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On the representation of distributive algebraic lattices. I

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Dedicated to Professor Béla Szőkefalvi-Nagy on his 70th birthday

E. T. SCHMIDT [4] proved that every distributive lattice is isomorphic with the lattice of all compact congruences of a lattice. The analogous question for distributive semilattices is a long-standing conjecture of lattice theory. In this paper we prove a theorem which can be considered as a further evidence to this conjecture. Our result is based on a theorem of P. Pudlák. Motivated by Schmidt's result, PUDLÁK [3] discovered another method suitable to attack the problem. He first proved that every distributive semilattice is the direct limit of its finite distributive subsemilattices. This reduces the conjecture to the following

Problem. Consider the category of finite distributive lattices where the morphisms are the one-to-one 0-preserving $\vee$-homomorphisms. Is there any functor R of this category to the category of finite lattices (with lattice embeddings) such that the following hold?

(α) For any distributive lattice D, there is an isomorphism $\varphi_D\colon D\cong \mathrm{Con}(R(D))$.

(β) Whenever D_1 has a one-to-one 0-preserving $\vee$-homomorphism δ to D_2, then $R(D_1)$ has a lattice embedding $R(\delta)$ to $R(D_2)$, such that

(γ) $R(\delta_{12}\delta_{23})=R(\delta_{12})R(\delta_{23})$ for all $\delta_{12}\colon D_1\to D_2$ and $\delta_{23}\colon D_2\to D_3$ satisfying the stipulations in (β), and,

(δ) if we denote by $\mathrm{Con}\,(R(\delta))$ the mapping of $\mathrm{Con}\,(R(D_1))$ to $\mathrm{Con}\,(R(D_2))$ induced by $R(\delta)$ (that is, the one, which maps $\Theta_1\in\mathrm{Con}\,(R(D_1))$ to the congruence generated by $\{(aR(\delta), bR(\delta))\in R(D_2)^2|(a,b)\in\theta_1\}$), then the following diagram is commutative

$$\begin{array}{ccc} D_1 & \xrightarrow{\quad\delta\quad} & D_2 \\ \downarrow{\scriptstyle\varphi_{D_1}} & & \downarrow{\scriptstyle\varphi_{D_2}} \\ \mathrm{Con}\,(R(D_1)) & \xrightarrow{\mathrm{Con}\,(R(\delta))} & \mathrm{Con}\,(R(D_2)) \end{array} .$$

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In case of an affirmative answer the conjecture would follow. Indeed, for any distributive semilattice D, we can choose a directed set $\{D_\gamma\}_{\gamma\in\Gamma}$ of finite distributive subsemilattices approaching to D. By (γ) and (δ), the $\mathrm{Con}\,(R(D_\gamma))$'s form the same directed set (up to commuting isomorphisms). Therefore, the direct limit of this set is D, too. On the other hand, the $R(D_\gamma)$'s, too, form a directed set, and the semilattice of all finitely generated congruences of their direct limit is the direct limit of the $\mathrm{Con}\,(R(D_\gamma))$'s.

Pudlák carried out a modification of this program, namely, he proved the analogous statement for distributive lattices and 0-preserving lattice embeddings in the place of distributive semilattices and 0-preserving $\vee$-embeddings, and obtained a new proof of Schmidt's theorem. We are interested in the question how much of Pudlák's theorem can be proved without imposing the restriction that the embeddings be lattice embeddings. It will be shown that two finite distributive semilattices with 0 have a simultaneous representation (that is, a representation satisfying (α), (β) and (δ)), provided one of them is a 0-subsemilattice of the other. In Part II of this paper we shall derive Pudlák's theorem from this result as well as Bauer's result on the representability of countable semilattices.

The main result of this part is the following

Theorem. *Let D_1 and D_2 be finite distributive lattices, and let $\delta: d\mapsto d^+$ be a one-to-one 0-preserving $\vee$-homomorphism of D_1 into D_2. Then there exist lattices L_1 and L_2 such that*

(α_1) $D_i\cong\mathrm{Con}\,(L_i)$, $i=1,2$ *(these isomorphisms will be denoted by φ_i),*

(β_1) *L_1 can be embedded to L_2 (by a one-to-one lattice isomorphism, to be denoted by λ),*

