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On how long interval is the empirical characteristic function uniformly consistent?

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In honour of Professor Béla Szőkefalvi-Nagy on his seventieth birthday

Introduction, results, and discussion

Let $X_1, X_2, \dots$ be a sequence of independent identically distributed d-dimensional random vectors, $d \geqq 1$, defined on a probability space $(\Omega, \mathcal{A}, P)$, with common distribution function $F(x)$, $x \in \mathbf{R}^d$, and characteristic function

$$C(t) = \int_{\mathbf{R}^d} e^{i\langle t, x\rangle}\, dF(x), \quad t = (t_1, \dots, t_d) \in \mathbf{R}^d,$$

where $\langle \cdot, \cdot \rangle$ stands for the inner product of $\mathbf{R}^d$. The n^{th} empirical characteristic function of the sequence is

$$C_n(t) = \frac{1}{n} \sum_{j=1}^{n} e^{i\langle t, X_j\rangle} = \int_{\mathbf{R}^d} e^{i\langle t, x\rangle}\, dF_n(x), \quad t = (t_1, \dots, t_d) \in \mathbf{R}^d,$$

where $F_n(x)$, $x \in \mathbf{R}^d$, denotes the empirical distribution function of $X_1, \dots, X_n$. By any advanced form of the strong law of large numbers, $\lim_{n\to\infty} C_n(t) = C(t)$ almost surely at each fixed $t \in \mathbf{R}^d$, but more than this is still trivial. Indeed, the d-variate Glivenko—Cantelli theorem for F_n and the d-variate continuity theorem of Lévy readily imply that

(1) $$\lim_{n\to\infty} \sup_{|t| \leqq T} |C_n(t) - C(t)| = 0$$

almost surely for any fixed positive $T < \infty$; that is, in statistical terminology, C_n is a strongly uniformly consistent estimator for C on any fixed bounded subset of $\mathbf{R}^d$.

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On the other hand, $C_n(t)$ is a d-variate almost periodic function for each n, at each $\omega \in \Omega$ where it is defined, and hence if

$$\Delta_n = \sup_{t \in \mathbf{R}^d} |C_n(t) - C(t)|$$

converges to zero at *only one single* $\omega \in \Omega$, then by Satz XXVI of Bochner [1] the limiting function $C(t)$ must be almost periodic. But then, by a simple extension of the corresponding univariate result (Corollary 1 of Theorem 3.2.3 of Lukacs [5]; here we use the Eindeutigkeitssatz (Satz XXXVII) of Bochner [1] instead of the corresponding univariate uniqueness theorem of Bohr), $C(t)$ must belong to a purely discrete F, i.e., it is of the form

$$C(t) = \sum_k q_k e^{i\langle t, \lambda_k \rangle}, \quad q_k \geqq 0, \quad \sum_k q_k = 1,$$

with a finite or infinite sequence of vectors λ_k. That Δ_n *does* converge to zero almost surely in such a case was pointed out by Feuerverger and Mureika [4] for univariate characteristic functions, i.e., for discrete real random variables, and later by Csörgő [3] for $d \geqq 1$.

So if we wish to say more than (1) in the general case, then we are lead to considering the quantities

$$\Delta_n(T_n) = \sup_{|t| \leqq T_n} |C_n(t) - C(t)|$$

for some sequence $\{T_n\}$ of positive numbers converging to infinity. This has been first done by Feuerverger and Mureika [4] in the univariate case, who showed that if $d=1$ and the singular part of C vanishes at infinity, then $\lim_{n\to\infty} \Delta_n(T_n)=0$ almost surely whenever $T_n = o((n/\log n)^{1/2})$. This result was improved by Csörgő [2, 3] ($d=1$ and $d \geqq 1$, respectively) who showed that $\lim_{n\to\infty} \Delta_n(T_n)=0$ almost surely for any characteristic function whenever $T_n = o((n/\log\log n)^{1/(2d)})$. The latter result is in fact an easy consequence of Kiefer's well-known d-variate extension of the Chung—Smirnov univariate law of the iterated logarithm for F_n. This familiar rate has made us think for a longer time that it was perhaps best possible, although its dependence on the dimension appeared strange. It is in fact very far from being best possible, and the final solution presented below is rather surprising.

