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A note on unitary dilation theory and state spaces

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Dedicated to the 70th anniversary of Professor B. Sz.-Nagy and to the 30th anniversary of his unitary dilation theorem

1. Introduction and preliminaires

The major breakthrough in dilation theory for contractions on a Hilbert space was the existence of a minimal unitary dilation obtained in 1953 by B. SZ.-NAGY [15]. Lately, the emphasis of dilation theory for contractions has been mainly on minimal isometric dilations (functional models, characteristic functions, intertwining lifting theorems, etc.). However, the natural abstract framework for certain problems in theoretical engineering (Markov realizations of wide sense stationary Gaussian random processes [6, 9—14]) are intimately related to unitary dilations of contractions. In this way very interesting and new problems arise, in which the emphasis lies entirely on unitary dilations. Here we present a solution to one of these problems.

We follow the notation and terminology in [17]. In particular by a dilation we mean a strong (or power) dilation in [8]. Throughout U is a unitary operator on $\mathfrak{K}$ and $\mathfrak{H}$ is a subspace of $\mathfrak{K}$ such that

$$\mathfrak{K} = \bigvee_{-\infty}^{\infty} U^n \mathfrak{H}. \tag{1.1}$$

For a subspace $\mathfrak{X}$ of $\mathfrak{K}$ we denote by $T_{\mathfrak{X}}$ the compression of U to $\mathfrak{X}$, that is $T_{\mathfrak{X}} = P_{\mathfrak{X}} U | \mathfrak{X}$. A *state space* $\mathfrak{X}$ (for $\mathfrak{H}$) is a subspace of $\mathfrak{K}$ such that $\mathfrak{H} \subseteq \mathfrak{X}$ and U is the minimal unitary dilation for $T_{\mathfrak{X}}$. An operator $T_{\mathfrak{X}}$ is a *state space operator* if $\mathfrak{X}$ is a state space for $\mathfrak{H}$. A state space $\mathfrak{X}$ is *minimal* if $\mathfrak{X}$ contains no strictly proper state space. Our problem is to obtain a classification of all minimal state spaces for $\mathfrak{H}$. This problem is equivalent to certain problems which naturally occur in engineering and Markov processes [6, 9—14]. There dilation theory is mentioned but not exploited. Here we shall fully exploit dilation theory to obtain all minimal

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state spaces. It is shown that the minimal state space problem is deeply related with infinite-dimensional Jordan model theory [1, 18—21] and the notion of property (P) in [3]. In this way new results will be given in Sections 3 and 4 and many known results will be derived in a simple manner in Section 2.

To complete this section some further notation is established. If $\mathfrak{M}$ is a subspace then $\mathfrak{M}^\perp$ is its orthogonal complement. For a subspace $\mathfrak{X}$ in $\mathfrak{K}$ we define $\mathfrak{X}_+$ and $\mathfrak{X}_-$ by

$$\mathfrak{X}_+ := \bigvee_{n\geqq 0} U^n\mathfrak{X} \quad \text{and} \quad \mathfrak{X}_- := \bigvee_{n\leqq 0} U^n\mathfrak{X}. \tag{1.2}$$

Let T be an operator in $\mathfrak{X}$; $\mathfrak{y}\subseteqq\mathfrak{X}$ is *cyclic* for T if $\mathfrak{X}=\bigvee_{n\geqq 0} T^n\mathfrak{y}$. A subspace $\mathfrak{W}$ is *semi-invariant* [4] for T if $\mathfrak{W}=\mathfrak{N}\ominus\mathfrak{M}$ where $\mathfrak{M}\subseteqq\mathfrak{N}$, and $\mathfrak{M}, \mathfrak{N}$ are both invariant subspaces for T. Obviously $\mathfrak{W}=\mathfrak{M}^\perp\ominus\mathfrak{N}^\perp$. Therefore $\mathfrak{W}$ is semi-invariant for T if and only if $\mathfrak{W}$ is semi-invariant for T^*. Finally, the following lemma is needed. Its proof follows from Proposition 1.3 in [4] and the geometry of dilation theory [17]. A proof is given in [6].

Lemma 1.1. *Let U be a unitary operator on $\mathfrak{K}$ and $\mathfrak{X}$ a subspace of $\mathfrak{K}$. The following statements are equivalent:*

(a) *U is a dilation of $T_\mathfrak{X}$.*
(b) *$\mathfrak{X}$ is semi-invariant for U.*
(c) $P_{\mathfrak{X}_-}\mathfrak{X}_+=P_\mathfrak{X}\mathfrak{X}_+=\mathfrak{X}$.
(d) $P_{\mathfrak{X}_+}\mathfrak{X}_-=P_\mathfrak{X}\mathfrak{X}_-=\mathfrak{X}$.