(δ_0) *every congruence of $L_1\lambda$ can be extended to L_2; and, therefore the mapping γ: $\mathrm{Con}\,(L_1)\to\mathrm{Con}\,(L_2)$, taking each $\Theta\in\mathrm{Con}\,(L_1)$ to its smallest extension, that is, to the congruence generated by $\{(a\lambda, b\lambda)|(a,b)\in\Theta\}$, is also a one-to-one 0-preserving $\vee$-homomorphism, furthermore*

(δ_1) *for all $d_i\in D_i$, $i=1,2$, δ maps d_1 to d_2 if and only if γ maps $d_1\varphi_1$ to $d_2\varphi_2$. In other words, γ represents δ.*

1. Proof of (α_1). We define L_1 (see E. T. Schmidt [5], pp. 82—87) as follows. Let B_1 be the Boolean lattice generated by D_1. Let M_1 consist of all triples $(x,y,z)\in B_1^3$ satisfying $x\wedge y=x\wedge z=y\wedge z$. Let L_1 be the set of all triples in M_1 also satisfying $x\in D_1$. Then L_1 is a lattice, too, under the ordering of B_1^3. It is proven in E. T. Schmidt [5] that $D_1\cong\mathrm{Con}\,(L_1)$. For further purposes we shall recall the proof here. We need a description of the operations of L_1. The meet operation is the same as in B_1^3. However, the joins in B_1^3, M_1 and L_1 are different. They will be denoted by $\vee, \vee_M, \vee_L$, respectively (or by $\vee, \vee_{M_1}, \vee_{L_1}$, where necessary). To describe them we introduce the following operators. $(x,y,z)\mapsto$

$\mapsto(x, y, z)^{\sim}$ acts on B_1^3 and maps (x, y, z) to the smallest element of M_1 above (x, y, z). $x\mapsto\bar{x}$ acts on B_1 and maps x to the smallest element of D_1 above x. Finally, $(x, y, z)\mapsto(x, y, z)^{\wedge}$ acts on M_1 and maps (x, y, z) to the smallest element of L_1 above (x, y, z). Now we have (see [5]),

$$(x, y, z)\vee_M(x', y', z') = (x\vee x', y\vee y', z\vee z')^{\sim},$$

$$(x, y, z)\vee_L(x', y'; z') = (x\vee x', y\vee y', z\vee z')^{\sim\wedge},$$

$$(x, y, z)^{\sim} = \big(x\vee(y\wedge z), y\vee(x\wedge z), z\vee(x\wedge y)\big) \quad \text{for} \quad (x, y, z)\in B_1^3,$$

$$(x, y, z)^{\wedge} = \big(\bar{x}, y\vee(\bar{x}\wedge z), z\vee(\bar{x}\wedge y)\big) \quad \text{for} \quad (x, y, z)\in M_1.$$

Now consider any congruence α of L_1. We shall prove that α is generated by a pair $((0, 0, 0), (x, 0, 0))\in L_1^2$. (Then $x\in D_1$, and hence $D_1\cong\mathrm{Con}\,(L_1)$.) To prove this claim, let $(x, y, z)\,\alpha\,(x', y', z')$. Then, forming the meets with $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$, respectively, we obtain

$$(x, 0, 0)\,\alpha\,(x', 0, 0), \quad (0, y, 0)\,\alpha\,(0, y', 0), \quad (0, 0, z)\,\alpha\,(0, 0, z').$$

Hence $(x, 0, 0)\vee_L(0, 1, 0)=(x, 1, 0)^{\sim\wedge}=(x, 1, x)^{\wedge}=(x, 1, x)$, and $(x', 0, 0)\vee_L(0, 1, 0)= =(x', 1, x')$, thus $(x, 1, x)\,\alpha\,(x', 1, x')$. Forming the meet of both sides with $(0, 0, 1)$, we get $(0, 0, x)\,\alpha\,(0, 0, x')$. Similarly, $(0, 0, y)\,\alpha\,(0, 0, y')$. Thus the congruence generated by $((x, y, z), (x', y', z'))$ contains the pairs

$$((0, 0, x),\ (0, 0, x')), \quad ((0, 0, y),\ (0, 0, y')), \quad ((0, 0, z),\ (0, 0, z')).$$

It is also generated by them. Indeed, under the congruence generated by these three pairs the following pairs are also related:

$$((x, 0, 0),\ (x', 0, 0)), \quad ((0, y, 0),\ (0, y', 0)), \quad ((0, 0, z),\ (0, 0, z')).$$

