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The RKNG (Rellich, Kato, Sz.-Nagy, Gustafson) perturbation theorem for linear operators in Hilbert and Banach space

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Dedicated to Professor Béla Szőkefalvi-Nagy on the occasion of his 70th Birthday 29 July 1983

In this paper I wish to discuss, including some hitherto unpublished comments, observations, and results, a fundamental perturbation theorem for linear operators, which I shall state as follows:

$$(*)\qquad \left.\begin{array}{l} A \text{ has a certain property and} \\ \|Bx\| \leqq a\|x\|+b\|Ax\|, \quad b<1 \end{array}\right\} \Rightarrow A+B \text{ has the same property.}$$

Two of the most important instances of this theorem are for the properties:

(1) selfadjointness in a Hilbert space;

(2) contraction semigroup generator in a Banach space.

The former is a special case of the latter.

Throughout I will assume, unless specified to the contrary, that B is a regular perturbation of A, that is, that $D(B)\supset D(A)$. In general, I will not consider form versions. Nor, except for a few comments, will I consider closure (e.g., the versions with $b=1$) versions.

1. Some history

Theorem $(*)$ for the selfadjointness property (1) usually goes by the name of the Rellich—Kato (sometimes, Kato—Rellich) Theorem. It is due originally to Lord Rayleigh and E. Schrödinger in their calculus of perturbations, which involved the formal assumption that an analytic perturbation $A(\varepsilon)$ of A yields analytic transformations of the eigenvalues and eigenvectors as well. A rigorous proof was first found by F. RELLICH [1]. The theorem was employed by KATO [2] in a fundamental way in an application to quantum mechanics.

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It is generally less well known that SZ.-NAGY [3] made an original contribution to this theorem (see the discussion below for more details). I made the extension to Banach space in [4], and with the apology that four letter theorems are now all the RAGE — you must know REED—SIMON [5] to understand this pun — I have here labeled it the RKNG Theorem.

I remember reading when I was young a newspaper account of an interview with a Nobel Prize winner, and his comment that (roughly) "we are all just bricklayers in the temple of science, some of us happen to arrive at the right time to turn the corners". Certainly that is also the case even with any theorem, be it four letter or not. Thus there are many many aspects of the RKNG Theorem that I will not elaborate on at all. Most of these related facts are available from the books REED—SIMON [5] and KATO [6] or from the recent literature, see for example the recent review by SOHR [7].

It should be pointed out that the initial workers and users of these theories were not primarily interested in the RKNG Theorem in its final "clean" form, but were instead motivated by its appearance in important problems in quantum physics, boundary values problems, and elsewhere.

Rigorous proofs of convergence eventually became of interest and Rellich managed to accomplish this in a series of papers [8] including the case of unbounded operators. For the case of bounded operators his results may be described as follows. For a convergent series depending on a real perturbation parameter ε,

$$A(\varepsilon) = A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \varepsilon^3 A_3 + \ldots,$$

$A(\varepsilon)$ will be a bounded selfadjoint operator with eigenvalues $\lambda(\varepsilon)$ depending on ε. If the unperturbed operator A_0 has an isolated eigenvalue $\lambda(0)$ of finite multiplicity, so will $A(\varepsilon)$, these will be close to $\lambda(0)$ and of the same multiplicity, provided that ε is small enough. Moreover the eigenvalues $\lambda(\varepsilon)$ and the associated eigenvectors can be expanded in series in terms of the eigenvalues of A and the A_i.

In particular, then, Rellich showed that if λ_0 is an isolated eigenvalue of finite multiplicity n of a selfadjoint transformation A_0, then there exists an ε-neighborhood of the origin and real valued functions $\lambda^{(1)}(\varepsilon), \ldots, \lambda^{(n)}(\varepsilon)$ and Hilbert space elements $f^{(1)}(\varepsilon), \ldots, f^{(n)}(\varepsilon)$ such that $\lambda^{(k)}(0)=\lambda_0$, $\lambda^{(k)}(\varepsilon)$ and $f^{(k)}(\varepsilon)$ are the eigenvalues and eigenvectors, respectively, of $A(\varepsilon)$. In addition (for all values of ε), the elements $f^{(k)}(\varepsilon)$ form an orthonormal sequence and, apart from the points $\lambda^{(k)}(\varepsilon)$, no member of the spectrum of $A(\varepsilon)$ lies in the neighborhood of λ_0.