2. Basic geometric results

In this section we develop a basic geometric structure for state spaces. The results in this section are not new. In a different form they are more or less contained in [9—14] and elsewhere. The proofs are presented for two reasons: first, for completeness; secondly and more importantly, to demonstrate the power of our approach. That is to demonstrate how minimal unitary dilation theory can be used to obtain simple proofs of these results. The following identities will be useful. If $\mathfrak{M}, \mathfrak{N}$ are subspaces then

$$\mathfrak{M} = (\overline{P_\mathfrak{M}\mathfrak{N}})\oplus(\mathfrak{M}\cap\mathfrak{N}^\perp). \tag{2.1}$$

If U is the minimal unitary dilation for $T_\mathfrak{X}$ then

$$\mathfrak{X} = \mathfrak{X}_+\ominus(\mathfrak{X}_-)^\perp = \mathfrak{X}_-\ominus(\mathfrak{X}_+)^\perp. \tag{2.2}$$

Equation (2.2) follows from (2.1) and Lemma 1.1. It is also a consequence of the geometry of minimal unitary dilation theory, [17], Ch. II.

Let $\mathfrak{X}$ be a state space for $\mathfrak{H}$. Following [9—14], $\mathfrak{X}$ is *observable* [constructible] if $\mathfrak{X}=\overline{P_{\mathfrak{X}}\mathfrak{H}_+}$ [$\mathfrak{X}=\overline{P_{\mathfrak{X}}\mathfrak{H}_-}$], respectively. From the dilation property $\mathfrak{X}$ is observable [constructible] if and only if $\mathfrak{H}$ is cyclic for $T_{\mathfrak{X}}$ [$T_{\mathfrak{X}}^*$], respectively. The *observable,* respectively *constructible part,* of $\mathfrak{X}$ is the subspace defined by

(2.3a) $$\mathfrak{X}_o := \bigvee_{n\geqq 0} T_{\mathfrak{X}}^n \mathfrak{H} = \overline{P_{\mathfrak{X}}\mathfrak{H}_+} = \overline{P_{\mathfrak{X}_-}\mathfrak{H}_+},$$

(2.3b) $$\mathfrak{X}_c := \bigvee_{n\geqq 0} T_{\mathfrak{X}}^{*n} \mathfrak{H} = \overline{P_{\mathfrak{X}}\mathfrak{H}_-} = \overline{P_{\mathfrak{X}_+}\mathfrak{H}_-}.$$

Since $\mathfrak{X}_o$ is an invariant subspace for $T_{\mathfrak{X}}$ and $\mathfrak{H}\subseteqq\mathfrak{X}_o$ it follows that U is the minimal unitary dilation for $T_{\mathfrak{X}_o}$. Obviously $\mathfrak{X}_o$ is an observable state space. In a similar manner it follows that $\mathfrak{X}_c$ is a constructible state space. The *observable and constructible part* of $\mathfrak{X}$ is

(2.4) $$\mathfrak{X}_{oc} := (\mathfrak{X}_o)_c := \bigvee_{n\geqq 0} T_{\mathfrak{X}_o}^{*n} \mathfrak{H} = \overline{P_{\mathfrak{X}_o}\mathfrak{H}_-}.$$

By construction $\mathfrak{X}_{oc}$ is a constructible state space. Using the observability of $\mathfrak{X}_o$ and the fact that $T_{\mathfrak{X}_o}$ is a dilation of $T_{\mathfrak{X}_{oc}}$, it is easy to verify that $\mathfrak{X}_{oc}$ is observable. Decomposing: $\mathfrak{X}=\mathfrak{X}_o\oplus\mathfrak{X}_{\bar{o}}$ and $\mathfrak{X}_o=\mathfrak{X}_{o\bar{c}}\oplus\mathfrak{X}_{oc}$ where $\mathfrak{X}_{\bar{o}}, \mathfrak{X}_{o\bar{c}}$ are the appropriate orthogonal subspaces yields:

(2.5) $$\mathfrak{X} = \mathfrak{X}_{o\bar{c}}\oplus\mathfrak{X}_{oc}\oplus\mathfrak{X}_{\bar{o}}.$$

Equation (2.5) implies that all state spaces contain a constructible and observable state space $\mathfrak{X}_{oc}$. In particular, a minimal state space is constructible and observable. If not, one could use (2.5) to obtain a smaller state space. Note $\mathfrak{X}=\mathfrak{X}_{oc}$ if and only if $\mathfrak{X}$ is observable and constructible. This proves half of

Proposition 2.1. [14] *Let $\mathfrak{X}$ be a state space. Then $\mathfrak{X}$ is constructible and observable if and only if $\mathfrak{X}$ is minimal.*

