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Analytic operator valued functions with prescribed local data

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Dedicated to B. Sz.-Nagy on the occasion of his seventieth birthday

This paper contains the operator generalization of the classical theorems of Mittag—Leffler and Weierstrass concerning construction of an analytic function with given local data which does not have additional singularities. The obtained results generalize earlier results of the authors on the finite dimensional case.

1. Introduction and main results

Let Ω be a domain in the complex plane $\mathbf{C}$. Consider the class Φ of all operator valued functions of the form $A(\lambda)=I+K(\lambda)$, $\lambda\in\Omega$, where $K(\lambda)$ is an analytic (in Ω) operator valued function whose values are compact operators acting in a Banach space B, with the additional property that at least one value of $A(\lambda)$ is an invertible operator. In particular, for every $A(\lambda)\in\Phi$ the spectrum $\sigma(A)=\{\lambda\in\Omega\,|\,A(\lambda)$ is not invertible$\}$ consists of isolated points in Ω. For any of these points $\lambda_0\in\sigma(A)$, in its deleted neighborhood the function $A(\lambda)^{-1}$ admits the form

$$A(\lambda)^{-1}=\sum_{j=-s}^{\infty}(\lambda-\lambda_0)^j M_j, \tag{1.1}$$

where $s>1$ is an integer and the operators $M_{-s}, M_{-s+1}, \dots, M_{-1}$ are finite dimensional (see [7]). Denote by $\mathrm{SP}\, A^{-1}(\lambda_0)$ the singular part $\sum_{j=-s}^{-1}(\lambda-\lambda_0)^i M_j$ of the Laurent series (1.1).

In this paper we shall solve the following problem: construct a function $A(\lambda)\in\Phi$ given its spectrum and the singular parts at each point of spectrum. The solution of this problem is given by the next theorem which is the main result.

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Theorem 1.1. *Let $\lambda_1, \lambda_2, \ldots$ be a sequence (finite or infinite) of different points in a domain $\Omega \subset \mathbf{C}$ with limit points (if any) on the boundary Γ of Ω. For each λ_i, $i=1, 2, \ldots$, let be given a rational operator function of the form*

$$M_i(\lambda) = \sum_{j=1}^{k_i} (\lambda - \lambda_i)^{-j} M_{ij}, \quad i = 1, 2, \ldots,$$

where M_{ij} are finite dimensional operators acting in B. Then there exists an analytic operator function $A(\lambda) \in \Phi$ such that $\sigma(A) = \{\lambda_1, \lambda_2, \ldots\}$ and $\mathrm{SP}\, A^{-1}(\lambda_i) = M_i(\lambda)$, $i=1, 2, \ldots$. Moreover, $A(\lambda)$ can be chosen so that $A(\lambda) - I \in \Sigma$ for every $\lambda \in \Omega$, where Σ is the algebra of all operators acting in B which are limits (in norm) of finite dimensional operators.

Theorem 1.1 is a generalization of Theorem 4.4 from [5], which in turn may be regarded as a generalization of the classical Mittag—Leffler theorem. The proof of Theorem 1.1 is given in the next section. It uses a theorem on triviality of cocycles (see [1, 4]). Note that using this theorem it is not difficult to construct a meromorphic function with given singular parts of Laurent series, as in Theorem 1.1. However, it requires additional work to ensure that this meromorphic function is the inverse of an analytic function, and this is the bulk of the proof of Theorem 1.1.

In the course of the proof of Theorem 1.1 we obtain also the following far-reaching generalization of Weierstrass' theorem (which states the existence of a scalar analytic function with prescribed zeros and prescribed multiplicities).

Theorem 1.2. *Let $\lambda_1, \lambda_2, \ldots$ be a sequence (finite or infinite) of different points in a domain $\Omega \subset \mathbf{C}$ with limit points (if any) on the boundary Γ of Ω. For every λ_j, let be given an operator polynomial of the form $P_j(\lambda) = I + \sum_{i=0}^{k_j} \lambda^i P_{ij}$, where P_{ij} are finite dimensional operators, such that $\sigma(P_j) = \{\lambda_j\}$. Then there exists an analytic (in Ω) operator valued function $A(\lambda)$ such that $A(\lambda) - I \in \Sigma$ for all $\lambda \in \Omega$, $\sigma(A) = \{\lambda_1, \lambda_2, \ldots\}$, and for every $j = 1, 2, \ldots$ the quotient $A(\lambda) P_j(\lambda)^{-1}$ is analytic and invertible at λ_j.*

As before, Σ stands for the algebra of norm limits of finite dimensional operators.

The above mentioned Weierstrass' theorem is obtained by taking $B = \mathbf{C}$ and $P_j(\lambda) = (\lambda - \lambda_j)^{k_j}$ in Theorem 1.2. Theorem 1.2 will be deduced from Theorem 1.1. The finite dimensional version of Theorem 1.2 was proved in [5].