Theorem 1. *For any d-variate characteristic function C, if $\lim_{n\to\infty} (\log T_n)/n = 0$ then $\lim_{n\to\infty} \Delta_n(T_n) = 0$ almost surely.*

Theorem 2. *If $\lim_{|t_k|\to\infty} |C(t_1, \ldots, t_k, \ldots, t_d)| = 0$ for some k, $1 \leqq k \leqq d$, and if $\overline{\lim}_{n\to\infty} (\log T_n)/n > 0$, then there exists a positive ε such that*

$$\overline{\lim_{n\to\infty}} P\{\Delta_n(T_n) \geqq \varepsilon\} > 0.$$

We see that the rate $T_n = \exp(o(n))$ is not only best possible in general for almost sure convergence, but if we take any faster sequence T_n then even stochastic convergence cannot be retained for any characteristic function vanishing at infinity along at least one path.

The proof of Theorem 2 implies that if $\log T_{n_k} \geqq \gamma n_k$, $k=1, 2, \ldots$, for a subsequence $\{n_k\}$ of the natural numbers and some $\gamma > 0$, then for any subsequence $\{m_k\}$ of $\{n_k\}$ the sequence

$$\sup_{|t| \leqq T_{m_k}} |C_{m_k}(t) - C(t)|$$

does not converge to zero in probability. Since the topology of stochastic convergence is metrisable, and since for every $T > 0$

$$P\{\sup_{|t| \leqq T} |C_{m_k}(t) - C(t)| > \varepsilon_k(T)\} > 0$$

with some $\varepsilon_k(T) > 0$ (the opposite could only occur in the case when $C(t) = \exp(i\langle t, \lambda\rangle)$ with some vector λ, i.e., when the distribution is degenerate at λ, but this case is excluded under the hypothesis of Theorem 2), the following somewhat sharper form of Theorem 2 is also true: *If* $\lim_{|t_k| \to \infty} |C(t_1, \ldots, t_k, \ldots, t_d)| = 0$ *for some* k, $1 \leqq k \leqq d$, *and if* $\log T_{n_k} \geqq \gamma n_k$, $k = 1, 2, \ldots$, *for a subsequence* $\{n_k\}$ *of positive integers and some* $\gamma > 0$, *then there is a positive* ε *such that*

$$P\{\sup_{|t| \leqq T_{n_k}} |C_{n_k}(t) - C(t)| \geqq \varepsilon\} \geqq \varepsilon$$

is satisfied for all k.

The proof in the positive direction is quite straightforward. Essentially it imitates that of the easier half of the continuity theorem in conjunction with the exponential inequality of Bernštein. Exactly the same approach was taken in [2, 3] for handling the much harder problem of weak convergence, or strong approximation of the process $n^{1/2}(C_n(\cdot) - C(\cdot))$. It was not realised then that this approach is also suitable for the easier problem of uniform consistency on long intervals. On the other hand, the proof of Theorem 2 shows that the behaviour of $\Delta_n(T_n)$ is intimately connected with an old number-theoretic problem. Indeed, our starting point will be Dirichlet's classic result in diophantine approximation.

Having Theorems 1 and 2 above, further questions can be posed which may be irrelevant from the statistical point of view but are interesting as purely probabilistic problems. Set $L_k = \overline{\lim}_{|t_k| \to \infty} |C(t_1, \ldots, t_k, \ldots, t_d)|$, $k = 1, \ldots, d$. Can T_n be faster than $\exp(o(n))$ if $L = \min(L_1, \ldots, L_d) > 0$ but the distribution is not purely discrete? In the positive direction we do not have anything more than Theorem 1. In the negative direction the hardest subcase seems to be the one when $L = 1$. Otherwise a slight modification of the proof of Theorem 2 below also gives the following result: *If* $0 < L < 1$ *and* $\overline{\lim}_{n \to \infty} (\log T_n)/n > \log(2\pi/\arccos L)$, *then* $\overline{\lim}_{n \to \infty} P\{\Delta_n(T_n) \geqq \varepsilon\} > 0$ *with some positive* ε.