This result applies only to isolated eigenvalues of finite multiplicity and is in general not true for those of infinite multiplicity. We shall return to this point in describing below Sz.-Nagy's results.

KATO [2, 9] employed this theory to establish the selfadjointness of the basic (Schrödinger) operators of quantum mechanics and to study their spectral properties.

This work is sufficiently extensive and well documented elsewhere in the physics and mathematics literature that I shall not attempt any comprehensive account here.

It is interesting, however, that one may illustrate and demonstrate this important scientific result for the case of the Hydrogen atom operator $H=-(1/2)\Delta-1/r$ quite readily from the fundamental theorem $(*)$ plus a basic inequality (Sobolev) of potential theory

$$\|u/r\| \leqq 2\|\text{grad } u\|,$$

the norm here being that of $L^2(R^3)$. For $A=-(1/2)\Delta$ it suffices to show that the perturbation $B=-1/r$ is relatively small with respect to A (that is what the norm inequality condition in $(*)$ is called). For functions u in $C_0^\infty(R^3)$ using the Sobolev inequality one has for any $\varepsilon>0$

$$\|u/r\| \leqq 2\|\text{grad } u\| \leqq \varepsilon^{-1}\|u\|+\varepsilon\|\Delta u\|$$

and thus the selfadjointness of the full Hamiltonian H from that of the bare Hamiltonian A. The selfadjointness of the latter is easily established by Fourier transform to a multiplication operator. We also used an integration by parts and the arithmetic-geometric mean inequality (the details may be found in [10]).

Finally I would like to mention that Kato employed $(*)$ (and the Sobolev inequality) in [2] not only for the nonrelativistic Schrödinger operators but also to establish the selfadjointness of the Dirac operator.

As mentioned above, although certainly known to specialists, it is not generally known that Sz.-Nagy [3, 11] made important contributions to this theory. Let us describe them here. The main result, in [3], Theorem I there, treats the behavior of an arbitrary, but isolated, part of a spectrum. Then in [3, Theorem III], Rellich's results for an isolated eigenvalue of finite multiplicity are reobtained. Also [3, Theorem II] estimates convergence conditions for the power series representation of the perturbed eigenvalues and eigenfunctions, the estimates obtained being somewhat sharper than those obtained by Rellich. Further, the approach of Rellich required the heavy function theory of Puiseux Series and the Weierstrass Vorbereitungsatz on zeros of a function of several variables.

In particular, Sz.-Nagy [3] showed the following. Let A_0 be selfadjoint and A_k, $k=1,2,3,\ldots$, be symmetric regular $(D(A_k)\supset D(A_0))$ perturbations satisfying

$$\|A_k f\| \leqq p^{k-1}(a\|f\|+b\|Af\|).$$

Then for $-p^{-1}<\varepsilon<p^{-1}$ the series $A(\varepsilon)\equiv A_0+\varepsilon A_1+\varepsilon^2 A_2+\ldots$ is selfadjoint for $-(p+b)^{-1}<\varepsilon<(p+b)^{-1}$. Moreover if an interval $\mu_1<\lambda<\mu_2$ contains an isolated part of A_0's spectrum, then the constantness of the spectral family $E_\lambda(\varepsilon)$ for $A(\varepsilon)$ follows from that of $E_\lambda(0)$ for A_0 for $-(p+a)^{-1}<\varepsilon<(p+b)^{-1}$ near the ends of the (μ_1,μ_2) interval, and in a neighborhood interior to the interval (μ_1,μ_2) the reduced perturbed operator $A(\varepsilon)[E_{\mu_2}(\varepsilon)-E_{\mu_1}(\varepsilon)]$ has power series expansion $\lambda_0(E_{\mu_2}(\varepsilon)-E_{\mu_1}(\varepsilon))+B_0+\varepsilon B_1+\varepsilon^2 B_2+\dots$ where the coefficients satisfy $\|B_k\| \leqq \frac{\mu_2+\mu_1}{2}\cdot$ $\cdot(\text{constant}_1)(p+\text{constant}_2)^{k-1}$.

Sz.-Nagy's approach in [3] uses the theory of the resolvent operator R_z and the associated spectral representations in the complex plane. In [11; first citation] he applies the theory to the ordinary differential equation boundary value problem $-py''=\lambda(1+\varepsilon\sigma)y$, $y(0)=y(l)=0$. In [11, second citation], some extensions to the theory of closed operators in Banach space are made, including several original ideas on the extending of the theory from bounded operators to relatively bounded ones. In [11; third citation] one finds a translation of the original paper [3].