Observe that in Theorem 1.2 one could replace the condition that P_{ij} are finite dimensional by the compactness of P_{ij}. Indeed, a polynomial $T(\lambda) = I + \sum_{i=0}^{k} \lambda^i T_i$

with compact T_i and $\sigma(T)=\{\lambda_0\}$ can be factored as follows:

$$T(\lambda) = T_0(\lambda)\left(I+\sum_{i=0}^{l}\lambda^i P_i\right)$$

with finite dimensional P_i and everywhere invertible $T_0(\lambda)$ (see Theorems 3.1 and 3.2 below). This allows us to reduce the problem to the case considered in Theorem 1.2.

2. Auxiliary results

In this section we shall prove Lemma 2.1 which will be used in the proof of Theorem 1.1, and is also of independent interest. As in Theorem 1.1, Ω stands for a domain in $\mathbf{C}$ with boundary Γ, and Σ denotes the algebra of all norm limits of finite dimensional operators acting in the Banach space B.

Lemma 2.1. *Let $\lambda_1, \lambda_2, \dots$ be a sequence (finite or infinite) of points in Ω with limit points (if any) in Γ. Let $Y_{j0}, Y_{j1}, \dots, Y_{j,k_{j-1}}$, $j=1, 2, \dots$, be given operators from Σ. Then there exists an analytic operator valued function $Y(\lambda)$ $(\lambda\in\Omega)$ with $Y(\lambda)-I\in\Sigma$ for all $\lambda\in\Omega$ and such that*

$$Y(\lambda_j) = I+Y_{j0}, \quad j = 1, 2, \dots \tag{2.1}$$

and

$$Y^{(k)}(\lambda_j) = Y_{jk}, \quad k = 1, \dots, k_j-1, \quad j = 1, 2, \dots. \tag{2.2}$$

If, in addition, $I+Y_{j0}$ is invertible for all $j=1, 2, \dots$, then the analytic operator function $Y(\lambda)$ can be chosen with the additional property that $Y(\lambda)$ is invertible for all $\lambda\in\Omega$.

We need some preparations for the proof of Lemma 2.1. A set $M\subset\mathbf{C}$ is called *finitely connected* if M is connected and $\mathbf{C}\setminus M$ consists of a finite number of connected components. We shall use later the fact that there is a sequence of finitely connected compacts $\Omega_1'\subset\Omega_2'\subset\dots$ such that $\Omega=\bigcup_{i=1}^{\infty}\Omega_i'$. The proof of this fact is not difficult and can be found in [4], Lemma 2.1.

The following lemma can be viewed as a local analogue of Lemma 2.1.

Lemma 2.2. *Let $\lambda_0\in\Omega$, and let Ω_0 be a finitely connected compact in Ω such that λ_0 lies in the unbounded component of $\mathbf{C}\setminus\Omega_0$. Let $X_0, X_1, \dots, X_k$ be given operators from the algebra Σ. Then for every $\varepsilon>0$ there exists an analytic (in Ω) operator valued function $L(\lambda)$ such that $L(\lambda)\in\Sigma$ for all $\lambda\in\Omega$, and*

(i) $L^{(i)}(\lambda_0) = X_i, \quad i = 0, \dots, k,$

(ii) $\|L(\lambda)\| \leqq \varepsilon$ *for* $\lambda\in\Omega_0$.

Proof. Put $M(\lambda)=\sum_{i=0}^{k}\frac{1}{i!}X_i(\lambda-\lambda_0)^i$, then $M^{(i)}(\lambda_0)=X_i$, $i=0, \dots, k$, and $M(\lambda)\in\Sigma$ for all $\lambda\in\Omega$. Let $\alpha=\max_{\lambda\in\Omega_0}\|M(\lambda)\|$. There exists a scalar polynomial $\varphi(\lambda)$ such that

$$|\varphi(\lambda)|\leqq\varepsilon\alpha^{-1} \quad \text{for} \quad \lambda\in\Omega_0, \tag{2.3}$$

$$\varphi^{(i)}(\lambda_0)=\begin{cases}1, & i=0\\ 0, & i=1, \dots, k.\end{cases} \tag{2.4}$$

Indeed, we seek for $\varphi(\lambda)$ in the form

$$\varphi(\lambda)=1-\psi(\lambda)(\lambda-\lambda_0)^k. \tag{2.5}$$

Let $\Omega_0'\subset\mathbf{C}$ be a simply connected compact such that $\Omega_0\subset\Omega_0'$ and $\lambda_0\notin\Omega_0'$. By Runge's theorem, there exists a polynomial $\psi(\lambda)$ with $\psi(\lambda_0)=0$ and $|\psi(\lambda)-(\lambda-\lambda_0)^{-k}|<\varepsilon\alpha^{-1}\beta^{-1}$, $\lambda\in\Omega_0'$, where $\beta=\max_{\lambda\in\Omega_0'}\{|\lambda-\lambda_0|^k\}$. With this $\psi(\lambda)$ in (2.5), the conditions (2.3) and (2.4) hold true. Now put $L(\lambda)=\varphi(\lambda)M(\lambda)$ to satisfy (i) and (ii).

