Acta Sci. Math., 45 (1983), 219—231

On partial asymptotic stability and instability. I (Autonomous systems)

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Dedicated to Professor Béla Szőkefalvi-Nagy on his 70th birthday

1. Introduction

Ljapunov's direct method is the most powerful tool for establishing also partial stability properties, i.e. stability properties regarding some variables only [1]—[3]. One finds, however, that in many applications it is very complicated to construct an appropriate Ljapunov function. For example, the derivative of the total mechanical energy of a holonomic mechanical system under the action of dissipative forces is negative definite with respect to velocities only, thus it cannot be used in the basic theorems to establish asymptotic stability or instability with respect to the generalized coordinates. The method of BARBASHIN and KRASOVSKIĬ [4], [5] and LASALLE's invariance principle [6] enable us to get asymptotic stability or instability by Ljapunov functions with semidefinite derivative. These methods have been extended to the study of partial stability [1], [7], [8]. However, in comparison with the stability investigations concerning all variables, a new difficulty appears: the extensions require the boundedness of all the uncontrolled coordinates along every solution. As it was shown in [9] by an example, this condition cannot be omitted. Our purpose is to replace this condition by such ones which can be checked directly, i.e. without a priori knowledge of the solutions.

We first study what we can state after having dropped the condition of boundedness of the uncontrolled coordinates. This allows us to locate the limit set of the vector function whose components are the controlled coordinates of a solution. Then we can find additional conditions on the *Ljapunov function* which assure the zero solution of an autonomous system to be partially asymptotically stable or

Received December 3, 1981.

The author wrote this paper while visiting the Institut de Mathématique Pure et Appliquée, Université Catholique de Louvain in Louvain-la-Neuve, in the spring of 1981.

unstable. Starting from the localization result mentioned above, in the continuation [10] of the present paper we give some additional conditions on the *right-hand side* of the system which imply the same properties. We apply our results to study stability properties of mechanical equilibrium in the presence of dissipative forces. As special cases we study motions of a material point along certain surfaces in a constant field of gravity.

2. Notations, definitions. Preliminaries

Consider the system of differential equations

$$\dot{x} = X(x, t), \tag{2.1}$$

where $t \in R_+ = [0, \infty)$, and $x = (x^1, \ldots, x^k)$ belongs to the space R^k with a norm $|x|$. Denote by $B_k(\varrho)$ the open ball in R^k with center at the origin and radius $\varrho > 0$, $\bar{B}_k(\varrho)$ its closure in R^k. Let a partition $x = (y, z)$ ($y \in R^m$, $z \in R^n$; $1 \leqq m \leqq k$, $n = k - m$) be given. Assume that the function X is defined on the set Γ_y:

$$\Gamma_y = G_y \times R_+ \quad (G_y = B_m(H) \times R^n; \quad 0 < H \leqq \infty),$$

it is continuous in x, is measurable in t, and satisfies the Carathéodory condition locally (i.e. for every compact set $K \subset R^k$ there is a locally integrable $h: R_+ \to R_+$ such that $|X(x, t)| \leqq h(t)$ for all $(x, t) \in K \times R_+$), so the local existence of solutions of initial value problems is assured, and every solution has a maximal extension. We denote by $x(t) = x(t; x_0, t_0)$ any solution with $x(t_0) = x_0$. If (2.1) is autonomous, i.e. X does not depend on t, we use the notation $x(t; x_0) = x(t; x_0, 0)$. We always assume that solutions are z-continuable [2], i.e. if $x(t) = (y(t), z(t))$ is a solution of (2.1) and $|y(t)| \leqq H' < H$ for $t \in [t_0, T)$, then $x(t)$ can be continued to the closed interval $[t_0, T]$.

The zero solution of (2.1) is said to be:

y-stable if for every $\varepsilon > 0$, $t_0 \in R_+$ there exists a $\delta(\varepsilon, t_0) > 0$ such that $|x_0| < \delta(\varepsilon, t_0)$ implies $|y(t; x_0, t_0)| < \varepsilon$ for $t \geqq t_0$;

asymptotically y-stable if it is y-stable, and for every $t_0 \in R_+$ there exists a $\sigma(t_0) > 0$ such that $|x_0| < \sigma(t_0)$ implies $|y(t; x_0, t_0)| \to 0$ as $t \to \infty$;

uniformly asymptotically y-stable if it is y-stable so that the number $\delta(\varepsilon, t_0)$ can be chosen independently of t_0, and there exists a $\sigma > 0$ such that $|y(t; x_0, t_0)| \to 0$ uniformly in $x_0 \in B_k(\sigma)$ as $t \to \infty$;

y-unstable if it is not y-stable.