I would not have known of these papers had not Professor Sz.-Nagy given them to me on a visit to Szeged in 1972.

My contribution [4] to the RKNG theorem came about in an indirect way. I was finishing my doctoral work in partial differential equations with L. E. Payne at the University of Maryland. Payne had already taken a position at Cornell University, and as I had some free time I was attending the lectures of S. Goldberg on operator theory. Goldberg was stuck at $b<1/2$ in a proof of the following basic defect index lemma.

Lemma [4, Lemma 1]. *Let T and B be linear operators with domains in a normed linear space X and ranges in a normed linear space Y. Suppose that T has a bounded inverse and that B is bounded with $\|B\|<b\|T^{-1}\|^{-1}$, $b<1$. Then* $\dim(Y/\overline{R(T)})=\dim(Y/\overline{R(T+B)})$.

Inasmuch as this lemma was desired by Goldberg for a proof of the basic index perturbation theorem that states that the full index of an operator is preserved under relatively small perturbations, I became interested and proved the above lemma, which also appears in the book [12] as Corollary V 1.3. Believing that the technique of proof should have wider value, upon stumbling onto Nelson's paper [13], Nelson also being stuck at $b<1/2$, I published [4] with its RKNG result for semigroup generators. Because Nelson was working in Banach space, so did I.

Theorem [4, Theorem 2]. *Let A be the infinitesimal generator of a contraction semigroup on the Banach space X, and let B be a dissipative operator with $D(B)\supset D(A)$. If there exist constants a and $b, b<1$, such that for all x in $D(A)$, $\|Bx\|\leqq a\|x\|+b\|Ax\|$ then $A+B$ is the infinitesimal generator of a contraction semigroup.*

In [14, first citation] I tried to place the doubling technique in a proper context with respect to the general Fredholm theory of linear operators in normed (not necessarily complete) spaces and in so doing gave a number of extensions of [4].

I also noted as an example of the method the following extension from $b<1/2$ to $b<1$ of a theorem of KATO [6, Theorem 3.4] on perturbation of sesquilinear forms.

Theorem [14, first citation, Theorem 4.1]. *Let* $t[u, v]$ *be a densely defined closed sectorial form with* $\operatorname{Re}(t[u, u]) \geqq 0$, *and let* $b[u, v]$ *satisfy*

$$|b[u, u]| \leqq a\|u\|^2 + b \operatorname{Re}(t[u, u]), \quad b < 1,$$

and $\operatorname{Re}(b[u, u]) \geqq -\operatorname{Re}(t[u, u])$ *for* u *in* $D(t) \subset D(b)$. *Then the resolvent* $R_\lambda(T_{t+b})$ *for the associated form operator exists for* $\operatorname{Re}\lambda < -ab^{-1}$ *and*

$$\|R_\lambda(T_t) - R_\lambda(T_{t+b})\| \leqq nb(b_n - b)^{-1}(-\operatorname{Re}\lambda)^{-1},$$

where n *is chosen such that* $b < b_n = (2^n - 1)/2^n$.

It should be noted that in extending the previous version from $b<1/2$ to $b<1$, I made an additional assumption that $\operatorname{Re}(t+b) \geqq 0$. Also for the record let me correct here two minor typographical errors: [14, p. 286, the last line] reads $\|(1-c_n)B_{n+1}\| < 2^{-(n+1)}\gamma(T) = \ldots$ and [14, p. 287, the first line] reads ... is a B_1 for $T + c_n B_{n+1}, \ldots$.

Shortly afterward, while on the faculty of the University of Minnesota, I attended a lecture by Ken-iti Sato who was visiting from Japan. Sato, who was chiefly interested in applications to probability and who, although there already for several months, I had not met, had proved the following theorem for $b<1/2$.

Theorem [14, second citation, Theorem 2.1]. *Let* A *be the infinitesimal generator of a nonnegative contraction semigroup on a Banach lattice* $\mathscr{B}$, *and let the perturbation* B *satisfy* $\|Bf\| \leqq a\|f\| + b\|Af\|$, $b<1$ *for all* f *in* $D(A) \subset D(B)$. *If* $A+B$ *is* $(\alpha+\beta)$*-weakly dispersive, then* $A+B$ *also generates a nonnegative contraction semigroup.*

During his lecture I wrote down the following "lemma": a weakly dispersive operator B is dissipative in at least one semi-inner-product, and based upon that assumption, immediately noted the extension to $b<1$ that is stated in the above theorem. Unfortunately, the just mentioned "lemma" resisted our efforts at its proof, causing the proofs in [14] to be longer than otherwise would have been necessary. Although the "Lemma" is true in a wide variety of cases and under many additional assumptions, its general validity or nonvalidity for all Banach lattices is to my knowledge not known.