Proof (only if). Assume $\mathfrak{X}$ is constructible and observable. Let $\mathfrak{W}$ be a state space contained in $\mathfrak{X}$. Lemma 1.1 implies $T_{\mathfrak{X}}$ is a dilation for $T_{\mathfrak{W}}$. Since $\mathfrak{H}$, and thus $\mathfrak{W}$, is cyclic for $T_{\mathfrak{X}}$ we have $T_{\mathfrak{W}}P_{\mathfrak{W}}=P_{\mathfrak{W}}T_{\mathfrak{X}}$. (This fact is well known [16], p. 1.) This identity implies $\mathfrak{W}$ is an invariant subspace for $T_{\mathfrak{X}}^*$. Hence $T_{\mathfrak{W}}^*=T_{\mathfrak{X}}^*|\mathfrak{W}$. The constructibility of $\mathfrak{X}$ yields

$$\mathfrak{X} = \bigvee_{n\geqq 0} T_{\mathfrak{X}}^{*n}\mathfrak{H} \subseteqq \bigvee_{n\geqq 0} T_{\mathfrak{X}}^{*n}\ \mathfrak{W} \subseteqq \mathfrak{W}.$$

Using $\mathfrak{W}\subseteqq\mathfrak{X}$ gives $\mathfrak{X}=\mathfrak{W}$ and completes the proof.

As noted earlier Proposition 2.1 is not new [14]. Ruckebusch's proof depends upon splitting subspaces and some results in [9]. Here this result was derived directly from dilation theory.

Let $\mathfrak{X}$ be a state space. Equation (2.5) demonstrates that $\mathfrak{X}_{oc}$ is a minimal state space. In a similar manner $\mathfrak{X}_{co}:=(\mathfrak{X}_c)_o$ is a minimal state space. From any state space $\mathfrak{X}$ we can obtain possibly two different minimal state spaces, $\mathfrak{X}_{oc}$ and $\mathfrak{X}_{co}$. Obviously $\mathfrak{K}$ is a state space. The minimal state spaces $\mathfrak{P}$ and $\mathfrak{F}$ are defined by $\mathfrak{P}:=\mathfrak{K}_{co}$ and $\mathfrak{F}:=\mathfrak{K}_{oc}$. A simple calculation shows that $\mathfrak{P}=\overline{P_{\mathfrak{H}_-}\mathfrak{H}_+}$ and $\mathfrak{F}=\overline{P_{\mathfrak{H}_+}\mathfrak{H}_-}$. Notice that $\mathfrak{P}$ $[\mathfrak{F}]$ is the minimal state space for $\mathfrak{H}$ contained in the past $\mathfrak{H}_-$ [future $\mathfrak{H}_+$] of $\mathfrak{H}$ respectively. Here as in [9—14] the spaces $\mathfrak{P}$ and $\mathfrak{F}$ play an important role in our theory.

Proposition 2.2. *If $\mathfrak{X}$ is an observable [a constructible] state space then $\mathfrak{X}_+\subseteqq\mathfrak{P}_+$ $[\mathfrak{X}_-\subseteqq\mathfrak{F}_-]$, respectively. In particular, if $\mathfrak{X}$ is minimal then $\mathfrak{X}_+\subseteqq\mathfrak{P}_+$ and $\mathfrak{X}_-\subseteqq\mathfrak{F}_-$.*

Proof. Assume $\mathfrak{X}$ is observable. Equations (2.1) and (2.2) give

$$\mathfrak{P}\oplus(\mathfrak{P}_+)^\perp = \mathfrak{P}_- = \mathfrak{H}_- = \mathfrak{P}\oplus[\mathfrak{H}_-\cap(\mathfrak{H}_+)^\perp].$$

So $(\mathfrak{P}_+)^\perp=\mathfrak{H}_-\cap(\mathfrak{H}_+)^\perp$. This, $\mathfrak{H}_-\subseteqq\mathfrak{X}_-$ and the observability of $\mathfrak{X}$ (i.e., $\mathfrak{X}=\overline{\mathfrak{P}_{\mathfrak{X}}\mathfrak{H}_+}$ by (2.3a)) implies that $\mathfrak{X}$ is orthogonal to $(\mathfrak{P}_+)^\perp$. Hence $\mathfrak{X}\subseteqq\mathfrak{P}_+$ and $\mathfrak{X}_+\subseteqq\mathfrak{P}_+$. A similar argument proves the other part.

The following will be useful.

Proposition 2.3. *Let $\mathfrak{X}$ be a state space and $\mathfrak{X}_c$ $[\mathfrak{X}_o]$ its constructible [observable] part, respectively. Then $\mathfrak{X}_{c+}=\mathfrak{X}_+$ and $\mathfrak{X}_{o-}=\mathfrak{X}_-$.*

Proof. Decomposing $\mathfrak{X}=\mathfrak{X}_c\oplus\mathfrak{X}_{\bar c}$ with (2.2) gives:

$$\mathfrak{X}_+ = \mathfrak{X}_c\oplus\mathfrak{X}_{\bar c}\oplus(\mathfrak{X}_-)^\perp \quad\text{and}\quad \mathfrak{X}_{c+} = \mathfrak{X}_c\oplus(\mathfrak{X}_{c-})^\perp. \tag{2.6}$$