(We have to compute as above.) Hence, computing modulo α,

$$(x, y, z) = ((x, 0, 0)\vee(0, y, 0)\vee(0, 0, z))^{\sim\wedge} =$$

$$= (x, 0, 0)\vee_L(0, y, 0)\vee_L(0, 0, z) \equiv (x', 0, 0)\vee_L(0, y', 0)\vee_L(0, 0, z') = (x', y', z').$$

The elements of the form $(0, 0, t)$ constitute a Boolean sublattice, thus the congruence generated by $((x, y, z), (x', y', z'))$ is generated by an ideal of $\{(0, 0, t)\,|\,t\in B_1\}$. Hence α is also generated by a pair $((0, 0, 0), (0, 0, t_\alpha))$ or, equivalently, by

$$((0, 0, 0),\ (0, 0, t_\alpha))\vee_L((0, 1, 0),\ (0, 1, 0)) = ((0, 1, 0),\ (\bar{t}_\alpha, 1, \bar{t}_\alpha)),$$

or by

$$((0, 1, 0),\ (\bar{t}_\alpha, 1, \bar{t}_\alpha))\wedge((1, 0, 0),\ (1, 0, 0)) = ((0, 0, 0),\ (\bar{t}_\alpha, 0, 0)),$$

as claimed. (For more details see [5].) Now consider the lattice of Figure 1. Let this lattice be denoted by L_2. We show that $D_2\cong\mathrm{Con}\,(L_2)$. First, however, let us give a more accurate description of this lattice. For a finite distributive lattice D let $M(D)$ (respectively, $L(D)$) denote the lattice formed from D analogously as M_1 (respectively, L_1) is formed from D_1. Furthermore, whenever D is a distributive lattice, let $B(D)$ denote the Boolean extension of D.

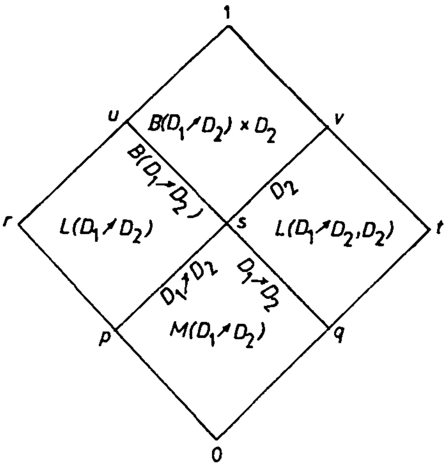

Figure 1

Finally, whenever D, D' are distributive lattices, $D \subseteq D'$, and the 0 and 1 of D are the same as those of D', then let $M(D', D)$ consist of all triples $(x, y, z) \in (D')^3$ satisfying $x \vee (y \wedge z) = y \vee (x \wedge z) = z \vee (x \wedge y)$ and let $L(D', D) = \{(x, y, z) | x \in D, (x, y, z) \in M(D', D)\}$. Now the meaning of $L(D_1 \nearrow D_2)$, $M(D_1 \nearrow D_2)$, $L(D_1 \nearrow D_2, D_2)$ and $B(D_1 \nearrow D_2) \times D_2$ of Figure 1 is clear. For the definition of $D_1 \nearrow D_2$, see [3].