Let $x = \varphi(t) = (\psi(t), \chi(t))$ be a solution of (2.1) defined on an interval $[t_0, \infty)$ ($t_0 \in R_+$; $\psi: [t_0, \infty) \to R^m$, $\chi: [t_0, \infty) \to R^n$). A point $q \in R^m$ is a *y-limit point* of the solution $x=\varphi(t)$ if there exists a sequence $\{t_i\}$ such that $t_i\to\infty$ and $\psi(t_i)\to q$ as $i\to\infty$. The *partial limit set* $\Omega_y(\varphi)$ of $x=\varphi(t)$ *with respect to* y is the set of all its y-limit points. It is easy to see that if ψ is bounded then $\Omega_y(\varphi)$ is non-empty, compact, connected and is the smallest closed set approached by $\psi(t)$ as $t\to\infty$.

The *complete trajectory* $\gamma(\xi)$ of (2.1) belonging to a non-continuable solution $\xi:(\alpha,\beta)\to R^k$ is defined by $\gamma(\xi)=\{\xi(t):\alpha<t<\beta\}$. It is known [11] that if (2.1) is autonomous then the set $E=\Omega_x(\varphi)\cap G_y$ is *semiinvariant* with respect to (2.1), which means that for every $x_0\in E$ the equation (2.1) has a solution ξ such that $\xi(0)=x_0$ and $\gamma(\xi)\subset E$.

A continuous function $V:\Gamma_y'\to R$, $\Gamma_y'=G_y'\times R_+$ $(G_y'=B_m(H')\times R^n;\ 0<H'<H)$ is a *Ljapunov function* of (2.1) if $V(0,t)\equiv 0$; V is locally Lipschitzian and

$$\dot V(x,t)=\limsup_{h\to 0+}\frac{V(x+hX(x,t),t+h)-V(x,t)}{h}\leqq 0$$

for all $(x,t)\in\Gamma_y'$. The function $\dot V$ is called the derivative of V with respect to (2.1).

We say that a function $a: R_+\to R_+$ belongs to the class $\mathcal{K}$ if it is continuous, strictly increasing and $a(0)=0$. A function $V:\Gamma_y'\to R$ is said to be *positive definite in* y or *positive y-definite* if there exists a function $a\in\mathcal{K}$ such that $a(|y|)\leqq V(y,z,t)$ for all $(y,z,t)\in\Gamma_y'$.

Let us given a continuous function $W:L\times R^q\times R_+\to R$, where $L\subset R^p$ is open, $p\geqq 1$, $q\geqq 0$ are integers. Following LASALLE's notation [6], for $c\in R$ we denote by $W_p^{-1}[c,\infty]$ the set of the points $u\in R^p$ for which there is a sequence $\{(u_i,v_i,t_i)\}$ such that $u_i\to u$, $|v_i|\to\infty$, $t_i\to\infty$, $W(u_i,v_i,t_i)\to c$ as $i\to\infty$. If $W:L\to R$ (i.e. $q=0$ and W does not depend on t), then $W_p^{-1}[c,\infty]$ is the inverse under W of c and is denoted by $W^{-1}(c)$ as usual.

We say that a function $x=(y,z): R_+\to R^k$ is *z-bounded* if $z: R_+\to R^n$ is bounded for $t\geqq 0$.

Now we can cite the following extensions of the Barbashin—Krasovskiĭ theorem to partial stability, which contain the original theorems as special cases $(y=x)$.

Theorem A (A. C. OZIRANER [9]). *Suppose that every solution of the autonomous system*

$$\dot x = X(x) \tag{2.2}$$

starting from a sufficiently small neighbourhood of the origin is z-bounded.

I. *If there exists a Ljapunov function* $V:G_y'\to R$ *such that*

(i) V *is positive y-definite and* $V(0)=0$;

(ii) *the set* $\{x: V(x)>0\}\cap\dot V^{-1}(0)$ *contains no complete trajectory of* (2.2), *then the zero solution of* (2.2) *is uniformly asymptotically stable.*

II *If there exists a Ljapunov function* $V:G_y'\to R$ *such that* (i′) $V(0)=0$, *and every neighbourhood of the origin contains a point* x *with* $V(x)<0$;

(ii′) *the set* $\{x: V(x)<0\}\cap\dot V^{-1}(0)$ *contains no complete trajectory of* (2.2), *then the zero solution of* (2.2) *is y-unstable.*

In [12] J. P. LaSalle pointed out that the essence of the Barbashin—Krasovskiĭ method consists in the location of the limit sets and formulated it in his "invariance principle": If $V: G'_y\to R$ is a Ljapunov function of (2.2), and $\varphi: R_+\to R^k$ is a solution of the same equation such that $|\varphi(t)|\leqq H''<H'$ for $t\geqq 0$, then $\Omega_x(\varphi)\subset\dot V^{-1}(0)\cap V^{-1}(c)$ with some $c\in R$.