Proof of Lemma 2.1. We shall seek for $Y(\lambda)$ in the form of an infinite product

$$Y(\lambda)=\prod_{j=1}^{\infty}(I+L_j(\lambda)). \tag{2.6}$$

Choose a non-decreasing sequence $\Omega_1'\subset\Omega_2'\subset\dots$ of finitely connected compacts whose union is Ω, and such that $\lambda_i\in\Omega_j'$ for $i=1, 2, \dots, j-1$, but λ_j lies in the unbounded component of $\mathbf{C}\setminus\Omega_j'$. Let $\varphi_j(\lambda)$ be a scalar function analytic in Ω with the following properties: $\varphi_j(\lambda_j)=1$; $\varphi_j^{(k)}(\lambda_j)=0$ for $k=1, \dots, k_j-1$, $\varphi_j^{(k)}(\lambda_i)=0$ for $k=0, \dots, k_j-1$ and $i\neq j$. Put $\alpha_j=\max_{\lambda\in\Omega_j'}|\varphi_j(\lambda)|$. By Lemma 2.2 there exist analytic operator functions (even operator polynomials) $M_j(\lambda)$, $j=1, 2, \dots$, such that $M_j(\lambda)\in\Sigma$ for all $\lambda\in\Omega$ and

$$M_j^{(k)}(\lambda_j)=Y_{jk},\ k=0, \dots, k_j-1,\ \|M_j(\lambda)\|\leqq\varepsilon_j\alpha_j^{-1} \quad \text{for} \quad \lambda\in\Omega_j',$$

where ε_j is any sequence of positive numbers for which the product $\prod_{k=1}^{\infty}(1+\varepsilon_k)$ converges. Define $L_j(\lambda)=\varphi_j(\lambda)M_j(\lambda)$, $j=1, 2, \dots$; with this definition of $L_j(\lambda)$ the product (2.6) converges uniformly in every Ω_j', and consequently $Y(\lambda)$ is analytic in Ω. Moreover, $L_j^{(k)}(\lambda_j)=Y_{jk}$ for $k=0, \dots, k_j-1$, and $L_j^{(k)}(\lambda_i)=0$ for $k=0, \dots, k_j-1$ and $i\neq j$. Consequently, equalities (2.1) and (2.2) are satisfied.

Suppose now that $I+Y_{j0}$ is invertible for all $j=1, 2, \dots$. In this case we shall look for $Y(\lambda)$ in the form

$$Y(\lambda)=\exp(X_1(\lambda))\cdot\exp(X_2(\lambda))\cdot\exp(X_3(\lambda))\cdot\dots. \tag{2.7}$$

We shall construct the operator functions $X_m(\lambda)$ by induction on m.

Choose a sequence of finitely connected compacts $\Omega_1' \subset \Omega_2' \subset \dots$ with $\bigcup_{i=1}^{\infty} \Omega_i' = \Omega$, and denote by $\Xi_{m1}, \dots, \Xi_{m,p_m}$ the bounded connected components of $\mathbf{C} \setminus \Omega_m'$. We shall assume that each Ξ_{mp}, $p=1, \dots, p_m$, contains a point μ_{mp} not belonging to Ω (otherwise consider $\Omega_m' \cup \Xi_{mp}$ in place of Ω_m'). We shall assume also that $k_i < m$ for every i such that $\lambda_i \in \Omega_m'$ (this can be arranged because the set of points $\lambda_i \in \Omega_m'$ with $k_i \geqq m$ is either empty or finite).

We construct now $X_1(\lambda)$. Put $X_{i1} = \ln(I + Y_{i0})$, where the branch of the logarithm is chosen so that $\ln 1 = 0$; then $X_{i1} \in \Sigma$. Let $\tilde{X}_1(\lambda)$ be a Σ-valued analytic function such that

$$\tilde{X}_1(\lambda_i) = X_{i1}, \quad i = 1, 2, \dots$$

(such $\tilde{X}_1(\lambda)$ exists in view of the already proved part of Lemma 2.1). Let $\varphi_{10}(\lambda)$ be an analytic (in Ω) scalar function with only zeros at $\{\lambda_1, \lambda_2, \dots\}$, which are simple. In particular the function $-\varphi_{10}(\lambda)^{-1}$ is analytic in Ω_1', therefore there exists a rational function $\psi_1(\lambda)$ with poles (if any) in $\mu_{11}, \dots, \mu_{1,p_1}$ such that

$$\max_{\lambda \in \Omega_1'} |\psi_1(\lambda) + \varphi_{10}(\lambda)^{-1}| \leqq \ln(1+\varepsilon_1)\{\max_{\lambda \in \Omega_1'} |\varphi_{10}(\lambda)|\}^{-1} \{\max_{\lambda \in \Omega_1'} \|\tilde{X}_1(\lambda)\|\}^{-1}.$$