Proofs

Theorem 1. Let $\varepsilon>0$ be artbitrarily small, $\varepsilon\leqq 2$, and choose $K=K(\varepsilon, F)$ so large that

$$\int_{|x|\geqq K} dF(x) < \frac{\varepsilon}{8}.$$

Writing $D_n(t)=B_n(t)-B(t)$, we have

$$\Delta_n(T_n) \leqq \sup_{|t|\leqq T_n} |D_n(t)| + \sup_{|t|\leqq T_n} |B_n(t)-C_n(t)| + \sup_{|t|\leqq T_n} |B(t)-C(t)|$$

with the truncated integrals

$$B(t) = \int_{|x|\leqq K} e^{i\langle t,x\rangle}\, dF(x),$$

$$B_n(t) = \int_{|x|\leqq K} e^{i\langle t,x\rangle}\, dF_n(x) = \frac{1}{n}\sum_{j=1}^{n} e^{i\langle t,X_j\rangle}\chi(|X_j|\leqq K),$$

where $\chi(A)$ denotes the indicator of the event A. The second term is

$$\frac{1}{n}\sup_{|t|\leqq T_n}\left|\sum_{j=1}^{n} e^{i\langle t,X_j\rangle}\chi(|X_j|>K)\right| \leqq \frac{1}{n}\sum_{j=1}^{n}\chi(|X_j|>K),$$

and these bounds converge almost surely to $\int_{|x|>K} dF(x)$ which is also a bound for the third term.

Let us cover the cube $[-T_n, T_n]^d$ by $N_n=([(8Kd^{3/2}T_n)/\varepsilon]+1)^d$ disjoint small cubes $\Delta_1, \ldots, \Delta_{N_n}$, the edges of each of which are of length $\varepsilon/(4Kd^{3/2})$, and let $t_1, \ldots, t_{N_n}$ be the centres of these cubes. Then

$$\sup_{|t|\leqq T_n} |D_n(t)| \leqq \max_{1\leqq k\leqq N_n} |D_n(t_k)| + \max_{1\leqq k\leqq N_n}\sup_{t\in\Delta_k} |D_n(t)-D_n(t_k)| \leqq \max_{1\leqq k\leqq N_n}|D_n(t_k)|+\frac{\varepsilon}{4},$$

for

$$|D_n(s)-D_n(t)| \leqq |B_n(s)-B_n(t)|+|B(s)-B(t)| \leqq$$

$$\leqq \frac{1}{n}\sum_{j=1}^{n}|\langle s-t, X_j\rangle|\chi(|X_j|\leqq K) + \int_{|x|\leqq K}|\langle s-t, x\rangle|\, dF(x) \leqq 2\,dK|s-t|, \quad s, t\in\mathbf{R}^d.$$

(In fact, the finer almost sure upper bound

$$4\int_{|x|\leqq K}\left|\sin\frac{\langle s-t, x\rangle}{2}\right| dF(x)$$

can be given here, but this is irrelevant in the present context, yielding the same result). Summing up;

$$\Delta_n(T_n) \leqq \max_{1\leqq k\leqq N_n}|D_n(t_k)|+\frac{\varepsilon}{2} \tag{2}$$

almost surely for large enough n, the threshold depending on ω. Now

$$p_n = P\left\{\max_{1 \leqq k \leqq N_n} |D_n(t_k)| > \frac{\varepsilon}{2}\right\} \leqq N_n \sup_{t \in \mathbf{R}^d} P\left\{|D_n(t)| > \frac{\varepsilon}{2}\right\} \leqq$$