Using $\mathfrak{X}_{c+}\subseteqq\mathfrak{X}_+$ implies:

$$(\mathfrak{X}_{c-})^\perp \subseteqq \mathfrak{X}_{\bar c}\oplus(\mathfrak{X}_-)^\perp. \tag{2.7}$$

To prove $\mathfrak{X}_{c+}=\mathfrak{X}_+$ it is sufficient to show that we have equality in (2.7). Assume x is in $\mathfrak{X}_{\bar c}\oplus(\mathfrak{X}_-)^\perp$ and x is orthogonal to $(\mathfrak{X}_{c-})^\perp$. Clearly x is in $\mathfrak{X}_{c-}$. Using $\mathfrak{X}_{c-}\subseteqq\mathfrak{X}_-$ places x in $\mathfrak{X}_-$. Hence x is in $\mathfrak{X}_{\bar c}$. This and (2.3b) verifies that x is orthogonal to $\mathfrak{H}_-$. Since x is in $\mathfrak{X}_{\bar c}$, (2.6) shows that x is orthogonal to $\mathfrak{X}_{c+}$. Combining:

$$x\perp(\mathfrak{H}_-\vee\mathfrak{X}_{c+}) \supseteqq (\mathfrak{H}_-\vee\mathfrak{H}_+) = \mathfrak{K}.$$

Therefore $x=0$ and there is equality in (2.7). The other part follows by duality.

One can easily derive Propositions 2.2 and 2.3 directly from the Lifting Theorem (Theorem 2.3 p. 66 in [17]). Let us show this for Proposition 2.3.

Alternate proof of Proposition 2.3. Since $\mathfrak{X}_c$ is an invariant subspace for $T^*_{\mathfrak{X}}$ we have $T^*_{\mathfrak{X}}Q^*=Q^*T^*_{\mathfrak{X}_c}$. Here Q^* is the operator mapping $\mathfrak{X}_c$ into

$\mathfrak{X}$ defined by $Q^*:=P_{\mathfrak{X}}|\mathfrak{X}_c$. Obviously $QT_{\mathfrak{X}}=T_{\mathfrak{X}_c}Q$ and $Q=P_{\mathfrak{X}_c}|\mathfrak{X}$. Notice that U is the minimal unitary dilation for both $T_{\mathfrak{X}}$ and $T_{\mathfrak{X}_c}$. By the Lifting Theorem there exists a contraction R on $\mathfrak{K}$ such that

$$Q = P_{\mathfrak{X}_c}R|\mathfrak{X}, \quad R\mathfrak{X}_+ \subseteqq \mathfrak{X}_{c+} \quad \text{and} \quad UR = RU.$$

Note $Qh=h$ for all h in $\mathfrak{H}$. Thus $Rh=h$ for all h in $\mathfrak{H}$. Since U commutes with R we have $R=I$; the identity. Hence $\mathfrak{X}_+\subseteqq\mathfrak{X}_{c+}$ and $\mathfrak{X}_+=\mathfrak{X}_{c+}$.

The following will be useful. Similar results are given in [14] and elsewhere.

Proposition 2.4. *If $\mathfrak{X}$ is a constructible [an observable] state space such that $\mathfrak{X}_+\subseteqq\mathfrak{P}_+$ [$\mathfrak{X}_-\subseteqq\mathfrak{F}_-$] then $T_{\mathfrak{X}}$ [$T_{\mathfrak{F}}$] is a quasi-affine transform of $T_{\mathfrak{P}}$ [$T_{\mathfrak{X}}$], respectively. In particular, if $\mathfrak{X}$ is a minimal state space then $T_{\mathfrak{X}}$ [$T_{\mathfrak{F}}$] is a quasi-affine transform of $T_{\mathfrak{P}}$ [$T_{\mathfrak{X}}$], respectively.*

Proof. Assume $\mathfrak{X}$ is constructible and $\mathfrak{X}_+\subseteqq\mathfrak{P}_+$. Applying Lemma 1.1 gives (for all $h\in\mathfrak{H}$ and $n\geqq0$),

(2.8) $P_{\mathfrak{X}}T_{\mathfrak{P}}^{*n}h = P_{\mathfrak{X}}P_{\mathfrak{P}_+}U^{*n}h = P_{\mathfrak{X}}U^{*n}h = T_{\mathfrak{X}}^{*n}h$ and $P_{\mathfrak{X}}T_{\mathfrak{P}}^{*}T_{\mathfrak{P}}^{*n}h = T_{\mathfrak{X}}^{*}P_{\mathfrak{X}}P_{\mathfrak{P}}^{*n}h.$

Let Q be the operator mapping $\mathfrak{P}$ into $\mathfrak{X}$ defined by $Q:=P_{\mathfrak{X}}|\mathfrak{P}$. Equation (2.8) and the constructibility of $\mathfrak{P}$ implies $QT_{\mathfrak{P}}^*=T_{\mathfrak{X}}^*Q$. This, the constructibility of $\mathfrak{X}$ and $Qh=h$ for all h in $\mathfrak{H}$ shows that Q has dense range in $\mathfrak{X}$. Obviously $T_{\mathfrak{P}}Q^*=Q^*T_{\mathfrak{X}}$ where $Q^*=P_{\mathfrak{P}}|\mathfrak{X}$. This, the observability of $\mathfrak{P}$ and $Q^*h=h$ for all h in $\mathfrak{H}$ shows that Q^* has dense range in $\mathfrak{P}$. Thus Q is a quasi-affinity. This completes the first part. The second part follows by a similar argument.