3. Theorems on general differential systems

First of all we have to locate the partial limit set of solutions of

$$\dot x = X(x) \tag{3.1}$$

with the knowledge of boundedness only of controlled coordinates. An easy but, so it appears, useful generalization of LaSalle's invariance principle is

Lemma 3.1. *Suppose that* $V: G'_y\to R$ *is a Ljapunov function of* (3.1) *and let* $x=\varphi(t)=(\psi(t),\chi(t))$ *be a solution of* (3.1) *such that* $|\psi(t)|\leqq H''<H'$ *for* $t\in R_+$. *Then either* a) $|\chi(t)|\to\infty$ *as* $t\to\infty$ *or* b) $v(t)=V(\varphi(t))\to v_0$ *as* $t\to\infty$; *the set* $\Omega_x(\varphi)$ *is not empty and is contained in* $\dot V^{-1}(0)\cap V^{-1}(v_0)$.

Proof. Assume that a) is not satisfied. Then, because of the boundedness of ψ, there exist $q\in R^m$, $r\in R^n$, $v_0\in R$ and a sequence $\{t_i\}$ such that $t_i\to\infty$, $\psi(t_i)\to q$, $\chi(t_i)\to r$ and $v(t_i)\to v_0$ as $i\to\infty$. Since $\Omega_x(\varphi)$ is semi-invariant with respect to (3.1), there exists a solution $x(t)=x(t;q,r)$ of (3.1) for which $\gamma(x)\subset\Omega_x(\varphi)$. Let $\bar t\in R_+$ be fixed. Then there exists a sequence $\{\bar t_i\}$ such that $\bar t_i\to\infty$, $(\psi(\bar t_i),\chi(\bar t_i))\to x(\bar t;q,r)$ and $v(\bar t_i)\to V(x(\bar t;q,r))$ as $i\to\infty$. Since v is decreasing, $V(x(\bar t;q,r))=v_0$, which completes the proof.

Denote by $P_y: R^k\to R^m$ the orthogonal projection from R^k into R^m, i.e. $P(x)=y$.

Lemma 3.2. *Let* V,φ *satisfy the conditions of Lemma* 3.1 *and assume, in addition, that* V *is bounded from below. Then*

$$\Omega_y(\varphi)\subset P_y(\Omega_x(\varphi))\cup V_m^{-1}[v_0,\infty],$$

where v_0 *is defined by* $v(t)=V(\varphi(t))\to v_0$ *as* $t\to\infty$.

Proof. Introduce the notations $L=\Omega_y(\varphi)$, $M=P_y(\Omega_x(\varphi))$, $N=L\setminus M$. Obviously, $M\subset L$; therefore, it is sufficient to prove that $N\subset V_m^{-1}[v_0,\infty]$. If $q\in N$, then there are a sequence $\{t_i\}$ and $v_0 \in R$ such that $t_i \to \infty$, $\psi(t_i) \to q$, $|\chi(t_i)| \to \infty$, $v(t_i) \to v_0$ as $i \to \infty$, i.e. $q \in V_m^{-1}[v_0, \infty]$. Since v is nonincreasing, v_0 is independent of q, thus $N \subset V_m^{-1}[v_0, \infty]$.

The lemma is proved.

Theorem 3.1. *Let $V: G_y' \to R$ be a positive y-definite Ljapunov function of* (3.1) *such that for every $c > 0$ the set $\dot{V}^{-1}(0) \cap V^{-1}(c)$ contains no complete trajectory. Then the zero solution of* (3.1) *is y-stable and for every solution $x(t) = (y(t), z(t))$ starting from a sufficiently small neighbourhood of the origin either* a) $V(x(t)) \to 0$ *(and, consequently, $|y(t)| \to 0$) or* b) $|z(t)| \to \infty$ *as $t \to \infty$.*

Proof. Since V is positive y-definite and $\dot{V}(x) \leqq 0$, the zero solution of (3.1) is y-stable (see [11], p. 15) and, a fortiori, every solution starting from some neighbourhood $B_k(\varrho)$ ($\varrho > 0$) of the origin is y-bounded. Let $x = \varphi(t) = (\psi(t), \chi(t))$ be such a solution. Suppose $|\chi(t)| \not\to \infty$ as $t \to \infty$. We have to prove that in this case $v(t) = V(\varphi(t)) \to 0$ as $t \to \infty$. By Lemma 3.1, $v(t) \to v_0 \geqq 0$, and there is a point $p \in R^k$ such that $p \in \Omega_x(\varphi) \subset \dot{V}^{-1}(0) \cap V^{-1}(v_0)$. The set $\Omega_x(\varphi)$ is semiinvariant with respect to (3.1); consequently, $\dot{V}^{-1}(0) \cap V^{-1}(v_0)$ contains a complete trajectory of (3.1). By the assumptions this implies $v_0 = 0$.

The proof is complete.