As to how they are glued together note that $L(D_1 \nearrow D_2)$ contains an ideal isomorphic with $D_1 \nearrow D_2$ (the set of elements $(x, 0, 0)$, $x \in D_1 \nearrow D_2$) and $M(D_1 \nearrow D_2)$ contains such a dual ideal. The mapping which is identical on the D_i's maps the ideal of $L(D_1 \nearrow D_2)$ in question isomorphically to this dual ideal of $M(D_1 \nearrow D_2)$. Further isomorphism maps an ideal of $L(D_1 \nearrow D_2, D_2)$ to another dual ideal of $M(D_1 \nearrow D_2)$. If we identify the elements corresponding to each other under these isomorphisms we get a partial lattice (the union of $[0, u]$ and $[0, v]$ on Figure 1). It can be made into a lattice by inserting a $B(D_1 \nearrow D_2) \times D_2$ to the top of the Figure, and making analogous identifications. ($B(D_1 \nearrow D_2) \times D_2$ has an ideal isomorphic with $B(D_1 \nearrow D_2)$. This will be identified with the dual ideal of $L(D_1 \nearrow D_2)$ consisting of all those elements which are greater than or equal to all elements used in the identification between $L(D_1 \nearrow D_2)$ and $M(D_1 \nearrow D_2)$. $B(D_1 \nearrow D_2) \times D_2$ also has an ideal isomorphic with D_2, to be used for the identification with the corresponding dual ideal of $L(D_1 \nearrow D_2, D_2)$.) We show that $D_2 \cong \mathrm{Con}(L_2)$. Consider any two elements of L_2. The congruence generated by them is obviously a join of four congruences α_i, $i=1, 2, 3, 4$ where α_1 (respectively, $\alpha_2, \alpha_3, \alpha_4$) is generated by a pair of elements in $L(D_1 \nearrow D_2)$ (respectively, $M(D_1 \nearrow D_2)$, $L(D_1 \nearrow D_2, D_2)$, $B(D_1 \nearrow D_2) \times D_2$). If we prove that all of these congruences are generated by subintervals of $[q, t]$ containing q, then we are done. Now the same calculations that proved that $\mathrm{Con}(L_1) \cong D_1$ show that α_1 is generated by a subinterval of $[p, s]$ containing p; the same computations in $M(D_1 \nearrow D_2)$ and in $L(D_1 \nearrow D_2, D_2)$ yield that α_1 is generated by a subinterval of $[q, s]$ containing q as well as by a subinterval of $[q, t]$ containing q. α_2 can also be generated by elements of $L(D_1 \nearrow D_2)$ which reduces the case of α_2 to that of α_1. The case of α_3 can be reduced to that of α_2, and, finally, the case of α_4 follows from the cases of α_1 and α_3.

2. Proof of (β_1). Preparing this proof it turned out that Theorem 1 of [3], which was intended to be used in the proof of (β_1), is still not general enough. We have to prove a stronger result (Lemma 1). The proof of this result goes along the lines of [3], Theorem 1, for completeness' sake, however, we repeat part of the details.

Let $B(D_1 \nearrow D_2)$ be the Boolean lattice generated by $D_1 \nearrow D_2$. Let B_i be the Boolean lattice generated by D_i, $i=1, 2$. Denote by B_1^b the Boolean lattice generated by D_1^b, where D_1^b denotes the lattice $D_1 \cup \{1\}$ with $x < 1$ for all $x \in D_1$. Now we know from [3] that $D_1 \nearrow D_2$ is the lattice obtained from $D_1^b * D_2$ (the 0—1-free product) by factorizing by the congruence generated by all pairs $(d \vee d^+, d^+)$ $d \in D_1$. Now factorizing $B_1^b * B_2$ by this congruence we get a Boolean lattice generated by $D_1 \nearrow D_2$. This Boolean lattice will be denoted by $B_1^b \nearrow B_2$. Clearly $B_1^b \nearrow B_2 = B(D_1 \nearrow D_2)$. It also contains B_1, the smallest Boolean lattice generated by D_1. This follows from [3], Theorem 1, for $D_1^b \nearrow D_2$ contains D_1. (Of course B_1, like D_1, does not contain the upper bound of $B_1^b \nearrow B_2$.)

In Section 1 we defined the operator $x \mapsto \bar{x}$ mapping the Boolean algebra $B(D)$ generated by the distributive lattice D to D by associating the least upper bound $\bar{x}$ in D with the element $x \in B$. Now B_1 is embedded to $B_1^b \nearrow B_2$. Therefore, for elements of B_1, there are two possibilities to define $x \mapsto \bar{x}$, namely within B_1 as the least upper bound of an element in D_1, and within $B_1^b \nearrow B_2$ as the least upper bound of an element in $D_1 \nearrow D_2$. We are going to show (and this is the crucial point of the proof) that these two definitions coincide.

This statement includes the main theorem of [3]. Indeed, from [3], Theorem 1 it follows that the smallest Boolean lattice generated by D_1 in $B_1^b \nearrow B_2$ intersects $D_1 \nearrow D_2$ in D_1. (This is not evident, we have to use GRÄTZER [1], Corollary 10.9., or more exactly a slight generalization of this Corollary as the units of D_1 and D_2 do not coincide, however, it can be proved.) The converse is also true: [3], Theorem 1 follows from the fact that the intersection of $B(D_1)$ and $D_1 \nearrow D_2$ in $B_1^b \nearrow B_2$ is D_1.