3. Consequences of general dilation theory

In this section we list several results which are trivial consequences of the previous section and unitary dilation theory [17]. Some of these results have not been noted before and would be difficult to obtain without dilation theory. Others (namely (1), and part of (2)) were previously obtained without consulting dilation theory.

(1) Let $\mathfrak{X}$ be a state space for $\mathfrak{H}$. Then $T_{\mathfrak{X}}^n\to0$ [$T_{\mathfrak{X}}^{*n}\to0$] in the strong operator topology as $n\to\infty$ if and only if $\bigcap_{n\geqq0} U^{*n}\mathfrak{X}_-=\{0\}$ [$\bigcap_{n\geqq0} U^n\mathfrak{X}_+=\{0\}$], respectively.

(2) Assume $\mathfrak{X}$ is a state space and $T_{\mathfrak{X}}$ is a completely nonunitary contraction. Then $T_{\mathfrak{X}}$ is a $C_{\cdot0}, C_{0\cdot}, C_{\cdot1}, C_{1\cdot}$ contraction if and only if its characteristic function is inner, $*$-inner, outer, $*$-outer, respectively.

Parts (1) and (2) are trivial consequences of [17], ch. II and ch. VI. Parts (3) and (4) follow from Propositions 2.1, 2.2, 2.4 with [17] ch. II.

(3) ([14]) The set of all minimal state spaces are of the same dimension. In particular, $\mathfrak{H}$ admits a finite-dimensional state space if and only if $\mathfrak{P}$ or $\mathfrak{F}$ is finite dimensional.

(4) Assume that $T_{\mathfrak{P}}$ $[T_{\mathfrak{F}}]$ is a $C_{\cdot 0}$ $[C_{0\cdot}]$ contraction and $\mathfrak{X}$ is a minimal state space; respectively. Then $T_{\mathfrak{X}}$ is a $C_{\cdot 0}$ $[C_{0\cdot}]$ contraction respectively. In particular if $T_{\mathfrak{P}}$ is a $C_{\cdot 0}$ and $T_{\mathfrak{F}}$ is a $C_{0\cdot}$ contraction then $T_{\mathfrak{X}}$ is a C_{00} contraction.

From [17] ch. VI sec. 5 we have

(5) Assume $\mathfrak{H}$ is finite dimensional and there exists a C_{00} contraction for a state space operator. Then all minimal state space operators are C_0 contractions with finite defect indices.

Our final remark follows from Section 2 and the Jordan model theory in [1, 18—21] (see also Lemma 4.1 below).

(6) Assume that there exists a C_0 contraction for a state space operator, then all minimal state space operators are C_0 contractions, quasi-similar, and have the same Jordan model.

4. Minimality and property (P)

The results in the rest of this paper are believed to be new. In this section we obtain a classification of all minimal state spaces when $T_{\mathfrak{P}}$ is a C_0 contraction with property (P) [3]. An example is given to demonstrate that the C_0 assumption is natural to the problem. Our approach depends heavily on the infinite Jordan model theory in [1, 18—21] and C_0 contractions with property (P) [3]. Throughout we follow the notation and terminology established there. If m is an inner function then $\mathfrak{H}(m):=H^2\ominus mH^2$ and $S(m)$ is the operator on $\mathfrak{H}(m)$ defined by $S(m)f=$ $=P_{\mathfrak{H}(m)}e^{it}f$ where $f\in\mathfrak{H}(m)$. A contraction T on $\mathfrak{X}$ has *property* (P), if A on $\mathfrak{X}$ is any injection such that $AT=TA$ then A is a quasi-affinity [3]. We begin with some results in [1, 3, 18—21], which we shall need.