A few years later a recent student of F. Browder in Chicago, B. Calvert from New Zealand, spent a postdoctoral one half year with me at the University of Colorado. Calvert had done his doctoral work on nonlinear semigroup generators in Banach lattices. Here is a nonlinear version of the RKNG theorem, which we needed for application to questions of multiplicative perturbation of generators.

Theorem [14, third citation, Lemma 1]. *Let A be m-accretive (φ) in a Banach space X, let B be single-valued with $D(B)\supset D(A)$ such that $A+B$ is accretive (ψ) such that $\psi(u, v, x+Bu, y+Bv)=\varphi(u, v, x\in Au, y\in Av)$. Suppose there exist constants a and b such that, for x_1 and x_2 in $D(A)$,*

$$\|Bx_1-Bx_2\| \leqq a\|x_1-x_2\|+b|Ax_1-Ax_2|, \quad b<1.$$

Then $A+B$ is m-accretive (ψ).

Here φ and ψ may be different duality maps and $|S|$ is the infinium of $\|s\|$ for all s in a set S. In the nonlinear case the technicalities in the statements cause some loss of interest in the results. We refer to Browder [15] for further information on the nonlinear semigroup literature.

For extensive treatments of the RKNG Theorem, selfadjoint operators, forms, and related matters, we point out, in addition to the books [5] and [6] already mentioned, two books almost totally devoted to such questions, Faris [16] and Chernoff [17].

2. The real Hilbert space case

The usual proofs of the RKNG Theorem utilize the defect index theories. In particular, for the instance of preservation of selfadjointness, it is interesting to ask if a direct proof can be worked out for real Hilbert space without employing the Von Neumann $R(A+B\pm iI)=H$ complex defect-zero criteria. A direct proof of the RKNG theorem for real spaces was worked out with my doctoral student D. K. Rao about ten years ago, and I would like to give it here since I have never seen it anywhere else.

Theorem (RKNG Theorem in real Hilbert space). *Let A be selfadjoint in a real Hilbert space and let B be a symmetric operator with $D(B)\supset D(A)$ and $\|Bx\|\leqq a\|x\|+b\|Ax\|$, $b<1$. Then $A+B$ is selfadjoint.*

Proof. It suffices to first prove the theorem for $b<1/2$. Then the doubling technique allows extension to $b<1$. Since $A+bB$ is closed and symmetric, it thus suffices to prove that for $\|Bx\|\leqq a\|x\|+\|A\|$, $b<1/2$, one has $D((A+bB)^*)\subset$ $\subset D(A+bB)$.

Let $y\in D((A+bB)^*)$. Then

$$\left|((A+bB)^*y,\ Ax)+k^2(y, x)\right| \leqq (\|(A+bB)^*y\|+\|y\|)\|x\|_1$$

where $(x, y)_1=(Ax, Ay)+k^2(x, y)$ is the inner product on $H_A\equiv D(A)$ with the A-norm. It will be advantageous to choose k such that $k>2ab(1-b)^{-1}$. The linear functional y_1^* on H_A induced from y via

$$y_1^*(x) = ((A+bB)^*y, Ax)+k^2(y, x)$$

is bounded and hence there exists some y^* in $D(A)$ such that $y_1^*(x)=(y^*,x)_1\equiv\equiv(Ay^*,Ax)+k^2(y^*,x)$. We now claim, remembering that $k>2ab(1-b)^{-1}$, that B may "inserted" so that one has the representation, for some y_B^* in $D(A)$;

$$y_1^*(x)\equiv(y^*,x)_1=\big((A+bB)y_B^*,Ax\big)+k^2(y_B^*,x).$$

Accepting this for the moment, for all x in $D(A^2)$ we then have

$$\big(y-y_B^*,((A+bB)A+k^2)x\big)=0.$$

For $k>2ab(1-b)^{-1}$ and $b<1/2$, the operator $(A+bB)A+k^2=(A^2+k^2)+bBA$ is onto because A^2+k^2 maps onto and bBA is a small perturbation. Accepting also this latter statement for the moment, we thus have y in $D(A)$ and thus $D((A+bB)^*)\subset D(A+bB)$.