$$\leqq MT_n^d \sup_{t \in \mathbf{R}^d} \left(P\left\{\frac{1}{n}\left|\sum_{j=1}^n R_j(t)\right| > \frac{\varepsilon}{4}\right\} + P\left\{\frac{1}{n}\left|\sum_{j=1}^n I_j(t)\right| > \frac{\varepsilon}{4}\right\}\right)$$

with some constant $M = M(\varepsilon, F, d)$, where the random variables

$$R_j(t) = (\cos\langle t, X_j\rangle)\chi(|X_j| \leqq K) - \int_{|x| \leqq K} \cos\langle t, x\rangle\, dF(x), \quad j = 1, \dots, n,$$

are independent, $|R_j(t)| \leqq 2$, $ER_j(t) = 0$, and

$$v^2(t) = ER_j^2(t) = \int_{|x| \leqq K} \cos^2\langle t, x\rangle\, dF(x) - \Big(\int_{|x| \leqq K} \cos\langle t, x\rangle\, dF(x)\Big)^2 \leqq 1.$$

The random functions $I_j(t)$, $j = 1, \dots, n$, are defined with the cosine function replaced by the sine; and hence these are also independent and identically distributed with $|I_j(t)| \leqq 2$, $EI_j(t) = 0$ and $EI_j^2(t) \leqq 1$. Therefore the Bernštein inequality ([6], Chapter X, §1, Lemma 1) gives

$$P\left\{\frac{1}{n}\left|\sum_{j=1}^n R_j(t)\right| > \frac{\varepsilon}{4}\right\} \leqq \begin{cases} 2e^{-\frac{\varepsilon n}{32}}, & \text{if } \varepsilon \geqq 2v^2(t), \\ 2e^{-\frac{\varepsilon^2 n}{64 v^2(t)}}, & \text{if } \varepsilon \leqq 2v^2(t). \end{cases}$$

Since $v^2(t) \leqq 1$ and $\varepsilon \leqq 2$, the probability in question is not greater than $2\exp(-\varepsilon^2 n/64)$, and the same holds for the other one with the I_j's. Thus

$$p_n \leqq 4MT_n^d e^{-\frac{\varepsilon^2 n}{64}}.$$

Let $\delta < \varepsilon^2/(64d)$. Then for large enough n, $T_n \leqq \exp(\delta n)$, and hence $\sum_{n=1}^{\infty} p_n < \infty$. The Borel—Cantelli lemma and (2) give the desired result.

Theorem 2. Since

$$\sup_{|(t_1, \dots, t_d)| \leqq T_n} |C_n(t_1, \dots, t_d) - C(t_1, \dots, t_d)| \geqq$$

$$\geqq \sup_{-T_n \leqq t_k \leqq T_n} |C_n(0, \dots, 0, t_k, 0, \dots, 0) - C(0, \dots, 0, t_k, 0, \dots, 0)|,$$

where $C_n(0, \dots, 0, t_k, 0, \dots, 0)$ is the empirical characteristic function of the k^{th} components of $X_1, \dots, X_n$ and $C(0, \dots, 0, t_k, 0, \dots, 0)$ is the common characteristic function of these components, it is clearly enough to prove the theorem in the univariate case. We assume therefore that $d = 1$, i.e., that $X = \{X_1, X_2, \dots\}$ are independent real random variables with common characteristic function $C(t)$, $-\infty < t < \infty$, with $\lim_{|t| \to \infty} |C(t)| = 0$.

Let

$$S_n(t) = S_n(t; X) = \sum_{j=1}^{n} e^{itX_j}.$$

Then $C_n(t)=n^{-1}S_n(t)$, and the theorem will easily follow from the following proposition of independent interest, in which there is no assumption whatsoever on the common characteristic function, or distribution, of the independent variables $X_1, X_2, \dots$.