Lemma 4.1. *Let T on $\mathfrak{X}$ be a C_0 contraction and $\hat{T}$ on $\hat{\mathfrak{X}}$ be any contraction.*

I) *The following statements are equivalent:*

a) *T is a quasi-affine transform of $\hat{T}$.*

b) *$\hat{T}$ is a quasi-affine transform of T.*

c) *T is a quasi-similar to $\hat{T}$.*

d) *$\hat{T}$ is a C_0 contraction and $\hat{T}$ has the same Jordan model as T.*

II) *If any of* a), b), c) *or* d) *is valid and T has property (P) then $\hat{T}$ is a C_0 contraction with property (P).*

III) *T has property (P) if and only if T^* has property (P). If $\mathfrak{W}$ is semi-invariant for T and T has property (P), then $T_{\mathfrak{W}}$ $(:=P_{\mathfrak{W}}T|\mathfrak{W})$ is a C_0 contraction with property (P).*

IV) *T has property (P) if and only if given any semi-invariant subspace $\mathfrak{W}$ for T such that $T_{\mathfrak{W}}$ and T have the same Jordan model then $\mathfrak{W}=\mathfrak{X}$.*

Proof. Part I follows from [1, 18—21]. Parts II and III are in [3]. Now for part IV. Assume T has property (P) and $T_{\mathfrak{W}}$ has the same Jordan model as T. Let $\mathfrak{W}=\mathfrak{M}\ominus\mathfrak{N}$ where $\mathfrak{M}$ and $\mathfrak{N}$ are invariant subspaces for T. Since $\mathfrak{N}$ is invariant for $T_{\mathfrak{M}}$ we have $T^*_{\mathfrak{W}}=T^*_{\mathfrak{M}}|\mathfrak{W}$. Let

$$\bigoplus_1^\infty S(m_i),\quad \bigoplus_1^\infty S(\tilde{m}_i),\quad \bigoplus_1^\infty S(\omega_i),\quad \bigoplus_1^\infty S(\tilde{\omega}_i)$$

be the Jordan models for $T_{\mathfrak{M}}, T^*_{\mathfrak{M}}, T_{\mathfrak{W}}, T^*_{\mathfrak{W}}$ respectively. Clearly $T^*_{\mathfrak{M}}X=XT^*_{\mathfrak{W}}$ where X is the identity operator injecting $\mathfrak{W}$ into $\mathfrak{M}$. Proposition 2 in [18] or [20] implies $\tilde{\omega}_i$ divides $\tilde{m}_i$ for all i. Furthermore, $TY=YT_{\mathfrak{M}}$ where Y is the identity operator injecting $\mathfrak{M}$ into $\mathfrak{X}$. Consulting Proposition 2 of [18] or [20] again implies m_i divides ω_i for all i. Combining, $\omega_i=m_i$ for all i. Therefore $T, T_{\mathfrak{M}}$ and $T_{\mathfrak{W}}$ all admit the same Jordan model. By Lemma 4.1.I there exists a quasi-affinity A mapping $\mathfrak{X}$ into $\mathfrak{M}$ such that $AT=T_{\mathfrak{M}}A=TA$. Since T has property (P): $\mathfrak{X}=\overline{A\mathfrak{X}}=\mathfrak{M}$ and $\mathfrak{W}=\mathfrak{X}\ominus\mathfrak{N}$ is invariant for T^*. There exists a quasi-affinity B mapping $\mathfrak{X}$ into $\mathfrak{W}$ such that $BT^*=T^*_{\mathfrak{W}}B=T^*B$. By Lemma 4.1.III, T^* also has property (P); consequently $\mathfrak{X}=\overline{B\mathfrak{X}}=\mathfrak{W}$. This completes half the proof of part IV.

The other half follows by contradiction. Assume that T does not have property (P). Then there exists an injection A on $\mathfrak{X}$ such that $TA=AT$ and $\overline{A\mathfrak{X}}\neq\mathfrak{X}$. Notice that $\overline{A\mathfrak{X}}$ is invariant for T. Lemma 4.1.I implies that $T|\overline{A\mathfrak{X}}$ and T have the same Jordan model. Since $\overline{A\mathfrak{X}}\neq\mathfrak{X}$ the proof is complete.

We begin with

Theorem 4.1. *Let $\mathfrak{H}$ admit a state space $\mathfrak{W}$ such that $T_{\mathfrak{W}}$ is a C_0 contraction with property (P). Then a state space $\mathfrak{X}$ is minimal if and only if $T_{\mathfrak{X}}$ is a C_0 contraction and has the same Jordan model as $T_{\mathfrak{F}}$ or $T_{\mathfrak{P}}$. In this case, all minimal state spaces $T_{\mathfrak{X}}$ are C_0 contractions with the same Jordan model as $T_{\mathfrak{F}}$ or $T_{\mathfrak{P}}$.*

Proof. First it is shown that $T_{\mathfrak{P}}$ is a C_0 contraction with property (P). Equation (2.5) shows that $\mathfrak{W}$ contains a minimal state space $\mathfrak{W}_{oc}$ which is semi-invariant for $T_{\mathfrak{W}}$. Lemma 4.1.III implies $T_{\mathfrak{W}_{oc}}$ is a C_0 contraction with property (P). By Proposition 2.4, $T_{\mathfrak{W}_{oc}}$ is a quasi-affine transform of $T_{\mathfrak{P}}$. By Lemma 4.1.II $T_{\mathfrak{P}}$ is a C_0 contraction with property (P).