Theorem A.I in the previous section is a corollary of Theorem 3.1. Indeed, by Theorem 3.1, the conditions of Theorem A.I imply that the zero solution of (3.1) is y-stable and $V(x(t; x_0)) \to 0$ as $t \to \infty$ for all $x_0 \in B_k(\sigma)$ with some $\sigma > 0$. Application of the classic covering theorem of Heine—Borel—Lebesgue gives that this convergence is uniform in $x_0 \in B_k(\sigma)$ (see [9]), which implies uniform asymptotical y-stability because V is positive y-definite.

The following two theorems show how to make use of the alternative given in Lemma 3.1 and Theorem 3.1 for getting sufficient conditions for partial asymptotic stability of the zero solution of an autonomous system.

Theorem 3.2. *Let the assumptions of Theorem* 3.1 *be satisfied. Suppose, in addition, that $V(y, z) \to 0$ uniformly in $y \in B_m(H')$ as $\dot{V}(y, z) \to 0$ and $|z| \to \infty$. Then the zero solution of* (3.1) *is uniformly asymptotically y-stable.*

Proof. In view of Theorem 3.1 and the remark following its proof, it is sufficient to prove that the function V tends to 0 along every solution starting from some neighbourhood of the origin. Denote by $\varphi(t) = (\psi(t), \chi(t))$ an arbitrary solution of (3.1) with $|\psi(t)| \leqq H'' < H'$ ($t \in R_+$) and let $v(t) = V(\varphi(t)) \to v_0$ as $t \to \infty$. Since $\dot{v}(t) = \dot{V}(\varphi(t)) \leqq 0$ for all $t \in R_+$ and the function v is bounded from below, there exists a sequence $\{t_i\}$ such that $t_i \to \infty$ and $\dot{V}(\varphi(t_i)) \to 0$ as $i \to \infty$. By Theorem 3.1 either $v_0 = 0$ or the sequence $\{z_i = \chi(t_i)\}$ diverges to infinity in norm as

$i\to\infty$. In the latter case we have $\dot V(\psi(t_i); z_i)\to 0$, $|z_i|\to\infty$ and, simultaneously, $V(\psi(t_i), z_i)\to v_0$ as $i\to\infty$. The last assumption of the theorem implies $v_0=0$, which completes the proof.

It may be pointed out that Theorem 3.2 improves certain results which can be obtained by the application of some basic theorems on partial uniform asymptotic stability (see [2, 13]) to autonomous systems. To illustrate this fact let us recall a theorem of K. Peiffer and N. Rouche [13, Th. IV]. For (3.1) it says that if there exists a positive y-definite Ljapunov function $V: G'_y\to R$ such that $V(x)\to 0$ uniformly in $x\in G'_y$ as $\dot V(x)\to 0$, then the zero solution of (3.1) is uniformly asymptotically y-stable. This obviously follows from Theorem 3.2. In fact, Theorem 3.2 improves this corollary as it is shown by the following example.

Consider the system

$$\dot x^1 = -x^1(1+(x^3)^2), \quad \dot x^2 = -x^2, \quad \dot x^3 = x^3-(x^3)^3 \tag{3.2}$$

and let $y=(x^1, x^2)$, $z=x^3$. The function $V(x^1, x^2, x^3)=(x^1)^2+(x^2)^2+(x^2)^2(x^3)^2$ is positive (x^1, x^2)-definite, its derivative with respect to (3.2) reads $\dot V(x^1, x^2, x^3)=-2(x^1)^2(1+(x^3)^2)-2(x^2)^2-2(x^2)^2(x^3)^4$. If $\dot V\to 0$ then

$$x^1\to 0, \quad x^2\to 0, \quad (x^2)^2(x^3)^4\to 0, \tag{3.3}$$

which do not imply that $\dot V\to 0$; therefore, the theorem of Pfeiffer and Rouche cannot be applied to this case. On the other hand, $|x^3|\to\infty$ and (3.3) together already imply $V\to 0$ and Theorem 3.2 can be applied.

C. Risito [7] proved that the statements of Theorem A (without uniformity) remain true if instead of (ii) and (ii′) one requires the following: the set $\{(y, z): y=0\}$ is invariant and the region $\dot V^{-1}(0)\setminus\{(y, z): y=0\}$ contains no complete positive semi-trajectory. Lemma 3.2 allows us to extend this result to the case when the uncontrolled coordinates are not supposed to be bounded.

Theorem 3.3. *Let $V: G'_y\to R$ be a positive y-definite Ljapunov function of* (3.1). *Suppose that for every $c>0$*

(i) *if the set $\dot V^{-1}(0)\cap V^{-1}(c)$ contains a complete trajectory then this trajectory is contained also in the set $\{(y, z): y=0\}$;*

(ii) $V_m^{-1}[c, \infty]\subset\{0\}$.