Proposition. *If $\mathcal{N}=\{n_k\}_{k=1}^{\infty}$ denotes an arbitrary nondecreasing sequence of natural numbers and if*

$$p_\alpha(\mathcal{N}) = \sup_{M>0} \inf_{K>0} \overline{\lim_{k\to\infty}} P\left\{ \sup_{K\le t\le \alpha^{n_k}} \frac{|S_{n_k}(t)|}{n_k} \ge M \right\}$$

then $p_\alpha(\mathcal{N})>0$ for every $\alpha>1$.

Indeed, taking for granted the validity of this Proposition, Theorem 2 can be proved as follows. By assumption there is a $\gamma>0$ such that $T_{n_k}\ge e^{\gamma n_k}$ for some subsequence $\{n_k\}$ of the positive integers. On applying the Proposition with $\alpha=e^\gamma>1$, we obtain an $M>0$ and a $\delta>0$ such that

$$P\left\{ \sup_{K\le t\le e^{\gamma n_k}} \frac{|S_{n_k}(t)|}{n_k} \ge M \right\} \ge \delta$$

for every $K>0$. Choosing K so large that $|C(t)|<M/2$ be satisfied for $t\ge K$ and then putting $\varepsilon=M/2$, we obtain

$$\overline{\lim_{k\to\infty}} P\left\{ \sup_{|t|\le T_{n_k}} |C_{n_k}(t)-C(t)| > \varepsilon \right\} \ge \overline{\lim_{k\to\infty}} P\left\{ \sup_{K\le t\le e^{\gamma n_k}} |C_{n_k}(t)| \ge M \right\} \ge \delta,$$

which is the desired result.

In order to prove the Proposition, define

$$\beta(\mathcal{N}) = \inf\{\alpha:\ p_\alpha(\mathcal{N}) > 0\}.$$

What we have to show is that $\beta(\mathcal{N})=1$. First we establish the following properties of $\beta(\mathcal{N})$:

(i) $\beta(\mathcal{N})\le 6$ for every $\mathcal{N}$,

(ii) if $\mathcal{M}=\{m_k\}_{k=1}^{\infty}$ is another sequence of positive integers with

(3) $$n_k-m_k = O(1), \quad k\to\infty,$$

then $\beta(\mathcal{N})=\beta(\mathcal{M})$.

(iii) if $2\mathcal{N}=\{2n_k\}_{k=1}^{\infty}$ then $(\beta(2\mathcal{N}))^2=\beta(\mathcal{N})$.

The proof of (i) is based on Dirichlet's theorem (see e.g. §2 of [7]) stating that if $y_1, \ldots, y_n$ are arbitrary real numbers, $K>0$ and $\alpha>1$, then there is an integer $t\in[K, K\alpha^n]$ such that with appropriate integers $v_1, \ldots, v_n$ the inequalities

$$|ty_j-v_j|<\frac{1}{\alpha}, \quad j=1, \ldots, n,$$

are satisfied simultaneously. Applying this with $\alpha=5$ and $y_j=x_j/2\pi$, $j=1, \ldots, n$, we get that for arbitrary real numbers $x_1, \ldots, x_n$ and $K>0$ there is an integer t, $K\leqq t\leqq K5^n$, such that

$$\left|\sum_{j=1}^{n} e^{itx_j}\right| \geqq \operatorname{Re}\left\{\sum_{j=1}^{n} e^{itx_j}\right\} = \sum_{j=1}^{n} \operatorname{Re} e^{i(tx_j-2\pi v_j)} \geqq \sum_{j=1}^{n} \operatorname{Re} e^{i\frac{2\pi}{5}} = n\cos\frac{2\pi}{5}.$$

Since for every fixed K, we have $K5^n<6^n$ for all sufficiently large n, it follows that

$$\inf_{K>0} \varlimsup_{k\to\infty} P\left\{\sup_{K\leqq t\leqq 6^{n_k}} \frac{|S_{n_k}(t)|}{n_k} \geqq \cos\frac{2\pi}{5}\right\} = 1.$$

This means that $p_6(\mathcal{N})=1$, and hence (i) is proved.