Put $X_1(\lambda) = (1 + \psi_1(\lambda)\varphi_{10}(\lambda))\tilde{X}_1(\lambda)$. Then $X_1(\lambda_i) = X_{i1}$, $i = 1, 2, \dots$, and $\|X_1(\lambda)\| \leqq \leqq \delta_1 = \ln(1+\varepsilon_1)$ for $\lambda \in \Omega_1'$. Here $\delta_1 > \delta_2 > \dots$ is a sequence of positive numbers chosen in advance.

Suppose Σ-valued analytic functions $X_1(\lambda), \dots, X_n(\lambda)$ are already constructed, with the following properties for $j = 1, \dots, n$:

$$X_j^{(k)}(\lambda_i) = 0, \quad k = 0, \dots, j-2, \quad i = 1, 2, \dots; \tag{2.8}$$

$$Y_{i,j-1} = [\exp(X_1(\lambda)) \dots \exp(X_j(\lambda))]_{\lambda=\lambda_i}^{(j-1)}, \quad k_i \geqq j; \tag{2.9}$$

$$\|X_j(\lambda)\| \leqq \delta_j \quad \text{for} \quad \lambda \in \Omega_j'. \tag{2.10}$$

(For $j=1$, replace Y_{i0} in (2.9) by $I + Y_{i0}$.) By the already proved part of Lemma 2.1, there exists a Σ-valued analytic function $\tilde{X}_{n+1}(\lambda)$ in Ω such that $\tilde{X}_{n+1}^{(k)}(\lambda_i) = 0$ for $k = 0, \dots, n-1$; and if $k_i \geqq n+1$, then $\tilde{X}_{n+1}^{(n)}(\lambda_i) = X_{i,n+1}$ $(i = 1, 2, \dots)$, where the operators $X_{i,n+1} \in \Sigma$ are chosen in such a way that

$$Y_{in} = [\exp(X_1(\lambda)) \exp(X_2(\lambda)) \dots (\exp(X_n(\lambda)) (\exp(\tilde{X}_{n+1}(\lambda)))]_{\lambda=\lambda_i}^{(n)}$$

for every λ_i with $k_i \geqq n+1$. A computation (using (2.8)) shows that one can put

$$X_{i,n+1} = [\exp(X_1(\lambda_i)) \dots \exp(X_n(\lambda_i))]^{-1} \cdot$$
$$\cdot \{Y_{in} - \sum_{\substack{\alpha + \dots + \alpha_n = n \\ \alpha_i \geqq 0}} [\exp(X_1(\lambda))]_{\lambda=\lambda_i}^{(\alpha_1)} \dots [\exp(X_n(\lambda))]_{\lambda=\lambda_i}^{(\alpha_n)}\}.$$

Now put $X_{n+1}(\lambda) = \varphi_{n+1}(\lambda)\tilde{X}(\lambda)$ where $\varphi_{n+1}(\lambda)$ is suitably chosen (as in the construction of $X_1(\lambda)$) scalar analytic function so that (2.8)—(2.10) hold also for $j = n+1$.

The condition (2.10) ensures uniform convergence of the infinite product (2.7) in every compact set in Ω, provided δ_n are chosen to tend sufficiently fast to zero. Equalities (2.8) and (2.9) ensure that $Y(\lambda)$ defined by (2.7) satisfies (2.1) and (2.2). Finally, for $\lambda \in \Omega_n'$ we have

$$\|\exp(X_n(\lambda))^{-1} - I\| \leqq \|I - \exp(X_n(\lambda))\| \cdot \|\exp(X_n(\lambda))\| \leqq (e^{\delta_n} - 1)e^{\delta_n},$$

and (assuming δ_n tend sufficiently fast to zero) the infinite product $\exp(X_n(\lambda))^{-1} \cdot \exp(X_{n-1}(\lambda))^{-1} \dots \exp(X_1(\lambda))^{-1}$ converges (necessarily to $Y(\lambda)^{-1}$) uniformly in every compact set in Ω. So $Y(\lambda)$ is invertible.

Lemma 2.1 is proved completely.

Note that Lemmas 2.1 and 2.2 remain true if the algebra Σ is replaced by the algebra of all compact operators.