Now assume $\mathfrak{X}$ is a minimal state space. Proposition 2.4 implies $T_{\mathfrak{X}}$ is a quasi-affine transform of $T_{\mathfrak{P}}$. By Lemma 4.1.I and the preceeding paragraph, $T_{\mathfrak{X}}$ is a C_0 contraction and has the same Jordan model as $T_{\mathfrak{P}}$.

Assume $T_{\mathfrak{X}}$ is a C_0 contraction with the same Jordan model as $T_{\mathfrak{P}}$. Lemma 4.1.I implies $T_{\mathfrak{X}}$ is a C_0 contraction with property (P). Equation (2.5) shows that $\mathfrak{X}$ contains a minimal state space $\mathfrak{X}_{oc}$ semi-invariant for $T_{\mathfrak{X}}$. Proposition 2.4 assigns the same Jordan model to both $T_{\mathfrak{X}_{oc}}$ and $T_{\mathfrak{P}}$. Hence $T_{\mathfrak{X}}$ and $T_{\mathfrak{X}_{oc}}$ have the same Jordan model. Lemma 4.1.IV gives $\mathfrak{X}=\mathfrak{X}_{oc}$ and completes the proof.

Our classification of all minimal state spaces is given in

Theorem 4.2. *Let $\mathfrak{H}$ admit a state space such that T is a C_0 contraction with property (P). Then there is a one to one correspondence between the set of all minimal state spaces for $\mathfrak{H}$ and the set of all invariant subspaces $\mathfrak{G}$ $[\mathfrak{I}]$ for U $[U^*]$ such that*

$$\mathfrak{F}_+ \subseteqq \mathfrak{G} \subseteqq \mathfrak{P}_+ \quad [\mathfrak{P}_- \subseteqq \mathfrak{I} \subseteqq \mathfrak{F}_-], \tag{4.1}$$

respectively. In this case, the set of all minimal state spaces for $\mathfrak{H}$ are $\{\overline{P_{\mathfrak{G}}\mathfrak{H}_-}\}$ $[\{\overline{P_{\mathfrak{I}}\mathfrak{H}_+}\}]$ where $\mathfrak{G}$ $[\mathfrak{I}]$ is an invariant subspace for U $[U^*]$ satisfying (4.1), *respectively.*

Proof. Assume $\mathfrak{X}$ is a minimal state space. Obviously $\mathfrak{X}_+$ is an invariant subspace for U. Proposition 2.2 implies that $\mathfrak{G}=\mathfrak{X}_+$ satisfies (4.1). By constructibility $\mathfrak{X}=\overline{P_{\mathfrak{X}_+}\mathfrak{H}_-}=\overline{P_{\mathfrak{G}}\mathfrak{H}_-}$.

Now assume $\mathfrak{G}$ is invariant for U and satisfies (4.1). Let $\mathfrak{X}=\overline{P_{\mathfrak{G}}\mathfrak{H}_-}$. Notice that $\mathfrak{G}$ is a state space and $\mathfrak{X}$ is its constructible part. Proposition 2.3 gives $\mathfrak{X}_+=\mathfrak{G}_+=\mathfrak{G}$. Proposition 2.4 implies $T_{\mathfrak{X}}$ is a quasi-affine transform of $T_{\mathfrak{P}}$. $T_{\mathfrak{P}}$ is a C_0 contraction with property (P) (see the proof of Theorem 4.1). Thus $T_{\mathfrak{X}}$ is a C_0 contraction, and by Lemma 4.1.I and Theorem 4.1, $\mathfrak{X}$ is a minimal state space. Hence the correspondence $\mathfrak{X}\leftrightarrow\mathfrak{G}$ $(\mathfrak{X}=\overline{P_{\mathfrak{G}}\mathfrak{H}_-})$ is bijective. This completes the proof of the first part. The second part follows in a similar manner.

Lemma 4.2. *Let T on $\mathfrak{X}$ be a C_0 contraction. If any one of the following statements holds then T has property (P):*

(a) *T is a weak contraction;*

(b) *T has finite multiplicity;*

(c) *T has finite defect indices.*

Lemma 4.2 follows from [2, 3, 21]. Recall that (c) implies (b) and (b) implies (a). Lemma 4.2, Theorem 4.2 and (5) in Section 3 gives

Corollary 4.2. *The conclusion of Theorem 4.2 is valid if any one of the following statements is true for any state space operator $T_{\mathfrak{X}}$.*

(i) *$T_{\mathfrak{X}}$ is a weak C_0 contraction;*

(ii) *$T_{\mathfrak{X}}$ is a C_0 contraction with finite multiplicity;*

(iii) *$T_{\mathfrak{X}}$ is a C_0 contraction with finite defect indices;*

(iv) *$T_{\mathfrak{X}}$ is a C_{00} contraction and $\mathfrak{H}$ is finite dimensional;*

(v) *$\mathfrak{H}$ is finite dimensional and $T_{\mathfrak{P}}$ or $T_{\mathfrak{F}}$ is a C_0 contraction.*

Obviously (v)$\Leftrightarrow$(iv)$\Rightarrow$(iii)$\Rightarrow$(ii)$\Rightarrow$(i).