Now suppose (3) and let $\alpha>\beta(\mathcal{N})$. If we choose α_1 in between, $\beta(\mathcal{N})<\alpha_1<\alpha$, then there exist an $M>0$ and a $\delta>0$ such that

$$\inf_{K>0} \varlimsup_{k\to\infty} P\left\{\sup_{K\leqq t\leqq \alpha_1^{n_k}} \frac{|S_{n_k}(t)|}{n_k} \geqq M\right\} = \delta.$$

But (3) implies that for all large enough k, $\alpha^{m_k}\geqq\alpha_1^{n_k}$ and

$$\frac{|S_{m_k}(t)|}{m_k} \geqq \frac{|S_{n_k}(t)|-|n_k-m_k|}{n_k+|n_k-m_k|} \geqq \frac{1}{2}\frac{|S_{n_k}(t)|}{n_k} - O\left(\frac{1}{n_k}\right), \quad t\in\mathbf{R},$$

and so

$$\inf_{K>0} \varlimsup_{k\to\infty} P\left\{\sup_{K\leqq t\leqq \alpha^{m_k}} \frac{|S_{m_k}(t)|}{m_k} \geqq \frac{M}{3}\right\} \geqq$$

$$\geqq \inf_{K>0} \varlimsup_{k\to\infty} P\left\{\sup_{K\leqq t\leqq \alpha_1^{n_k}} \frac{|S_{n_k}(t)|}{n_k} \geqq M\right\} = \delta > 0.$$

This means that $p_\alpha(\mathcal{M})>0$. Since this is true for all $\alpha>\beta(\mathcal{N})$, we can conclude that $\beta(\mathcal{M})\leqq\beta(\mathcal{N})$. Reversing the role of $\mathcal{N}$ and $\mathcal{M}$, we obtain the opposite inequality, and hence (ii) is also proved.

Turning now to the proof of (iii), we introduce the following subsequences of the original X sequence:

$$X^{(1)} = \{X_1, X_4, X_7, X_{10}, \ldots\}, \quad X^{(2)} = \{X_2, X_5, X_8, X_{11}, \ldots\},$$

$$Y^{(1)} = \{X_2, X_3, X_5, X_6, X_8, X_9, X_{11}, X_{12}, \ldots\},$$

$$Y^{(2)} = \{X_1, X_3, X_4, X_6, X_7, X_9, X_{10}, X_{12}, \ldots\},$$

and

$$Y^{(3)} = \{X_1, X_2, X_4, X_5, X_7, X_8, X_{10}, X_{11}, \ldots\}.$$

Let $\alpha<\beta(\mathcal{N})$. For each k;

$$S_{2n_k}(t;\ Y^{(3)})=S_{n_k}(t;\ X^{(1)})+S_{n_k}(t;\ X^{(2)}),$$

whence

$$p_{\sqrt{\alpha}}(2\mathcal{N}) = \lim_{M\downarrow 0} \lim_{K\to\infty} \varlimsup_{k\to\infty} P\left\{\sup_{K\le t\le(\sqrt{\alpha})^{2n_k}} \frac{|S_{2n_k}(t;\ Y^{(3)})|}{2n_k} \ge M\right\} \le$$

$$\le \lim_{M\downarrow 0} \lim_{K\to\infty} \varlimsup_{k\to\infty} \left(P\left\{\sup_{K\le t\le\alpha^{n_k}} \frac{|S_{n_k}(t;\ X^{(1)})|}{2n_k} \ge \frac{M}{2}\right\}+\right.$$

$$\left.+P\left\{\sup_{K\le t\le\alpha^{n_k}} \frac{|S_{n_k}(t;\ X^{(2)})|}{2n_k} \ge \frac{M}{2}\right\}\right) \le$$