Then the zero solution of (3.1) *is asymptotically y-stable.*

Proof. As it was shown in the proof of Theorem 3.1, the zero solution starting from some neighbourhood of the origin is y-bounded. Let $x=\varphi(t)=(\psi(t), \chi(t))$ be such a solution. We have to prove that $|\psi(t)|\to 0$ as $t\to\infty$, i.e. $\Omega_y(\varphi)=\{0\}$. The function $v(t)=V(\varphi(t))$ is nonincreasing and nonnegative, hence $v(t)\to v_0\geqq 0$. If $v_0=0$ then the statement is true because V is positive y-definite. Assume that

$v_0>0$ and there exists a q such that $0\neq q\in\Omega_y(\varphi)$. By Lemma 3.2 and condition (ii) there is an $r\in R^n$ such that $p=(q,r)\in\Omega_x(\varphi)$. The set $\Omega_x(\varphi)$ is semiinvariant, hence there exists a solution $\xi:(-\infty,\infty)\to R^k$ of (3.1) for which $\xi(0)=p$ and $\gamma(\xi)\subset\subset\Omega_x(\varphi)$. In view of Lemma 3.1 $\gamma(\xi)$ is contained also in the set $\dot V^{-1}(0)\cap V^{-1}(v_0)$. On the other hand, $q\neq 0$; therefore $\gamma(\xi)$ is not contained in the set $\{x: y=0\}$, in contradiction to condition (i) of the theorem. This means that either $v_0=0$ or $\Omega_y(\varphi)=0$, which completes the proof.

Corollary 3.1. *Let condition* (i) *in Theorem* 3.3 *be satisfied. Suppose, in addition, that there is a number* $\varrho>0$ *such that* $0<|\bar y|<\varrho$ *implies*

$$\lim_{y\to\bar y,\,|z|\to\infty} V(y,z)=\infty. \tag{3.4}$$

Then the zero solution of (3.1) *is asymptotically y-stable.*

The alternative given in Lemma 3.1 can be used also for getting sufficient conditions for the instability of the zero solution of (3.1).

Theorem 3.4. *Suppose that there is a Ljapunov function* $V: G'_y\to R$ *of* (3.1) *satisfying the following properties:*

(i) *for every* $\delta>0$ *there exists* $x_0(\delta)\in B_k(\delta)$ *with* $V(x_0(\delta))<0$;

(ii) *there is an* ε_0 $(0<\varepsilon_0<H')$ *such that for every* $c<0$ *the set*

$$\dot V^{-1}(0)\cap V^{-1}(c)\cap(\bar B_m(\varepsilon_0)\times R^n) \tag{3.5}$$

contains no complete trajectory.

Then for every δ $(0<\delta<\varepsilon_0)$ *either* a) *every curve* $\gamma_y^+(x_0(\delta))\colon t\mapsto y(t;x_0(\delta))$ $(t\in R_+)$ *leaves the ball* $B_m(\varepsilon_0)$ *in finite time, or* b) $|z(t;x_0(\delta))|\to\infty$ *as* $t\to\infty$.

Proof. If the statement is not true, then for some δ_0 $(0<\delta_0<\varepsilon_0)$ there exists a solution

$$\varphi(t)=(\psi(t),\chi(t))\quad(\varphi(0)=x_0(\delta_0)) \tag{3.6}$$

such that

$$|\psi(t)|\leqq\varepsilon_0\quad(t\in R_+);\quad v(t)=V(\varphi(t))\to v_0<0\quad(t\to\infty), \tag{3.7}$$

the limit set $\Omega_x(\varphi)$ is not empty and is contained in the set (3.5) for $c=v_0$. On the other hand, $\Omega_x(\varphi)$ is semiinvariant, thus (3.5) contains a complete trajectory in contradiction to condition (ii) of the theorem.

The proof is complete.

Corollary 3.2. *Let the conditions of Theorem* 3.4 *be satisfied. Suppose, in addition, that*

$$\liminf_{|z|\to\infty} V(y,z)\geqq 0 \tag{3.8}$$

uniformly in $y\in\bar B_m(\varepsilon_0)$. *Then the zero solution of* (3.1) *is y-unstable.*

Theorem 3.5. *Let the conditions of Theorem 3.4 be satisfied. Suppose, in addition, that the Ljapunov function V is bounded from below on the set $\bar{B}_m(\varepsilon_0)\times R^n$, and*

$$\liminf_{|z|\to\infty,\, \dot{V}\to 0} V(y, z) \geqq 0 \tag{3.9}$$

uniformly in $y\in\bar{B}_m(\varepsilon_0)$. Then the zero solution of (3.1) *is unstable with respect to y.*

Proof. We shall prove that for all $\delta>0$ every curve $\gamma_y^+(x_0(\delta))$ defined in Theorem 3.4 leaves the ball $B_m(\varepsilon_0)$. If it is not true then for some δ_0 $(0<\delta_0<\varepsilon_0)$ the solution (3.6) possesses properties (3.7), and, by Theorem 3.4, $|\chi(t)|\to\infty$ as $t\to\infty$. Similarly to the proof of Theorem 3.2 it can be proved that (3.9) implies $v_0=0$. This contradiction completes the proof.