The following shows that one cannot remove the C_0 assumption in Theorem 4.2.

Example 4.1. Here we will construct a system U, $\mathfrak{H}, \mathfrak{K}$ and $\mathfrak{G}$ such that $\mathfrak{G}$ is an invariant subspace for U satisfying (4.1) and $\mathfrak{X}=\overline{P_{\mathfrak{G}}\mathfrak{H}_-}$ is a not a minimal state space. To this end, let $z:=e^{it}$ and U be the bilateral shift on $L^2=\mathfrak{K}$. (Here $L^2=L^2(0,2\pi)$ and $Uf=zf$ for f in L^2.) It is easy to see that $T_{\mathfrak{K}}=U$ has the property (P) (cf. [8], Problem 115). Let $\mathfrak{H}$ equal the one dimensional space spanned by e^z. The space $\mathfrak{G}$ will be defined later. Finally

$$U_+ := U|H^2 \quad \text{and} \quad K^2 := L^2\ominus(zH^2). \tag{4.2}$$

We begin the proof of our counter example. Since e^z is outer:

$$\mathfrak{H}_+ = \mathfrak{F}_+ = H^2 \quad \text{and} \quad \mathfrak{K} = L^2 = \bigvee_{-\infty}^{\infty} U^n\mathfrak{H}. \tag{4.3}$$

Consulting [5] implies e^z is cyclic for U_+^*. A simple calculation gives:

$$\mathfrak{F} := \overline{P_{\mathfrak{H}_+}\mathfrak{H}_-} = \bigvee_{n\geqq 0} U_+^{*n}e^z = H^2. \tag{4.4}$$

Therefore $\mathfrak{F}=\mathfrak{F}_+=H^2$. Recall $\mathfrak{P}=\overline{P_{\mathfrak{H}_-}\mathfrak{H}_+}$. Note e^z has an inverse in L^∞. Hence $\mathfrak{H}_-=\overline{e^zK^2}=e^zK^2$.

Obviously $\mathfrak{P}\subseteqq\mathfrak{H}_-$. We claim

$$\mathfrak{P} = \mathfrak{H}_- = e^zK^2. \tag{4.5}$$

Let e^zh^* be any element in $\mathfrak{H}_-$ that is orthogonal to $\mathfrak{P}$. (Here h is in H^2 and h^* is its complex conjugate.) Using $\mathfrak{P}=\overline{P_{\mathfrak{H}_-}H^2}$ shows that e^z is orthogonal to hH^2. Hence e^z is orthogonal to h_iH^2 where h_i is the inner part of h. Equivalently e^z is in $H^2\ominus h_iH^2$. Since e^z is cyclic for U_+^* and $H^2\ominus h_iH^2$ is invariant for U_+^* we have $h=0$. Therefore $e^zh^*=0$ and (4.5) holds.

Equation (4.5) gives $\mathfrak{P}_+=L^2$. Let ψ be any nonconstant inner function. Let $\mathfrak{G}=\psi^*H^2$. Then $H^2=\mathfrak{F}_+\subseteqq\mathfrak{G}\subseteqq\mathfrak{P}_+=L^2$. Consulting [5] implies ψe^z is also cyclic for U_+^*. A calculation gives:

$$\mathfrak{X} = \overline{P_{\mathfrak{G}}\mathfrak{H}_-} = \overline{P_{\psi^*H^2}e^zK^2} = \psi^*(\overline{P_{H^2}\psi e^zK^2}) = \psi^*\big[\bigvee_{n\geqq 0} U_+^{*n}(\psi e^z)\big] = \psi^*H^2.$$

Hence $\mathfrak{X}=\mathfrak{G}=\psi^*H^2$. Obviously $\mathfrak{X}$ is not a minimal state space. It strictly contains the minimal state space $\mathfrak{F}=H^2$. The example is now complete.

Remark 4.1. In Example 4.1 the space $\mathfrak{H}$ only admits two minimal state spaces $\mathfrak{P}$ and $\mathfrak{F}$.

Classifications of all minimal state spaces are given in [9, 10, 13, 14] and elsewhere. It was shown in [11] that the proofs given there were not correct. In fact, Example 4.1 can be used to demonstrate that these results are not valid for certain infinite dimensional vector spaces. Recently [11] corrected some of the proofs in [9, 10, 13, 14] and showed that the classification of all minimal state spaces in [9, 10, 13, 14] (and elsewhere) were indeed valid for certain $\mathfrak{H}$ and U. It turns out that the results in [11] are equivalent to Corollary 4.2 part (v). However the methods in [11] do not extend to the general case Theorem 4.2. Here we have shown that property (P) plays an important role in obtaining all state spaces. Property (P) also plays an important role in deterministic systems theory [6] and other problems in operator theory [3, 21].

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