3. Local spectral data of an analytic operator function

In this section we shall present some facts about local spectral data of an analytic function $A(\lambda) \in \Phi$ which are relevant to the proof of Theorem 1.1. As for the case of matrix functions (see [5]), one can define the local spectral data of $A(\lambda) \in \Phi$ in several forms: one-sided (right and left eigenpairs, local divisors and singular subspaces) and two-sided (the singular part of the Laurent expansion of $A(\lambda)^{-1}$). The results concerning the relationship between the various kinds of spectral data are the same as in the finite dimensional case, with essentially the same proofs (see [5] for details). So we shall focus on the kinds of spectral data which will be used in the proof of Theorem 1.1.

Let $A(\lambda) \in \Phi$, and let $A(\lambda)^{-1} = \sum_{j=-s}^{\infty} (\lambda - \lambda_0)^j M_j$ be the Laurent series of $A(\lambda)$ in a deleted neighbourhood of $\lambda_0 \in \sigma(A)$. The finite dimensional subspace

$$\operatorname{Im} \begin{bmatrix} M_{-s} & 0 & \dots 0 \\ M_{-s+1} & M_{-s} & \dots 0 \\ \vdots & \vdots & \vdots\ \vdots \\ M_{-1} & M_{-2} & \dots M_{-s} \end{bmatrix} \subset B^s$$

is called the (right) *singular subspace* of $A(\lambda)$ at λ_0. (Sometimes in this definition it is convenient to consider $A(\lambda)^{-1} = \sum_{j=-s'}^{\infty} (\lambda - \lambda_0)^j$ with $s' > s$ and $M_{-s'} = \dots \dots = M_{-s+1} = 0$; so the singular subspace becomes a subspace in $B^{s'}$.) An operator polynomial of the form $P(\lambda) = I + \sum_{i=0}^{k} \lambda^i P_i$, where P_i are finite dimensional operators, is called a (right) *local divisor* of $A(\lambda)$ at λ_0, if $\sigma(P) = \{\lambda_0\}$ and the operator function $A(\lambda)P(\lambda)^{-1}$ is analytic and invertible at λ_0. The local divisor of $A(\lambda)$

at λ_0 is unique up to multiplication from the left by an everywhere invertible operator polynomial $S(\lambda)$ such that $S(\lambda)-I$ is finite dimensional for all λ.

The next result provides the relationships between singular subspaces and local divisors. We need the following definition of a special left inverse (cf. [6]). Let B_f be a finite dimensional vector space, and let $Z: B_f \to B$, $T: B_f \to B_f$ be linear operators such that for some s the operator

$$Q_s(Z, T) \stackrel{\text{def}}{=} \begin{bmatrix} Z \\ ZT \\ \vdots \\ ZT^{s-1} \end{bmatrix} : B_f \to B^s$$

is left invertible. A left inverse of $Q_s(Z, T)$ is called *special* if its kernel is of the form $\{(x_1, \dots, x_s) \in B^s \mid x_i \in W_i,\ i=1, \dots, s\}$, where $W_1 \supset W_2 \supset \dots \supset W_s$ is a non-increasing sequence of (closed) subspaces in B. If T is invertible, a special left inverse always exists. Indeed, since $\dim B_f < \infty$, one works in the proof of existence of a special left inverse with subspaces which have a finite dimensional complement, and then the proof given in the finite dimensional case ($\dim B < \infty$) applies (see Lemma 2.1 in [6]).

Theorem 3.1. *The singular subspace R of an analytic operator function $A(\lambda) \in \Phi$ at $\lambda_0 \neq 0$ determines a local divisor $P(\lambda)$ of $A(\lambda)$ at λ_0 by the formula*

$$P(\lambda) = I - ZT^{-s}(V_1\lambda^s + V_2\lambda^{s-1} + \dots + V_s\lambda),$$

where $Z: R \to B$ is the projector on the last coordinate in $R \subset B^s$, $T: R \to R$ is defined by the formula $T(x_1, \dots, x_s) = (\lambda_0 x_1, \lambda_0 x_2 + x_1, \dots, \lambda_0 x_s + x_{s-1})$, $(x_1, \dots, x_s) \in R$, and $[V_1 \dots V_s]$ is a special left inverse of

$$\begin{bmatrix} Z \\ ZT^{-1} \\ \vdots \\ ZT^{-(s-1)} \end{bmatrix}.$$

Conversely, if $P(\lambda)$ is a local divisor of $A(\lambda) \in \Phi$ at λ_0, then

$$\text{(3.1)} \qquad \operatorname{Ker} \begin{bmatrix} P(\lambda_0) & 0 & \dots 0 \\ P'(\lambda_0) & P(\lambda_0) & \dots 0 \\ \vdots & \vdots & \vdots \\ \frac{1}{(s-1)!} P^{(s-1)}(\lambda_0) & \frac{1}{(s-2)!} P^{(s-2)}(\lambda_0) & \dots P(\lambda_0) \end{bmatrix}$$

is the singular subspace of $A(\lambda)$ at λ_0.