4. Applications to mechanical systems

Consider a holonomic mechanical system of r degrees of freedom with time-independent constraints under the action of potential and dissipative forces. The Lagrangian equations of motions are

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}}-\frac{\partial T}{\partial q} = -\frac{\partial P}{\partial q}+Q, \tag{4.1}$$

where the following notations [11] are used: the column vectors $q, \dot{q}\in R^r$ consist of generalized coordinates and velocities, respectively (v^T denotes the transposed of $v\in R^r$); the potential energy $P: q\mapsto P(q)\in R$ is continuously differentiable, $P(0)=0$; $T=T(q, \dot{q})=(1/2)\dot{q}^T A(q)\dot{q}$ is the kinetic energy, where the symmetric matrix function $A: q\mapsto A(q)\in R^{r\times r}$ is continuously differentiable; the continuous function $Q: (q, \dot{q})\mapsto Q(q, \dot{q})\in R^r$ is the resultant of non-energic and dissipative forces with complete dissipation, i.e. there exists a function $c\in\mathcal{K}$ such that $Q^T(q, \dot{q})\dot{q}\leqq -c(|\dot{q}|)$ for all $q, \dot{q}\in R^r$. Assume that $q=\dot{q}=0$ is an equilibrium of (4.1) and the motions starting from a neighbourhood of this equilibrium depend continuously on the initial coordinates and velocities.

L. Salvadori [11] proved that if the equilibrium at $q=0$ is isolated then it is asymptotically stable if P has a minimum there, and unstable if P has no minimum there. By means of a simple example with one degree of freedom K. Peiffer [14] showed that this theorem would be false without the condition that the equilibrium at $q=0$ is isolated. Applying our results we give sufficient conditions for asymptotic stability and instability of the equilibrium at $q=0$ (possibly non-isolated) with respect to velocities or some generalized coordinates.

For $q \in R^r$ denote by $\lambda(q)$ $(\Lambda(q))$ the smallest (largest) eigenvalue of the symmetric matrix $A(q)$. We can estimate the kinetic energy as follows

$$\frac{1}{2}\lambda(q)|\dot q|^2 \leqq T(q, \dot q) \leqq \frac{1}{2}\Lambda(q)|\dot q|^2 \quad (q, \dot q \in R^r). \tag{4.2}$$

For every $q \in R^r$ the matrix $A(q)$ is positive definite (i.e. $\lambda(q) > 0$) (see [11], p. 362); therefore, (4.1) can be rewritten into the equivalent normal form

$$\dot x = X(x) \quad (x = \mathrm{col}\,(q, \dot q)).$$

As is known (see [11], p. 358), the derivative of the total mechanical energy $H = T + P$ with respect to (4.1) is $\dot H(q, \dot q) = Q(q, \dot q)\dot q \leqq 0$; consequently, H is a Ljapunov function of (4.1). The dissipation is complete, hence for arbitrary $c \in R$ we have

$$\dot H^{-1}(0) \cap H^{-1}(c) = \{\mathrm{col}\,(q, \dot q)\colon\ P(q) = c,\ \dot q = 0\};$$

therefore, the complete trajectories of (4.1) contained in this set are the equilibria $q = q_0$, $\dot q = 0$ for which $P(q_0) = c$.

Sometimes we shall use a partition $q = \mathrm{col}\,(\hat q, \tilde q)$ of the vector of generalized coordinates, where $\hat q \in R^s$, $\tilde q \in R^{r-s}$, $0 \leqq s \leqq r$ (if $s = 0$ then $\tilde q = q$, and the conditions and statements concerning $\hat q$ are to be dropped).

In his first paper on partial stability, V. V. Rumjancev [1] proved that in the absence of any potential forces the equilibrium $q = \dot q = 0$ of (4.1) is asymptotically $\dot q$-stable provided that there are some constants λ_0, Λ_0 such that

$$0 < \lambda_0 \leqq \lambda(q) \leqq \Lambda(q) \leqq \Lambda_0.$$

The following two corollaries generalize this result to the case of the presence of potential forces.

Corollary 4.1. *Suppose that the potential energy $P(\hat q, \tilde q)$ is positive $\hat q$-definite and the region $\{q\colon P(q) > 0\}$ contains no equilibria. If for some H'*

$$\lim_{|\tilde q| \to \infty} \lambda(\hat q, \tilde q) = \infty \tag{4.1}$$

uniformly in $\hat q \in \bar B_s(H')$, then the equilibrium $q = \dot q = 0$ of (4.1) is asymptotically $\dot q$-stable.

Proof. We will apply Corollary 3.1 to equation (4.1) and Ljapunov function H with $y = \dot q$. Condition (4.3) and inequality (4.2) imply that H is positive $\hat q$-definite as well, hence the equilibrium is $\hat q$-stable (see [11], p. 15) and we can assume that $\hat q$ belongs to a bounded set. Consequently, in (4.3) we can change $|\tilde q| \to \infty$ into $|q| \to \infty$ and, in view of (4.2), we have $T(q, \dot q) \to \infty$ as $|q| \to \infty$ for every $\dot q \neq 0$. This means, that all conditions of Corollary 3.1 are fulfilled.