There are several ways to verify the two details in the above.

For the first, from $y_1^*(x)=(Ay^*,Ax)+k^2(y^*,x)$ we may "insert" the (bBy_B^*,Ax) term as follows. The linear functional $-(bBz_0,Ax)$ induced by any vector z_0 in $D(A)$ is bounded on H_A immediately by Schwarz's inequality and the fact that B is A-small with $ab<k(1-b)/2$. Thus there exists z_1 in $D(A)$ such that $(-bBz_0,Ax)==(z_1,x)_1$ and $\|z_1\|_1<\frac{1+b}{2}\|z_0\|_1$. Repeating this with z_1 we are led to the sequence z_n given by $(z_{n+1},x)_1=-(bBz_n,Ax)$. Thus we may write the "telescoped sum"

$$(z_0,x)_1=\sum_{m=0}^{n}\big[\big((A+bB)z_m,Ax\big)+k^2(z_m,x)\big]+(z_{n+1},x)_1.$$

Let $\sum_{m=0}^{\infty}z_m=z$, and note that $\sum_{m=1}^{n}\|z_m\|<\sum_{m=1}^{n}(2^{-1}(1+b))^m\|z_0\|_1$, which guarantees absolute convergence (for $b<1$) of both $\sum_{m=0}^{\infty}z_m$ and $\sum_{m=0}^{\infty}Az_m$. By the relative boundedness $\sum_{m=0}^{\infty}\|(A+bB)z_m\|$ is similarly seen to converge and hence $\sum_{m=0}^{n}(A+bB)z_m$ converges to $(A+bB)z$ because $\sum_{m=0}^{n}z_m\to z$ and $A+bB$ is closed. Letting y^* be z_0 and y_B^* be z we thus have shown that $(y^*,x)_1=((A+bB)y_B^*,Ax)+k^2(y_B^*,x)$.

For the second detail, we note that the perturbation bBA satisfies the relative bounds

$$b\|BAz\|\leqq ab\|Ax\|+b^2\|A^2x\|\leqq(abk^{-1}2^{-1/2}+b^2)\|(A^2+k^2)x\|\leqq$$
$$\leqq((1-b)2^{-3/2}+b^2)\|(A^2+k^2)x\|.$$

Because the coefficient of the right hand side is less than one for $b<1/2$, the surjectivity of A^2+k^2 implies that of A^2+k^2+bBA by the basic defect index lemma of [4] mentioned in the section above; more precisely and quickly, by its unbounded version [14, first citation, Theorem 2.4].

There are no doubt shorter and sharper versions of the proof given above for the RKNG Theorem for Real Hilbert Space, perhaps also a proof by complexifying and using the complex version. The latter was not evident to us at the time and in any case the proof given above holds as well for the complex case. We did not investigate the above proof in any generality (e.g., Banach space, nonlinear versions).

Although the important quantum mechanical application occurs in a complex Hilbert space setting, there are a number of suppositions that go into that choice of scalars, and the issue is not completely settled. See my remarks in [18]. Along these lines it would perhaps be interesting to have the RKNG theorem for the quaternion field, perhaps also for the Clifford algebra.

3. Regular positive perturbations

One would like to have a specialized RKNG *theorem of the form: if B is a symmetric regular positive perturbation of positive selfadjoint A, then $A+B$* is selfadjoint (or at least essentially selfadjoint). The idea is that the regularity of B with respect to A means that B is A-bounded, and that even though the relative bound b may be greater than one, it doesn't matter. There also are a number of compelling physical examples for such a theorem. I conjectured in 1972, in some discussions with K. Jörgens who was visiting me in Boulder for the year, that such a result should hold, and spent some time on it that year.

By translation the question is equivalent to that for the more general situation of two semibounded selfadjoint operators, one regular with respect to the other, and also equivalent to the more special case of A and B both bounded below by one, $D(B)\supset D(A)$. One can write down lots of operator-theoretic conditions for the selfadjointness (better: essential selfadjointness) of $A+B$. For example, for strongly positive selfadjoint A and B, the latter a regular perturbation, $D(A^{1/2})\subset \subset D(B^{1/2})$ by forms so that

$$A+B = A^{1/2}[I+A^{-1/2}B^{1/2}\cdot B^{1/2}A^{-1/2}]A^{1/2} = A^{1/2}[I+TT^*]A^{1/2}.$$

