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The behavior of the Riesz representation theorem with respect to order and topology

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Dedicated with admiration to Béla Sz.-Nagy on the occasion of his 70th birthday

The celebrated representation theorem of Frederick Riesz, the revered teacher of both Béla Sz.-Nagy and myself, can be stated in the following form: every Radon measure on an open interval of the real line is the derivative in the sense of distributions of a function which is locally of bounded variation. In the present note I want to give some precisions about how this correspondence between functions locally of bounded variation and Radon measures behaves with respect to the topologies and the order structures on the two spaces involved.

1. Let I be a non-empty open interval $]a, b[$ of the real line $\mathbf{R}$, which may be finite or infinite, i.e., $-\infty \leqq a < b \leqq \infty$. We shall only consider real-valued functions and real measures in this note. We denote by $\mathscr{V}(I)$ the vector space of all functions $f\colon I \to \mathbf{R}$ whose total variation

$$q_{\alpha\beta}(f) = \sup_{\Delta} \sum_{j=1}^{l} |f(x_j) - f(x_{j-1})|$$

is finite on every compact subinterval $[\alpha, \beta]$ of I; here the least upper bound is taken with respect to all subdivisions

$$\Delta\colon\ \alpha = x_0 < x_1 < \ldots < x_l = \beta \tag{1}$$

of $[\alpha, \beta]$. Each $q_{\alpha\beta}$ is a semi-norm. We consider $\mathscr{V}(I)$ equipped with the non-Hausdorff locally convex topology defined by the family $(q_{\alpha\beta})$ of semi-norms, where $[\alpha, \beta]$ varies in I.

A preorder compatible with the vector space structure of $\mathscr{V}(I)$ is defined if we take for the cone P of positive elements the set of all increasing functions. Then $P \cap (-P)$ consists of the constants and is, incidentally, also the closure of $\{0\}$

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in the topology defined above. Every $f \in \mathscr{V}(I)$ is the difference of two increasing functions, i.e., $\mathscr{V}(I)=P-P$, and therefore $f(x-0)$ and $f(x+0)$ exist at every $x \in I$. Any two elements $f, g \in \mathscr{V}(I)$ have a least upper bound $\sup(f, g)$ which is determined up to an additive constant. In particular the positive variation $\Phi = = \mathrm{var}^+ f = \sup(f, 0)$ of $f \in \mathscr{V}(I)$ is defined up to an additive constant; it is an increasing function such that $\Phi - f$ is increasing and such that, whenever g is an increasing function for which $g-f$ is increasing, then $g-\Phi$ is increasing; it is given explicitly by the formula

$$\Phi(\beta)-\Phi(\alpha) = \sup_{\Delta} \sum_{j=1}^{l} \max\left(f(x_i)-f(x_{i-1}), 0\right),$$

where $a<\alpha<\beta<b$, and the least upper bound is taken with respect to all subdivisions (1). Similarly, the negative variation $\Psi = \mathrm{var}^- f = \sup(-f, 0)$ is defined, and $F=\Phi+\Psi=\mathrm{var} f=\sup(f, -f)$ is the absolute variation of f.

Since each $f \in \mathscr{V}(I)$ is locally integrable, we can associate with it the Radon measure T_f which has density f with respect to Lebesgue measure, i.e., which is given by

$$\langle T_f, \varphi \rangle = \int_I \varphi(x) f(x)\, dx$$

for all functions φ belonging to the space $\mathscr{K}(I)$ of continuous functions with compact support in I.

With $f \in \mathscr{V}(I)$ we can associate a second Radon measure, the Stieltjes measure S_f, defined by the Stieltjes integral

$$\langle S_f, \varphi \rangle = \int_I \varphi(x)\, df(x)$$

for all $\varphi \in \mathscr{K}(I)$. The inequality

$$|\langle S_f, \varphi \rangle| \leqq q_{\alpha\beta}(f) \cdot \max |\varphi(x)|, \tag{2}$$

valid for all $\varphi \in \mathscr{K}(I)$ with $\mathrm{Supp}\, \varphi \subset [\alpha, \beta]$, shows that S_f is indeed a Radon measure. Clearly, $f \mapsto S_f$ is a linear map from $\mathscr{V}(I)$ into the space $\mathscr{M}(I)=\mathscr{K}'(I)$ of all Radon measures on I. For $a<\alpha\leqq\beta<b$ we have

$$\begin{aligned} &S_f(]\alpha, \beta[) = f(\beta-0)-f(\alpha+0), \quad S_f([\alpha, \beta]) = f(\beta+0)-f(\alpha-0), \\ &S_f(]\alpha, \beta]) = f(\beta+0)-f(\alpha+0), \quad S_f([\alpha, \beta[) = f(\beta-0)-f(\alpha-0). \end{aligned} \tag{3}$$

The integration by parts formula

$$\int_I \varphi(x)\, df(x) = -\int_I \varphi'(x) f(x)\, dx$$

holds in particular for all φ belonging to the space $\mathcal{D}(I)$ of all infinitely differentiable functions with compact support in I. Rewriting it in the form

$$\langle S_f, \varphi\rangle = -\langle T_f, \varphi'\rangle = \langle \partial T_f, \varphi\rangle,$$

we see that S_f is the derivative ∂T_f in the sense of distributions of T_f.