The proof of Theorem 3.1 is the same as the proof of Theorem 2.4 in [5]. Note that T is invertible in view of the condition $\lambda_0 \neq 0$. Note also that T maps R into R, as easily seen from the definition of the singular subspace R. The case $\lambda_0 = 0$ can be easily reduced to the case $\lambda_0 \neq 0$ by considering $A(\lambda + a)$ in place of $A(\lambda)$, for some $a \in \mathbf{C} \setminus \{0\}$.

The number s in (3.1), which is the least positive integer such that $(\lambda - \lambda_0)^s A(\lambda)^{-1}$ is analytic at λ_0, is determined by the local divisor $P(\lambda)$ as follows:

$$s = \min \{ j > 0 \mid \dim \operatorname{Ker} \mathcal{N}_j = \dim \operatorname{Ker} \mathcal{N}_{j+1} \}$$

where $\mathcal{N}_j$ is the matrix in (3.1) with j in place of s.

A function $A(\lambda) \in \Phi$ is a *right divisor* of a function $B(\lambda) \in \Phi$ if $B(\lambda) = C(\lambda) A(\lambda)$ for some $C(\lambda) \in \Phi$. The description of divisibility in terms of singular subspaces and local divisors is given by the following Theorem, the proof of which is analogous to the proof of Theorems 1.4 and 2.5 in [5].

Theorem 3.2. *The following statements are equivalent:*

(i) *$A(\lambda) \in \Phi$ is a right divisor of $B(\lambda) \in \Phi$;*

(ii) *$\sigma(A) \subset \sigma(B)$, and for any $\lambda_0 \in \sigma(A)$ the local divisor of $A(\lambda)$ is in turn a right divisor of a local divisor of $B(\lambda)$ at λ_0;*

(iii) *$\sigma(A) \subset \sigma(B)$, and the singular subspace of $A(\lambda)$ at any $\lambda_0 \in \sigma(A)$ is contained in the singular subspace of $B(\lambda)$ at λ_0.*

In particular, $A(\lambda) \in \Phi$ and $B(\lambda) \in \Phi$ are right divisors of each other if and only if $\sigma(A) = \sigma(B)$, and $A(\lambda)$ and $B(\lambda)$ have the same local divisors at each $\lambda_0 \in \sigma(A)$, or equivalently, if the singular subspaces of $A(\lambda)$ and $B(\lambda)$ at each $\lambda_0 \in \sigma(A) = \sigma(B)$ coincide.

Let us remark (this remark will not be used in the proof of Theorem 1.1) that Theorems 3.1 and 3.2 can be also stated in terms of Jordan chains of a function $A(\lambda) \in \Phi$ corresponding to λ_0. By definition, the vectors $y_0, \ldots, y_{k_0 - 1} \in B$ form a Jordan chain of $A(\lambda)$ corresponding to λ_0 if $\sum_{i=0}^{k} \frac{1}{i!} A^{(i)}(\lambda_0) y_{k-i} = 0, k = 0, \ldots, k_0 - 1$. The q Jordan chains $y_0^{(j)}, \ldots, y_{k_j - 1}^{(j)}, j = 1, \ldots, q$, of $A(\lambda)$ corresponding to λ_0 are said to be a *canonical set* if the eigenvectors $y_0^{(1)}, \ldots, y_0^{(q)}$ are linearly independent and the sum $\sum_{p=1}^{q} k_p$ is maximal possible (cf. [7]). Every Jordan chain $y_0, \ldots, y_{k_0 - 1}$ of $A(\lambda)$ corresponding to λ_0 is a linear combination of the canonical set: namely,

$$y_m = \sum_{j=1}^{q} \alpha_j y_m^{(j)}, \quad m = 0, \ldots, k_0 - 1, \quad \text{for some} \quad \alpha_j \in \mathbf{C}.$$

As in the finite dimensional case (see Theorem 2.4 in [5]) one can prove that each of the three local characteristics of an analytic function $A(\lambda) \in \Phi$ at λ_0 — singular subspace, local divisor, canonical set of Jordan chains — determines the other two. In fact, a canonical set of Jordan chains appeared implicitly in Theorem 3.1. Namely, there exists an invertible operator $S\colon R\to\mathbf{C}^r$ such that (in the notation of Theorem 3.1) the operators $ZS^{-1}\colon \mathbf{C}^r\to B$ and $STS^{-1}\colon \mathbf{C}^r\to\mathbf{C}^r$ have the following structure, in the standard orthonormal basis in $\mathbf{C}^r$:

$$ZS^{-1}=[y_0^{(1)}\dots y_{k_1-1}^{(1)}y_0^{(2)}\dots y_{k_2-1}^{(2)}\dots y_0^{(q)}\dots y_{k_q-1}^{(q)}],$$

where $y_0^{(j)},\dots,y_{k_j-1}^{(j)}$, $j=1,\dots,q$, is a canonical set of Jordan chains of $A(\lambda)$ corresponding to λ_0; $STS^{-1}=J_1\oplus\dots\oplus J_q$, where J_i is the Jordan block of size $k_i\times k_i$ with eigenvalue λ_0.