Corollary 4.2. *If*

(i) $P(q) \geqq 0$ $(q \in R^r)$;

(ii) *the set* $\{q: P(q) > 0\}$ *contains no equilibria;*

(iii) $\lambda(q) \geqq \lambda_0$ $(0 < \lambda_0 = \text{const.})$;

(iv) $\lim_{|q| \to \infty} P(q) \leqq \infty$ *exists;*

(v) *there are* $d \in \mathcal{K}$ *and* $H' > 0$ *such that*

$$Q(q, \dot{q})\dot{q} \leqq -d(T(q, \dot{q})) \quad (q \in R^r, \ |\dot{q}| \leqq H'), \tag{4.4}$$

then the equilibrium $q = \dot{q} = 0$ *of* (4.1) *is asymptotically* $\dot{q}$*-stable.*

Proof. If $P(q) \to \infty$ as $|q| \to \infty$, then the generalized coordinates are bounded along every motion, and the statement follows from Theorem A in Section 2. Suppose the limit of P is finite. By Theorem 3.1 the equilibrium $q = \dot{q} = 0$ is $\dot{q}$-stable, and for every motion $(q(t), \dot{q}(t))$ starting from some neighbourhood of $q = \dot{q} = 0$ either $h(t) = H(q(t), \dot{q}(t)) \to 0$ or $|q(t)| \to \infty$ as $t \to \infty$. In the second case $P(q(t))$ has a finite limit, thus $T(q(t), \dot{q}(t)) \to T_0$ as $t \to \infty$. If $T_0 > 0$, then by condition (v)

$$\frac{d}{dt} h(t) = Q(q(t), \dot{q}(t))\dot{q}(t) \leqq -d(T_0) < 0 \quad (t \in R_+),$$

which is impossible, because h is non-negative. Therefore, in both cases $T(q(t), \dot{q}(t)) \to 0$ $t \to \infty$. According to (iii) and (4.2) this implies $|\dot{q}(t)| \to 0$ as $t \to \infty$, which completes the proof.

Condition (iv) is rather restrictive, but if we know more of the behaviour of generalized coordinates we can weaken it, as its only role is to assure the existence of the (finite or infinite) limit of the potential energy along motions. For example, if $P(\hat{q}, \tilde{q})$ is positive $\hat{q}$-definite then we can assume that $\hat{q} \in \bar{B}_s(H')$ with some constant H'. Suppose that $P(\hat{q}, \tilde{q}) \to P_*(\hat{q})$ as $|\tilde{q}| \to \infty$. If $P_*(\hat{q})$ is constant then (iv) is satisfied. The case of changing $P_*(\hat{q})$ can be treated by the application of the further development of the Barbashin—Krasovskiĭ method in another direction [10].

As to condition (v), it is obviously fulfilled if there is a Λ_0 such that $\Lambda(q) \leqq \Lambda_0$ for all $q \in R^r$ (namely, $d(u) = c(u/\Lambda_0)$).

If we know a priori that the generalized coordinates are bounded, then conditions (iii)—(v) can be dropped, and the statement is a consequence of Oziraner's theorem (Theorem A in Section 2). But it is worth noticing that the conditions of Corollary 4.2 can be satisfied even if the generalized coordinates are not bounded. This can be shown by the system of one degree of freedom described by the equation

$$\ddot{q} + q^3 = 0 \quad (q, \dot{q} \in R)$$

found by K. Peiffer (see [11], p. 115) in order to prove that complete dissipation does not imply stability in case $P(q) \leqq 0$.

Applying Corollary 3.1 to equation (4.1) and Ljapunov function H with $y=\hat{q}$ we obtain

Corollary 4.3. *If*

(i) *P is positive $\hat{q}$-definite;*

(ii) *the set $\{q\colon \hat{q}\neq 0\}$ contains no equilibria;*

(iii) *for every $\hat{q}_0\neq 0$,* $\lim\limits_{\hat{q}\to\hat{q}_0,\,|\tilde{q}|\to\infty} P(\hat{q},\tilde{q}) = \infty,$

then the equilibrium $q=\dot{q}=0$ is asymptotically $\hat{q}$-stable.

Assuming that the equilibrium $q=0$ is isolated with respect to the region $\{q\colon P(q)<0\}$ and P has no minimum there, W. T. Koiter [15] proved that the equilibrium is unstable. The special case $q=\hat{q}$ of our following corollary shows that it is, in fact, q-unstable.