The representation theorem of Frederick Riesz now states that the map $f \mapsto S_f$ is surjective from $\mathcal{V}(I)$ onto $\mathcal{M}(I)$ [2]. It follows from the formulas (3) that $S_f = 0$ if and only if there exists a constant $C \in \mathbf{R}$ such that $f(x-0) = f(x+0) = C$ at each $x \in I$. It follows furthermore from (3) that if μ is a positive measure, one can find an increasing function $f \in \mathcal{V}(I)$ for which $S_f = \mu$. It is obvious that, conversely, if f is increasing, then S_f is positive.

2. If $\mu = S_f$, we do not have necessarily $|\mu| = S_{\mathrm{var}\, f}$. Indeed, take $I = \mathbf{R}$ and let $f(x) = 0$ for $x \neq 0$ and $f(0) = 1$. Then $\mathrm{var}^+ f$ is the Heaviside function Y, $\mathrm{var}^- f$ is the function taking the value 0 for $x \leqq 0$ and the value 1 for $x > 0$, and

$$\mathrm{var}\, f = \begin{cases} 0 & \text{if} \quad x < 0, \\ 1 & \text{if} \quad x = 0, \\ 2 & \text{if} \quad x > 0. \end{cases}$$

Thus $S_{\mathrm{var}\, f} = 2\delta$ but $S_f = |S_f| = 0$. The situation improves if we suppose that $f(x)$ is between $f(x-0)$ and $f(x+0)$:

Theorem 1. *If $f \in \mathcal{V}(I)$, then*

$$S_f^+ \leqq S_{\mathrm{var}^+ f}, \; S_f^- \leqq S_{\mathrm{var}^- f}, \; |S_f| \leqq S_{\mathrm{var}\, f}.$$

If we assume furthermore that $f(x)$ is between $f(x-0)$ and $f(x+0)$ for every $x \in I$, then the sign of equality is valid in the three inequalities.

Proof. Set $\Phi = \mathrm{var}^+ f$, $\Psi = \mathrm{var}^- f$ and $F = \mathrm{var}\, f$. Both Φ and $\Phi - f$ are increasing functions, hence $S_\Phi \geqq 0$ and $S_\Phi \geqq S_f$, i.e., $S_\Phi \geqq S_f^+$. One sees similarly that $S_\Psi \geqq S_f^-$. Thus $S_F = S_\Phi + S_\Psi \geqq S_f^+ + S_f^- = |S_f|$.

Assume next that f is continuous and let μ be a positive measure on I such that $\mu \geqq S_f$. There exists an increasing function g such that $S_g = \mu$. It follows from the above remarks that we can find an increasing function h such that $S_h = \mu - S_f = S_{g-f}$ and which satisfies

(4) $$h(x-0) = g(x-0) - f(x), \quad h(x+0) = g(x+0) - f(x)$$

at every $x \in I$. Since g is increasing, we have

(5) $$g(x-0) - f(x) \leqq g(x) - f(x) \leqq g(x+0) - f(x)$$

at each $x\in I$. Formulas (4) and (5) imply that $g-f$ is an increasing function. It follows from the definition of $\mathrm{var}^+ f$ that $g-\Phi$ is increasing. Hence $\mu=S_g\geqq S_\Phi$, and so S_Φ is indeed the measure $S_f^+=\sup(S_f, 0)$. One can prove in exactly the same way that $S_\Psi=S_f^-=\sup(-S_f, 0)$. Thus we have also $S_F=S_\Phi+S_\Psi=$ $=S_f^+ +S_f^-=|S_f|$.