Note that $T=A^{-1/2}B^{1/2}$ is a densely defined bounded operator and that $C=I+TT^*$ is a strongly positive essentially selfadjoint bounded densely defined operator. One has then the situation

$$\overline{A+B}\subset A^{1/2}\bar{C}A^{1/2}\subset (A+B)^*$$

where the middle term $A^{1/2}\bar{C}A^{1/2}$ may be seen to be the Friedrichs extension of $A+B$. Thus $A+B$ is selfadjoint iff C maps $D(A^{1/2})$ onto itself iff $D((A+B)^*)\cap \cap D(A^{1/2})$ is contained in $D(B)$, and $A+B$ is essentially selfadjoint iff $\overline{A^{1/2}C}=A^{1/2}\bar{C}$ iff $(A+B)^2$ is densely defined.

Unfortunately the general proposition is false, and in this final section I want to discuss why, and present some counter examples. These were obtained in May 1979 during visits with J. Weidmann in Frankfurt and with T. Kato, who provided the coup de grâce, in Paris. They are important to have, and as far as I know, are not in the published literature.

But before I conclude, I would like to mention some ways in which such questions were, and in some cases may be, avoided.

The usual dodge is to accept form sums. Then for example if A and B are positive selfadjoint and $D(A^{1/2})\cap D(B^{1/2})$ is dense, the "form sum" $A+B$ is selfadjoint. One does not even need a regular perturbation. But that is the point: one also does not in general get a "regular" sum selfadjoint on $D(A)$.

A second dodge is to invoke a positivity assumption on the product term $\langle Ax, Bx\rangle$ rather than on A and B. For example, for A selfadjoint and B a symmetric regular perturbation, the condition $\operatorname{Re}(Ax, Bx)\geqq 0$ implies (Okazawa [19]) that $A+B$ is selfadjoint. An extension of this result is that one needs (Gustafson and Rejto [20]) only $\operatorname{Re}(Ax, Bx)\geqq -(a\|x\|^2+b\|Ax\|\,\|Bx\|)$ for $b<1$ to conclude $A+cB$ selfadjoint for all $c\geqq 0$. Further discussions of these methods may be found in [7].

A third set of variations involves closure assumptions. For example, for a regular perturbation B the selfadjointness of $A+cB$ fails only when the closure of $A+cB$ fails.

Let us conclude with some counter examples.

First, it is not sufficient that just A and the sum of $A+B$ be positive. Consider any closed symmetric positive nonselfadjoint operator T, let $A=|T|$ and $B=T-|T|$. Then B is a regular symmetric perturbation of positive selfadjoint A, with relative bound 2, $A+B$ is closed symmetric positive but not selfadjoint.

Second, a version of the result mentioned above from [20] is: for A essentially selfadjoint and B a symmetric A-bounded regular perturbation of A, if the set of values (Au, Bu) for all u in $D(A)$ is contained in some half-plane not containing $(-\infty, 0]$ then $A+B$ is essentially selfadjoint. To see that the half line $(-\infty, 0]$ must be excluded, let $Ay=-y''$ in $L^2(0,\infty)$ with $D(A)=\{y\in L^2(0,\infty)$ such that y and y' are absolutely continuous, $y\in C_0^\infty[0,\infty), y(0)=0\}$, and let $By=y''+iy'-y$ on $D(A)$. A is essentially selfadjoint and $A+B$ has no selfadjoint extensions at all. It is easy to verify that (Au, Bu) values lie in the second quadrant.

Counter examples to A and B positive essentially selfadjoint operators, one regular with respect to the other, yet $A+B$ not essentially selfadjoint, may be obtained as follows. Consider $Ay=-(py')'$, $B=(-qy')'$, with $p(x)=x^4+\sin^2(1/x)$ and $q(x)=x^4+\cos^2(1/x)$ for $0<x<1$ and continued smoothly to 1 as $x\to\infty$. A and B are essentially selfadjoint in $L^2(0,\infty)$ on the domain $C_0^\infty(0,\infty)$; but $A+B$ fails to be essentially selfadjoint on that domain. One may use other p and

q and also one can replace the oscillatory parts by such discretized oscillating functions such as $s(x)=1$ for $(2n+1)^{-1} \leqq x < (2n)^{-1}$, 0 otherwise, $c(x)=1-s(x)$.

The idea is that the wave functions must go to zero as $x \to 0$ so well for A and B to accomodate the oscillation, whereas for $A+B$; in which the oscillation has cancelled itself, the wave functions need not be so good and $A+B$ will need a larger domain of selfadjointness.

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