(This is evident.) Now consider any element x of $B(D_1)$ in $D_1 \nearrow D_2$. Then $\bar{x}$ formed in $D_1 \nearrow D_2$ is x. Now applying $B(D_1) \cap (D_1 \nearrow D_2) = D_2$, that is, using [3], Theorem 1 we have that $\bar{x}$ formed in D_2 is x, too. This shows that the statement whose proof we promised is, indeed stronger than [3], Theorem 1. Now let $\overline{x}$ be the least upper bound of $x \in B_1$ in D_1 and let $\bar{x}$ be the least upper bound of x in $D_1 \nearrow D_2$. Obviously $\overline{x} \leqq \bar{x}$.

Lemma 1. *For all* $x \in B_1, \overline{x} \leqq \bar{x}$.

Before proving Lemma 1, we have to solve the word problem of $B_1 \nearrow D_2$, where $B_1 \nearrow D_2$ denotes the lattice generated by $B_1 \cup D_2$ in $B_1^b \nearrow B_2$. A solution will be given in the following lemma.

Let Θ denote the congruence generated by the pairs $(d^+, d \vee d^+)$, $d \in D_1$, in $B_1 \nearrow D_2$. Let Q_1 denote the set of atoms of B_1. Let $\mathscr{I}(k)$ be the subset $\{j | k \not\equiv j^+\}$ of Q_1, if k is an irreducible of D_2. (There is a homomorphism of Q_1 to P_1 corresponding to the embedding $D_1 \to B_1$. For any k, $\mathscr{I}(k)$ goes to an ideal of P_1 under this homomorphism; P_i denotes the set of join-irreducibles of D_i, $i = 1, 2$; j^+ denotes $\bar{j}^+$.)

Lemma 2. *For arbitrary elements* $f, g \in B_1 \nearrow D_2, f \equiv g \pmod{\Theta}$ *iff, for all* k, $f(k) \equiv g(k) \pmod{\Theta(\mathscr{I}(k))}$ *where* $\Theta(\mathscr{I}(k))$ *is the congruence generated by the ideal* $\mathscr{I}(k)$.

The proof is analogous with that of [3], Theorem 2, and it will be omitted.

Now we go on to prove Lemma 1. We have to show $\overline{x} \leqq \bar{x}$. As in [3], elements of $B_1 \nearrow D_2$ will be represented by antitone functions from P_2 to B_1. It is enough to show that for all $b \in B_1, f_b \leqq f$ implies $f_{\bar{b}} \leqq f$ in $B_1 \nearrow D_2$ where f_b (respectively, $f_{\bar{b}}$) is the function identically b (respectively, $\bar{b}$) and $f \in D_1 \nearrow D_2$. It suffices to show this statement for b irreducible, as the operation $b \mapsto \bar{b}$ preserves joins.

Now let j be irreducible and assume that $f_j \leqq f \pmod{\theta}$. Then, for all k, $j \leqq f(k) \pmod{\theta(\mathscr{I}(k))}$. Hence, we have either $j \leqq f(k)$ (and then also $\bar{j} \leqq f(k)$ as $f(k)$ is in D_1) or $j \equiv j \wedge f(k) \pmod{\theta(\mathscr{I}(k))}, j \not\leqq f(k)$. In the latter case $j \wedge f(k) = 0$, thus $j \equiv 0 \pmod{\theta(\mathscr{I}(k))}$, that is, $k \not\equiv j^+$, whence $k \not\equiv \bar{j}^+$, that is, $\bar{j} \equiv 0 \pmod{\theta(\mathscr{I}(k))}$. In either case $\bar{j} \leqq f(k) \pmod{\theta(\mathscr{I}(k))}$, whence $f_{\bar{j}} \leqq f \pmod{\Theta}$ completing the proof of Lemma 1.

Now we return to the proof of (β_1) and show that $L(D_1)$ is a sublattice of $L(D_1 \nearrow D_2)$. Consider the elements (x, y, z) of $L(D_1 \nearrow D_2)$ with $x, y, z \in B_1$ (hence $x \in D_1$, by [3], Theorem 1). These triples form a $\wedge$-subsemilattice of $L(D_1 \nearrow D_2)$. But, because of Lemma 1, the join of two such triples is the same as their join in $L(D_1)$:

$$(x, y, z) \vee_{L(D_1)} (x', y', z') = (x \vee x', y \vee y', z \vee z')^{\sim\wedge(L(D_1))}$$

$$(x, y, z) \vee_{L(D_1 \nearrow D_2)} (x', y', z') = (x \vee x', y \vee y', z \vee z')^{\sim\wedge(L(D_1 \nearrow D_2))}$$

Now the operation $\tilde{}$ does not depend upon, in which lattice the triple is considered, and Lemma 1 shows that the same is true for $\hat{}$.