Consider now a pure jump function $f\in\mathscr{V}(I)$. Such a function is determined by two families $(l_x)_{x\in I}$, $(r_x)_{x\in I}$ of real numbers such that for every compact subset K of I we have $\sum_{x\in K}|l_x|<\infty$ and $\sum_{x\in K}|r_x|<\infty$. Writing $j_x=l_x+r_x$ and taking $a<\alpha<\beta<b$, the function f is given up to an additive constant by the formula

$$f(\beta)-f(\alpha) = r_\alpha+\sum_{\alpha<x<\beta} j_x+l_\beta.$$

One has $f(x)-f(x-0)=l_x$, $f(x+0)-f(x)=r_x$, i.e., f is the pure jump function with left jump l_x and right jump r_x at x. If we define $l_x^+=\max(l_x, 0)$, $l_x^-=$ $=\max(-l_x, 0)$, and similarly for r_x^+, r_x^-, then $\Phi=\mathrm{var}^+ f$ is the pure jump function with jumps l_x^+, r_x^+, $\Psi=\mathrm{var}^- f$ is the pure jump function with jumps l_x^-, r_x^-, and $F=\mathrm{var} f$ is the pure jump function with jumps $|l_x|$, $|r_x|$. The corresponding Stieltjes measures are $S_f=\sum_{x\in I} j_x\delta_x$, $S_\Phi=\sum_{x\in I}(l_x^+ +r_x^+)\delta_x$, $S_\Psi=\sum_{x\in I}(l_x^- +r_x^-)\delta_x$ and $S_F=\sum_{x\in I}(|l_x|+|r_x|)\delta_x$. On the other hand, $|S_f|=\sum_{x\in I}|j_x|\delta_x$. If we assume that $f(x)$ is between $f(x-0)$ and $f(x+0)$, then $|j_x|=|l_x|+|r_x|$ and so $S_F=|S_f|$.

Every function $f\in\mathscr{V}(I)$ can be decomposed into a sum $f=f_C+f_J$ of a continuous function $f_C\in\mathscr{V}(I)$ and a pure jump function $f_J\in\mathscr{V}(I)$; and this decomposition is unique up to an additive constant. If f satisfies the condition that $f(x)$ is between $f(x+0)$ and $f(x-0)$, then so does f_J since f has the same jumps as f_J. The measure S_{f_J} is concentrated on the countable set of points where the jumps of f are non-zero; i.e., S_{f_J} is atomic. By virtue of (3) the measure S_{f_C} of every countable set is zero, i.e., S_{f_C} is diffuse. It follows from what has been said above and from [1], Chap. V, §5, n° 10, Proposition 15 that

$$|S_f| = |S_{f_C}|+|S_{f_J}| = S_{\mathrm{var} f_C}+S_{\mathrm{var} f_J}. \tag{6}$$

Since the functions $\mathrm{var} f_C-f_C$ and $\mathrm{var} f_J-f_J$ are increasing, so is $\mathrm{var} f_C+\mathrm{var} f_J-f$ and therefore also $\mathrm{var} f_C+\mathrm{var} f_J-\mathrm{var} f$. Thus

$$S_{\mathrm{var} f_C}+S_{\mathrm{var} f_J} \geqq S_{\mathrm{var} f}=S_F. \tag{7}$$

Combining (6) and (7) we obtain $|S_f|\geqq S_F$. The opposite inequality has already been proved as the first assertion of the theorem, hence $|S_f|=S_F$.

Finally we have $S_\Phi=(1/2)(S_F+S_f)=(1/2)(|S_f|+S_f)=S_f^+$ and $S_\Psi=$ $=(1/2)(S_F-S_f)=(1/2)(|S_f|-S_f)=S_f^-$.

3. To conclude, I want to show that the map $f\mapsto S_f$ behaves in the best possible way with respect to the topologies involved. I announced this result earlier ([2], Theorem 2) with a somewhat terse proof. The "good" proof is based on the following simple, general observation:

Theorem 2. *Let X be a locally compact, paracompact topological space. The semi-norms $\mu \mapsto |\mu|(K)$, where K runs through the compact subsets of X, define on the space $\mathcal{M}(X)$ of Radon measures the strong topology $\beta(\mathcal{M}(X), \mathcal{K}(X))$.*

Proof. Let us denote by $\mathcal{T}$ the topology defined by the semi-norms $\mu \mapsto |\mu|(K)$.

(a) Let V be a neighborhood of 0 for the topology $\mathcal{T}$. We may assume that V is of the form $\{\mu \in \mathcal{M}(X): |\mu|(K) \leqq \varepsilon\}$, where $\varepsilon > 0$ and K is a compact subset of X. Let L be a compact neighborhood of K. The set

$$B = \{\varphi \in \mathcal{K}(X)\colon \operatorname{Supp} \varphi \subset L, |\varphi(x)| \leqq 1/\varepsilon\}$$

is bounded in $\mathcal{K}(X)$. Consider an arbitrary μ in the polar B° of B. If φ is a positive function in B and $\psi \in \mathcal{K}(X)$ is such that $|\psi| \leqq \varphi$, then ψ also belongs to B, and it follows from [1], Chap. III, § 1, n° 5, formula (9) that

$$\langle |\mu|, \varphi \rangle = \sup_{\substack{|\psi| \leqq \varphi \\ \psi \in \mathcal{K}(X)}} \langle \mu, \psi \rangle \leqq 1.$$

By [1], Chap. III, § 1, n° 2, Lemme 1 there exists $\varphi \in B$ such that $\chi_K \leqq \varepsilon\varphi$. Thus

$$|\mu|(K) = |\mu|(\chi_K) \leqq \varepsilon \langle |\mu|, \varphi \rangle \leqq \varepsilon$$

and therefore $\mu \in V$. We have proved that V contains the strong neighborhood B° of 0.