Corollary 4.4. *Suppose that for some $H', \varepsilon_0, \lambda_0$ $(0<\varepsilon_0<H', \lambda_0>0)$ the following conditions are satisfied:*

(i) *for every δ $(0<\delta<\varepsilon_0)$ there is a $q_0(\delta)\in B_r(\delta)$ with $P(q_0(\delta))<0$;*

(ii) $\lambda(\hat{q},\tilde{q})\geqq\lambda_0>0$ $(|\hat{q}|\leqq H', \tilde{q}\in R^{r-s})$;

(iii) *the region $\{q\colon P(q)<0,\ |\hat{q}|<\varepsilon_0\}$ contains no equilibria.*

Then for every δ $(0<\delta<\varepsilon_0)$ either a) *the curve $\gamma_{\hat{q}}^{+}(q_0(\delta),0)\colon t\mapsto\hat{q}(t;\, q_0(\delta),0)$ $(t\in R_+)$ leaves the ball $B_s(\varepsilon_0)$ in finite time, or* b) $|\tilde{q}(t;\, q_0(\delta),0)|\to\infty$ *as* $t\to\infty$.

Proof. We can apply Theorem 3.4 with $V=H, y=\hat{q}$, observing that condition (ii) precludes the possibility of $|\dot{q}(t)|\to\infty$ without $|\tilde{q}(t)|\to\infty$ as $t\to\infty$.

Now, Corollary 3.2 yields

Corollary 4.5. *Let all conditions of Corollary* 4.4 *be satisfied. Suppose, in addition, that*

(iv) $\liminf\limits_{|\tilde{q}|\to\infty} P(\hat{q},\tilde{q}) \geqq 0$ *uniformly in* $\hat{q}\in\bar{B}_s(H')$.

Then the equilibrium $q=\dot{q}=0$ of (4.1) *is $\hat{q}$-unstable.*

Examples. Finally, in order to illustrate the results of this section we study the stability properties of the mechanical system of two degrees of freedom introduced and investigated by K. Peiffer and N. Rouche [13]. Consider a point of mass equal to 1 moving in a constant field of gravity in the inertial frame of reference $Oxyz$; Oz directed vertically upward. Suppose the point is constrained to move on a surface of equation $z=(1/2)y^2(1+x^2)$ and furthermore, that it is subjected to viscous friction. The total mechanical energy is $H=T+P$ where

$$T=\frac{1}{2}\dot{x}^2+\frac{1}{2}\dot{y}^2+\frac{1}{2}y^2[\dot{y}(1+x^2)+\dot{x}xy]^2,\quad P=\frac{g}{2}y^2(1+x^2)$$

and g is the acceleration in the gravity field. Let the dissipative forces be defined by the formulas

$$Q^1 = -\alpha \frac{\partial T}{\partial \dot{x}} \quad Q^2 = -\alpha \frac{\partial T}{\partial \dot{y}}. \tag{4.5}$$

By Rumjancev's theorem ([1], see also [11], p. 15) the equilibrium $x=y=\dot{x}=\dot{y}=0$ is $(y, \dot{x}, \dot{y})$-stable, but the coordinate x may be even unbounded. Therefore, although the system is autonomous, the earlier theorems of Barbashin—Krasovskiĭ type cannot be applied to establish asymptotic stability with respect to $(y, \dot{x}, \dot{y})$. Peiffer and Rouche proved that the stability is asymptotic with respect to $\dot{x}$. Applying Corollary 4.3 with $\hat{q}=y$ we obtain that the equilibrium is asymptotically y-stable even under arbitrary nonlinear friction with total dissipation (the special form (4.5) of the dissipative forces is not needed).

Note that by the use of the Ljapunov—Malkin theorem on the critical case of the stability investigations by first approximations (see [16], p. 113) one can prove that the equilibrium is stable with respect to all variables, the stability is asymptotic with respect to $(y, \dot{x}, \dot{y})$, and for every motion starting from some neighbourhood of the equilibrium $x(t) \to x_0 = \text{const.}$ as $t \to \infty$.

Our theorems allow us to investigate the general case when the point is constrained to move on a surface of equation $z=f(x, y)$ $(f(0,0)=0)$. Corollaries 4.3 and 4.5 yield conditions on the potential energy $P=gf(x, y)$ assuring the equilibrium $x=y=0$ to be asymptotically y-stable or y-unstable. We illustrate this by two simple examples.

Let

$$f(x, y) = \begin{cases} (1/2)y^2(1+x^2) + e^{-1/|x|}\sin^2(1/x^2) & (x \neq 0) \\ 0 & (x = 0). \end{cases}$$

By Corollary 4.3, the equilibrium $x=y=\dot{x}=\dot{y}=0$ is asymptotically y-stable in spite of the fact that the region $\{(x, y)\colon P(x, y) > 0\}$ contains equilibria (see condition (ii) in Theorem A in Section 2).

As Corollary 4.5 shows, in the case of $f(x, y) = y^3/(1+x^2)$ the equilibrium $x=y=\dot{x}=\dot{y}=0$ is y-unstable.

Acknowledgement. The author is very grateful to Professor K. Peiffer for many useful discussions.

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