4. Proof of the main theorem

The following result, which will be used in the proof of Theorem 1.1, is a particular case of Theorem 2.1 in [4], see also [1], and may be regarded as a theorem on triviality of cocycles. Given a compact set $\Omega_0\subset\Omega$, we denote by $\mathrm{GL}_s(\Omega_0)$ the set of all operator valued functions $G(\lambda)$ which are analytic and invertible in some neighborhood U_G of Ω_0 (the neighbourhood depending on the function) and such that $G(\lambda)-I\in\Sigma$ for every $\lambda\in U_G$.

Proposition 4.1. *There exists a sequence of compacts* $\Omega_1\subset\Omega_2\subset\dots\subset\Omega$, $\bigcup_{i=1}^{\infty}\Omega_i=\Omega$ *with the following property: For every sequence of analytic operator functions* $G_m(\lambda)\in\mathrm{GL}_s(\Omega_m)$, $m=1,2,\dots,$ *there exists a sequence* $D_m(\lambda)\in\mathrm{GL}_s(\Omega_m)$ *such that*

$$G_m(\lambda)=(D_{m+1}(\lambda))^{-1}D_m(\lambda),\quad \lambda\in\Omega_m,\quad m=1,2,\dots. \tag{4.1}$$

We are ready now to prove Theorem 1.1.

Proof of Theorem 1.1. We shall break the proof into two steps.

a) Let $R_{k_i}\subset B^{k_i}$ be the singular subspace determined by $M_i(\lambda)$:

$$R_{k_i}=\mathrm{Im}\begin{bmatrix} M_{i,k_i} & 0 & \dots & 0\\ M_{i,k_i-1} & M_{i,k_i} & \dots & 0\\ \vdots & \vdots & & \vdots\\ M_{i1} & M_{i2} & \dots & M_{i,k_i}\end{bmatrix}.$$

We shall construct first an analytic operator function $\tilde A(\lambda)$ with $\sigma(\tilde A)=\{\lambda_1,\lambda_2,\dots\}$ and corresponding singular subspaces $R_{k_1},R_{k_2},\dots,$ and such that $\tilde A(\lambda)-I\in\Sigma$ for every $\lambda\in\Omega$.

Let $\Omega_1\subset\Omega_2\subset\dots$ be the sequence of compacts as in Proposition 4.1. Observe that each Ω_m contains only a finite number of λ_i's; let S_m be the (finite) set of indices i such that $\lambda_i\in\Omega_m$, $m=1,2,\dots$. It is not difficult to see that there exists a finite dimensional subspace $B_m \subset B$ and a direct complement B'_m to B_m in B such that $M_{ij}B_m \subset B_m$ and $M_{ij}B'_m = 0$ for $j=1, \dots, k_i$ and $i \in S_m$. Using Theorem 4.1 of [5], for every $m=1, 2, \dots$ construct an analytic (in Ω) operator function of the form $A_m(\lambda) = I + K_m(\lambda)$; where $K_m(\lambda)B_m \subset B_m$ and $K_m(\lambda)B'_m = 0$ for every $\lambda \in \Omega$, such that $\sigma(A_m(\lambda)) = \{\lambda_i \mid i \in S_m\}$, and R_{k_i} is the singular subspace of $A_m(\lambda)$ corresponding to λ_i, for every $i \in S_m$. Let $G_m(\lambda) = A_{m+1}(\lambda)(A_m(\lambda))^{-1}$; Theorem 3.2 ensures that $G_m(\lambda)$ is invertible in Ω_m, $m=1, 2, \dots$. Applying Proposition 4.1, find a sequence $D_m(\lambda) \in \mathrm{GL}_s(\Omega_m)$, $m=1, 2, \dots$, with the property (4.1). Then $D_m(\lambda)A_m(\lambda) = D_{m+1}(\lambda)A_{m+1}(\lambda)$, $\lambda \in \Omega_m$; so in fact the function $\tilde{A}(\lambda) = D_m(\lambda)A_m(\lambda)$ $(\lambda \in \Omega_m)$ is defined and analytic in Ω. Clearly, $\tilde{A}(\lambda) - I \in \Sigma$ for every $\lambda \in \Omega$; moreover, $\sigma(\tilde{A}) = \{\lambda_1, \lambda_2, \dots\}$ with corresponding singular subspaces $R_{k_1}, R_{k_2} \dots$.

b) We construct $A(\lambda)$ in the form $A(\lambda) = X(\lambda)\tilde{A}(\lambda)$; where $X(\lambda)$ is an everywhere invertible analytic (in Ω) operator function such that $X(\lambda) - I \in \Sigma$ for all $\lambda \in \Omega$, and $\tilde{A}(\lambda)$ is the operator valued function constructed in the part a).