(b) Conversely, let W be a strong neighborhood of 0. We may assume that $W = B^\circ$ where B is a bounded subset of $\mathcal{K}(X)$. Since X is paracompact, by [1], Chap. III, § 1, n° 1, Proposition 2 (ii) there exists a compact set $K \subset X$ and a number $\gamma > 0$ such that $\operatorname{Supp} \varphi \subset K$ and $|\varphi(x)| \leqq \gamma$ for $\varphi \in B, x \in X$. The set

$$V = \{\mu \in \mathcal{M}(X)\colon |\mu|(K)| \leqq 1/\gamma\}$$

is a $\mathcal{T}$-neighborhood of 0. If $\mu \in V$ and $\varphi \in B$, then $|\varphi| \leqq \gamma\chi_K$ and therefore

$$|\langle \mu, \varphi \rangle| \leqq \left|\langle |\mu|, |\varphi| \rangle\right| \leqq \gamma \cdot |\mu|(\chi_K) = \gamma \cdot |\mu|(K) \leqq 1.$$

Thus $\mu \in B^\circ$ and we have proved that $V \subset W$.

If (K_ν) is a family of compact subsets of X such that each compact subset of X is contained in some K_ν, then the strong topology on $\mathcal{M}(X)$ is also defined by the family of semi-norms $\mu \mapsto |\mu|(K_\nu)$.

If X is locally compact but not paracompact, then the semi-norms $\mu \mapsto |\mu|(K)$ define the quasi-strong topology ([1], Chap. III, § 1, Exerc. 8) on $\mathcal{M}(X)$.

4. We are now ready to prove the result referred to above.

Theorem 3. *The surjective linear map $f \mapsto S_f$ from $\mathscr{V}(I)$ onto $\mathscr{M}(I)$ is a strict morphism if we equip $\mathscr{M}(I)$ with the strong topology.*

Proof. (a) We first prove that the map is continuous. Let B be a bounded subset of $\mathscr{K}(I)$. There exists a compact subinterval $[\alpha, \beta]$ of I and a number $\gamma > 0$ such that $\operatorname{Supp} \varphi \subset [\alpha, \beta]$ and $|\varphi(x)| \leqq \gamma$ for $\varphi \in B$, $x \in I$. Define a neighborhood of 0 in $\mathscr{V}(I)$ by

$$V = \{f \in \mathscr{V}(I): q_{\alpha\beta}(f) \leqq 1/\gamma\}.$$

If $f \in V$ and $\varphi \in B$, then by inequality (2) we have $|\langle S_f, \varphi \rangle| \leqq 1$, i.e., S_f belongs to the neighborhood B° of 0 in $\mathscr{M}(I)$.

(b) Now we prove that the map is open. Let V be a neighborhood of 0 in $\mathscr{V}(I)$ which we may assume to be of the form

$$V = \{f \in \mathscr{V}(I): q_{\alpha\beta}(f) \leqq \varepsilon\},$$

where $K = [\alpha, \beta] \subset I$ and $\varepsilon > 0$. Let W be the neighborhood $\{\mu \in \mathscr{M}(I): |\mu|(K) \leqq \varepsilon\}$ of 0 in $\mathscr{M}(I)$. Given $\mu \in W$, we want to find an f in V such that $\mu = S_f$; then we will have proved that the image of V contains W.

The existence of such an f is implicit in the proof of Theorem 1 of [2] but we can also proceed as follows. Let $\mu = \mu^+ - \mu^-$, then $|\mu| = \mu^+ + \mu^-$. There exist increasing functions g and h such that $\mu^+ = S_g$, $\mu^- = S_h$. Then by virtue of the formulas (3)

$$q_{\alpha\beta}(g) \leqq g(\beta+0) - g(\alpha-0) = \mu^+(K)$$

and similarly $q_{\alpha\beta}(h) \leqq \mu^-(K)$. Setting $f = g - h$ we have $S_f = S_g - S_h = \mu^+ - \mu^- = \mu$ and

$$q_{\alpha\beta}(f) \leqq q_{\alpha\beta}(g) + q_{\alpha\beta}(h) \leqq \mu^+(K) + \mu^-(K) = |\mu|(K) \leqq \varepsilon,$$

i.e., $f \in V$.

References

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