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On contractive ϱ-dilations

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Dedicated to Professor B. Sz.-Nagy on the occasion of his seventieth birthday

Let T be a (bounded linear) operator on a Hilbert space $\mathfrak{H}$ and ϱ a positive number. We say that W is a ϱ-dilation of T if W is an operator on a Hilbert space $\mathfrak{K}\supset\mathfrak{H}$ and

(1) $$T^n h = \varrho P W^n h \quad (h\in\mathfrak{H},\ n = 1, 2, \ldots)$$

where P denotes the orthogonal projection of $\mathfrak{K}$ onto $\mathfrak{H}$. $\mathscr{C}_\varrho$ denotes the class of those operators which have *unitary* ϱ-dilations.

The study of unitary ϱ-dilations and $\mathscr{C}_\varrho$ classes was initiated by B. Sz.-Nagy and C. Foiaş [4] and continued by a number of authors. (See [3] also for further references and [2], [5], [6] for some recent results.)

Studying operators of $\mathscr{C}_\varrho$ classes, sometimes (non-unitary) contractive ϱ-dilations can be succesfully used [1]. So the dilation space and the ϱ-dilation themselves remain "near enough" to the initial space and operator, respectively. In this note we show that, for any $T\in\mathscr{C}_\varrho$, there exists a contractive ϱ-dilation with certain additional properties. Moreover, any other contractive (especially unitary) ϱ-dilation of T is a 1-dilation of a contractive ϱ-dilation of T with such properties.

Theorem. *Let $T\in\mathscr{C}_\varrho$ and let W be any contraction satisfying* (1). *Introduce the notations*

$$\mathfrak{K}_+ = \bigvee_{n=0}^{\infty} W^n\mathfrak{H}, \quad \mathfrak{L} = \mathfrak{H}\vee(W|\mathfrak{K}_+)^*\mathfrak{H}$$

and define the contraction C on $\mathfrak{L}$ by $C=Q(W|\mathfrak{L})$, where Q denotes the orthogonal projection of $\mathfrak{K}_+$ onto $\mathfrak{L}$. Then W is a 1-dilation of C; C is a ϱ-dilation of T; and

(2) $$C^2h = CTh \quad (h\in\mathfrak{H}),$$ (2*) $$C^{*2}h = C^*T^*h \quad (h\in\mathfrak{H}),$$

(3) $$\mathfrak{L} = \mathfrak{H}\vee C\mathfrak{H},$$ (3*) $$\mathfrak{L} = \mathfrak{H}\vee C^*\mathfrak{H}.$$

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Proof. We introduce the notation

$$V = W|\mathfrak{K}_+ \quad (V\colon \mathfrak{K}_+ \to \mathfrak{K}). \tag{4}$$

For $h, g \in \mathfrak{H}$ and $n=0, 1, 2, \ldots$ (4) and (1) imply that

$$(V^*(V^*-T^*)h, W^n g) = (h, W^{n+2}g)-(T^*h, W^{n+1}g) = 0.$$

This fact, (4) and the definition of $\mathfrak{K}_+$ show that $\mathfrak{K}_+ \perp (V^{*2}-V^*T^*)h \in \mathfrak{K}_+$ and consequently

$$V^{*2}h = V^*T^*h \quad (h \in \mathfrak{H}). \tag{5}$$

So (4) and the definition of $\mathfrak{L}$ show that $\mathfrak{L}$ is an invariant subspace of V^*.

Now we are going to prove by induction that W is a 1-dilation of C, i.e.

$$C^n h = QW^n h \quad (h \in \mathfrak{L},\ n = 1, 2, \ldots). \tag{6}$$

For $n=1$, (6) is clear from the definition of C. If (6) is true for some positive integer n, then for $h, g \in \mathfrak{L}$ we have

$$(QW^{n+1}h, g) = (VW^n h, g) = (W^n h, V^*g) = (QW^n h, V^*g) =$$
$$= (C^n h, V^*g) = (WC^n h, g) = (QWC^n h, g) = (C^{n+1}h, g)$$

and this proves (6).

For $h \in \mathfrak{H}$ and $n=1, 2, \ldots$ we have $PC^n h = PQW^n h = PW^n h = (1/\varrho)T^n h$, thus C is a ϱ-dilation of T.