$$\le \lim_{M\downarrow 0} \lim_{K\to\infty} \varlimsup_{k\to\infty} P\left\{\sup_{K\le t\le\alpha^{n_k}} \frac{|S_{n_k}(t;\ X^{(1)})|}{n_k} \ge M\right\}+$$

$$+\lim_{M\downarrow 0} \lim_{K\to\infty} \varlimsup_{k\to\infty} P\left\{\sup_{K\le t\le\alpha^{n_k}} \frac{|S_{n_k}(t;\ X^{(2)})|}{n_k} \ge M\right\} = 0+0 = 0,$$

where, at the last step, we used $\alpha<\beta(\mathcal{N})$. Thus $\alpha<\beta(\mathcal{N})$ implies $\sqrt{\alpha}\le\beta(2\mathcal{N})$. Therefore $\beta(\mathcal{N})\le(\beta(2\mathcal{N}))^2$.

Now let $\alpha<\beta(2\mathcal{N})$. Clearly,

$$S_{n_k}(t;\ X^{(1)}) = \frac{1}{2}\{S_{2n_k}(t;\ Y^{(2)})+S_{2n_k}(t;\ Y^{(3)})-S_{2n_k}(t;\ Y^{(1)})\}.$$

Hence, similarly as above,

$$p_{\alpha^2}(\mathcal{N}) = \lim_{M\downarrow 0} \lim_{K\to\infty} \varlimsup_{k\to\infty} P\left\{\sup_{K\le t\le(\alpha^2)^{n_k}} \frac{|S_{n_k}(t;\ X^{(1)})|}{n_k} \ge M\right\} \le$$

$$\le \lim_{M\downarrow 0} \lim_{K\to\infty} \varlimsup_{k\to\infty} \left(P\left\{\sup_{K\le t\le\alpha^{2n_k}} \frac{|S_{2n_k}(t;\ Y^{(2)})|}{2n_k} \ge \frac{M}{3}\right\}+\right.$$

$$\left.+P\left\{\sup_{K\le t\le\alpha^{2n_k}} \frac{|S_{2n_k}(t;\ Y^{(3)})|}{2n_k} \ge \frac{M}{3}\right\}+P\left\{\sup_{K\le t\le\alpha^{2n_k}} \frac{|S_{2n_k}(t;\ Y^{(1)})|}{2n_k} \ge \frac{M}{3}\right\}\right) =$$

$$= 0+0+0 = 0,$$

i.e., $\alpha<\beta(2\mathcal{N})$ implies $\alpha^2\le\beta(\mathcal{N})$. Therefore the opposite inequality $(\beta(2\mathcal{N}))^2\le\le\beta(\mathcal{N})$ also follows, and hence we have (iii).

Having now the three properties of $\beta(\mathcal{N})$, the proof of our Proposition is easy. For a positive integer m, set

$$\left[\frac{1}{2^m}\mathcal{N}\right] = \left\{\left[\frac{n_k}{2^m}\right]\right\}_{k=1}^{\infty}.$$

Since for fixed m,

$$2^m\left[\frac{n_k}{2^m}\right] - n_k = O(1), \quad k \to \infty,$$

we obtain by property (ii) that

$$\beta\left(2^m\left[\frac{1}{2^m}\mathcal{N}\right]\right) = \beta(\mathcal{N}),$$

and, by an m-fold application of property (iii), that

$$\left(\beta\left(2^m\left[\frac{1}{2^m}\mathcal{N}\right]\right)\right)^{2^m} = \beta\left(\left[\frac{1}{2^m}\mathcal{N}\right]\right).$$

Thus, by property (i),

$$\beta(\mathcal{N}) = \left(\beta\left(\left[\frac{1}{2^m}\mathcal{N}\right]\right)\right)^{\frac{1}{2^m}} \leqq 6^{\frac{1}{2^m}},$$

and since this holds for any integer $m \geqq 1$, the equality $\beta(\mathcal{N}) = 1$ follows.

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