3. Proof of (δ_0) and (δ_1). (δ_0) is a consequence of (δ_1), thus we need only prove (δ_1). Let $d\in D_1$. (δ_1) says that $d\delta\varphi_2=d\varphi_1\gamma$. Now, $d\varphi_1$ is the congruence generated by $((0,0,0),(d,0,0))$ in L_1. λ takes this pair to the interval $[p,r]$. Let these elements there be denoted by $(0,0,0)_{[p,r]}$ and $(d,0,0)_{[p,r]}$ (Figure 1). Then the congruence $d\varphi_1\gamma$ is generated by this pair. With analogous notations, it is also generated by $((0,0,0)_{[q,s]},(0,0,d)_{[q,s]})$. $(L(D_1\nearrow D_2,D_2)$ was defined such that the first component must be in D_2. Therefore, when we glue it by a $D_1\nearrow D_2$ to $M(D_1\nearrow D_2)$, the third (or second) component must denote the elements used in the gluing. That is $[q,s]$ is the interval $[(0,0,0),(0,0,1)]$ of $L(D_1\nearrow D_2,D_2)$. Omitting the subscript $[q,s]$, let us meet the pair $((0,0,0);(0,0,d))$ with $(0,1,0)$ and join the result with $(1,0,0)$; so we get

$$(0,1,0)\equiv(\bar{d},1,\bar{d})\ (\mathrm{mod}\ d\varphi_1\gamma),\quad (0,0,0)\equiv(\bar{d},0,0)\ (\mathrm{mod}\ d\varphi_1\gamma),$$

and both pairs generate $d\varphi_1\gamma$, where $\bar{d}$ denotes the least upper bound of $d\in D_1$ $(\subseteq D_1\nearrow D_2)$ in D_2 $(\nsubseteq D_1\nearrow D_2)$. On the other hand, $d\delta=d^+$, thus $d\delta\varphi_2$ is generated by $((0,0,0),(d^+,0,0))$. We only have to prove $\bar{d}=d^+$ in $D_1\nearrow D_2$ for all $d\in D_1$. Recall that $\bar{d}$ denotes the least upper bound of d in D_2. It suffices to show that $d_1\leqq d_2$ $(d_i\in D_i,\ i=1,2)$ implies $d_1^+\leqq d_2$. Besides, if we prove it for d_1 irreducible, then it is true for arbitrary d_1. This follows from the fact that $^+$ preserves joins. Now assume that $f_{d_1}\leqq f_{d_2}\ (\mathrm{mod}\ \Theta)$ that is, for all $k\in P_2$

$$d_1\leqq f_{d_2}(k)\quad (\mathrm{mod}\ \theta(I(k))),$$

where f_{d_1} represents the element d_1, that is, f_{d_1} takes the value d_1 identically and f_{d_2} is the characteristic function of d_2:

$$f_{d_2}(k)=\begin{cases}1 & \text{if}\quad k\leqq d_2,\\ 0 & \text{otherwise.}\end{cases}$$

Now $d_1\vee f_{d_2}(k)\equiv f_{d_2}(k)$ means that for all k, the value of the function

$$d_1\vee f_{d_2}(k)=\begin{cases}1 & \text{if}\quad k\leqq d_2,\\ d_1 & \text{otherwise,}\end{cases}$$

is congruent with $f_{d_2}(k)$ modulo $\Theta(\mathscr{I}(k))$. Now let us go out to $B_1\nearrow B_2$ and form the meet with

$$\bar{f}_{d_2}(k)=\begin{cases}0 & \text{if}\quad k\leqq d_2,\\ 1 & \text{otherwise;}\end{cases}$$

then we obtain that for all $k \not\leqq d_2$, $k \in P_2$;

$$d_1 \equiv 0 \pmod{\theta(I(k))},$$

that is, $d_1 \in \mathscr{I}(k)$, in other words $k \not\leqq d_1^+$. Thus $\{k | k \in P_2, k \leqq d_1^+\} \subseteqq \{k | k \in P_2, k \leqq d_2\}$. Hence $d_1^+ \leqq d_2$, as claimed.

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