If $h, g \in \mathfrak{L}$, then by (6) and (4)

$$(C^*h, g) = (h, Cg) = (h, Wg) = (h, Vg) = (V^*h, g).$$

Since $\mathfrak{L}$ is invariant for V^*, we have

$$C^* = V^*|\mathfrak{L}. \tag{7}$$

This fact, (4) and the definition of $\mathfrak{L}$ show that (3*) is true. Moreover, (5) and (7) imply (2*).

For $h, g \in \mathfrak{H}$ and $n=0$ or 1 we have

$$(C^2h - CTh, C^{*n}g) = (C^{n+2}h - C^{n+1}Th, g) = (1/\varrho)(T^{n+2}h - T^{n+2}h, g) = 0,$$

and so by (3*), $C^2h - CTh \perp \mathfrak{L}$ $(h \in \mathfrak{H})$, consequently (2) is true.

In order to prove (3), suppose that $g \in \mathfrak{L}$, $g \perp \mathfrak{H}$ and $g \perp C\mathfrak{H}$. In this case, by (2), $g \perp C^n\mathfrak{H}$ for $n=0, 1, \ldots$. Now for every $h \in \mathfrak{H}$ we have

$$(g, W^n h) = (g, QW^n h) = (g, C^n h) = 0,$$

consequently, by the definition of $\mathfrak{K}_+$, $\mathfrak{K}_+ \perp g \in \mathfrak{K}_+$. This implies $g=0$. So the proof is complete.

The following two remarks show that the dilation space $\mathfrak{L}$ is "not too large".

Remark 1. $\mathfrak{L}=\mathfrak{H}$ if and only if $\varrho=1$ or $T^2=0$.

Proof. If $\mathfrak{L}=\mathfrak{H}$, then $Cg=(1/\varrho)Tg$ $(g\in\mathfrak{H})$ and so we have for every $h\in\mathfrak{H}$

$$(1/\varrho)T^2h = PC^2h = PC(1/\varrho)Th = P(1/\varrho^2)T^2h = (1/\varrho^2)T^2h.$$

This implies that $\varrho=1$ or $T^2=0$.

In order to prove the converse implication, suppose first that $\varrho=1$. In this case for $f, g\in\mathfrak{H}$ we have

$$(Ch-Th, g) = 0, \quad (Ch-Th, C^*g) = (C^2h-CTh, g) = 0.$$

Thus, by (3*) and (3), $\mathfrak{L}\perp Ch-Th\in\mathfrak{L}$, consequently $C|\mathfrak{H}=T$; and so by (3), $\mathfrak{L}=\mathfrak{H}$.

Suppose now that $T^2=0$. In this case for $h, g\in\mathfrak{H}$ we have

$$((C-(1/\varrho)T)h, C^*g) = (C^2h, g)-(1/\varrho)(CTh, g) = (1/\varrho)(T^2h, g)-(1/\varrho^2)(T^2h, g) = 0.$$

This means that $(C-(1/\varrho)T)h\perp C^*\mathfrak{H}$. Since $(C-(1/\varrho)T)h\perp\mathfrak{H}$ is also true, so by (3*), $(C-(1/\varrho)T)h\perp\mathfrak{L}$, consequently $Ch=(1/\varrho)Th$ and now (3) implies $\mathfrak{L}=\mathfrak{H}$.

Remark 2. For every $h\in\mathfrak{H}$, $Th=0$ implies $Ch=0$ and $T^*h=0$ implies $C^*h=0$.

Proof. If $Th=0$ then for every $g\in\mathfrak{H}$

$$0 = (Th, g) = \varrho(PCh, g) = \varrho(Ch, g)$$

and using (2)

$$0 = (Th, C^*g) = (CTh, g) = (C^2h, g) = (Ch, C^*g).$$

These mean that $Ch\perp\mathfrak{H}$ and $Ch\perp C^*\mathfrak{H}$, so by (3*), $Ch\perp\mathfrak{L}$ and consequently $Ch=0$.

The second implication can be proved in the same way, by using T^* in place of T and C^* in place of C.

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