0 \text{ and } \int_0^1 (f'(x))^2\,dx \leqq 1.$$

The celebrated Strassen's (functional) law of iterated logarithm says:

Theorem D. [13] *The net* $\{W_T(x);\ 0\leqq x\leqq 1\}$ *is relatively compact in* $C(0,1)$ *with probability* 1 *and the set of its limit points is* $\mathcal{S}$.

It is an interesting question to characterize the limit points of $\eta(x, T)$ as $T\to\infty$. DONSKER and VARADHAN [5] solved this problem. Here we intend to present a result characterizing the limit points of the net

$$y_T(x) = y(x) = b_T^{-1}\eta(0, xT) \quad (0\leqq x\leqq 1).$$

Since $y_T(x)$ $(0\leqq x\leqq 1)$ for any fixed $T$ is a non-decreasing function, its limit points must be also non-decreasing. Introduce the following

Definition. Let $\mathcal{M}\subset\mathcal{S}$ be the set of non-decreasing elements of $\mathcal{S}$.

Then we formulate our

Theorem 3. *The net* $\{y_T(x);\, 0\leqq x\leqq 1\}$ *is relatively compact in* $C(0,1)$ *with probability* 1 *and the set of its limit points is* $\mathcal{M}$.

Proof. This result is a trivial consequence of Theorems A and D.

It looks more interesting to characterize jointly the limit points of the vectors $\{w_T(x), y_T(x);\, 0\leqq x\leqq 1\}$ $(T\to\infty)$. Intuitively it is clear enough that $y_T(x)$ must be constant in an interval where $w_T(x)\neq 0$. Hence in order to characterize the set of possible limit points it is natural to introduce the following

Definition. Let $\mathcal{N}$ be the set of those two-dimensional vector valued functions $h(x)=(f(x), g(x))$ $(0\leqq x\leqq 1)$ for which

(i) $f$ and $g$ are absolutely continuous in $(0, 1)$ with respect to the Lebesgue-measure,

(ii) $f(0)=g(0)=0$,

(iii) $g$ is non-decreasing,

(iv) $f(x)g'(x)\equiv 0$ $(0<x<1)$,

(v) $\int_0^1 (f'(x)+g'(x))^2 dx \leqq 1$.

Now we have

Theorem 4. *The net* $\{w_T(x), y_T(x);\, 0\leqq x\leqq 1\}$ $(T\to\infty)$ *is relatively compact in* $C(0,1)\times C(0,1)$ *with probability* 1 *and the set of its limit points is* $\mathcal{N}$.

This Theorem is again a simple consequence of Theorems A and D. Theorem 4 clearly implies the following interesting

Consequence. *The net* $\{w_T(1), y_T(1)\}=\{b_T^{-1}W(T), b_T^{-1}\eta(0,T)\}$ *is relatively compact in the plane* $R^2$ *with probability* 1 *and the set of its limit points is the triangle*

$$T=\{(x,y):\ -1\leqq x\leqq 1,\quad 0\leqq y\leqq |x-1|\}$$

*which, in turn, also implies*

$$\limsup_{T\to\infty} b_T^{-1}\big(\eta(0,T)+|W(T)|\big) = 1 \quad a.s.$$

Remark. Theorem 1 shows that our Theorems 3 and 4 as well as the above Consequence remain true if we investigate the properties of the random walk $S(1), S(2), \ldots$ of the introduction instead of a Wiener process. The invariance principles of Csáki—Révész [2] and Révész [12] shows that these results can be extended for more general random walks.

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