For a fixed λ_i, there exists an operator function $\tilde{A}_i(\lambda)$, analytic in a neighborhood U_i of λ_i, with the properties that $\tilde{A}_i(\lambda) - I$ is finite dimensional for $\lambda \in U_i$ and $\mathrm{SP}\, \tilde{A}_i^{-1}(\lambda_i) = \sum_{j=1}^{k_i} (\lambda - \lambda_i)^{-j} M_{ij}$ (see Theorem 4.4 in [5]). Then the singular subspaces of $\tilde{A}(\lambda)$ and $\tilde{A}_i(\lambda)$ corresponding to λ_i coincide. By Theorem 3.2, the operator function $Z_i(\lambda) \stackrel{\text{def}}{=} \tilde{A}_i(\lambda)A(\lambda)^{-1}$ is analytic and invertible in U_i. Moreover, $Z_i(\lambda) - I \in \Sigma$; $\lambda \in U_i$. Write $Z_i(\lambda) = \sum_{j=0}^{\infty} (\lambda - \lambda_i)^j Z_{ij}$, and let $X(\lambda)$ be an everywhere invertible analytic (in Ω) operator function such that $X(\lambda) - I \in \Sigma$, $\lambda \in \Omega$, and

$$\frac{1}{j!} X^{(j)}(\lambda_i) = Z_{ij}, \quad j = 0, \dots, k_i, \quad i = 1, 2, \dots .$$

The existence of such $X(\lambda)$ is ensured by Lemma 2.1. Now put $A(\lambda) = X(\lambda)\tilde{A}(\lambda)$.

Let us check that the requirements of Theorem 1.1 are satisfied with this choice of $A(\lambda)$. Indeed,

$$A(\lambda) - I = X(\lambda)(\tilde{A}(\lambda) - I) + X(\lambda) - I \in \Sigma,$$

for all $\lambda \in \Omega$. For every λ_i, in a neighbourhood of λ_i we have

$$A(\lambda) = X(\lambda)\tilde{A}(\lambda) = X(\lambda)(Z_i(\lambda))^{-1} \cdot Z_i(\lambda)\tilde{A}(\lambda).$$

Now because of the choice of $X(\lambda)$ we obtain $Z_i(\lambda)(X(\lambda))^{-1} = (\lambda - \lambda_i)^{k_i+1} U_i(\lambda) + I$ for some operator function $U_i(\lambda)$ which is defined and is analytic in a neighborhood of λ_i. Hence

$$\mathrm{SP}\, A^{-1}(\lambda_i) = \sum_{j=1}^{k_i} (\lambda - \lambda_i)^{-j} M_{ij}, \quad i = 1, 2, \dots,$$

and Theorem 1.1 is proved completely.

Finally, observe that in view of Theorem 3.1, the part a) of the proof of Theorem 1.1 provides the proof for Theorem 1.2.

In conclusion let us remark that Theorem 1.2 can be stated also in terms of singular subspaces, as well as in terms of canonical set of Jordan chains, in the same way as in the finite dimensional case (see [5]). We state it in terms of canonical set of Jordan chains: Let $\lambda_1, \lambda_2, \ldots$ be a sequence as in Theorem 1.2, and for every λ_i, let be given a set of vectors in B:

$$y_{01}^{(i)}, \ldots, y_{k_{1i}-1,1}^{(i)};\ y_{02}^{(i)}, \ldots, y_{k_{2i}-1,2}^{(i)};\ \ldots;\ y_{0q}^{(i)}, \ldots, y_{k_{qi}-1,q}^{(i)} \tag{4.2}$$

(q depends on i) with linearly independent vectors $y_{01}^{(i)}, \ldots, y_{0q}^{(i)}$. Then there exists an analytic (in Ω) operator valued function $A(\lambda)$ such that $A(\lambda)-I\in\Sigma$ for all $\lambda\in\Omega$, $\sigma(A)=\{\lambda_1, \lambda_2, \ldots\}$, and for every $i=1, 2, \ldots$, the set (4.2) is a canonical set of Jordan chains of $A(\lambda)$ corresponding to λ_i. The proof of this statement is obtained immediately from Theorem 1.2, taking into account the remark at the end of Section 3.

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