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Upper estimates for the eigenfunctions of higher order of a linear differential operator

V. KOMORNIK

Dedicated to Professor Béla Szőkefalvi-Nagy on the occasion of his 70th birthday

In several problems of the spectral theory of non-selfadjoint differential operators it occurs the need to estimate the eigenfunctions of higher order of these operators (cf. [3], [4], [5], [7], [8], [11]). These results were proved in general by the application of the mean value formulas of TITCHMARSH [2], MOISEEV [6] and JOÓ [7]. For the case of the Schrödinger operator, exact estimates were obtained in [7]. However, in case the differential operator is of order $n \geqq 3$, the mean value formula becomes rather complicated (see [6]), and it seems to be hard to obtain exact estimates by its application. In this paper, we choose another approach: using the method of variation of constants instead of the mean value formula, we trace the difficulties back to the investigation of some concrete determinants. As the result of these considerations, we obtain the formula of Theorem 1. This formula actually equals the mean value formula in case of the Schrödinger operator, but differs from it in general.

Using this formula, we extend the upper estimates of [7] to the case of an arbitrary linear differential operator. We obtain estimates not only for the eigenfunctions, but also for their derivatives. These results are formulated in Theorem 2.

Let $G \subset \mathbf{R}$ be an arbitrary open interval and consider the formal differential operator

$$Lu = u^{(n)} + p_1 u^{(n-1)} + \ldots + p_n u, \tag{1}$$
$$p_1, \ldots, p_n \in L^1_{\mathrm{loc}}(G) \text{ are arbitrary complex functions.}$$

Let λ be a complex number. The function $u_{-1}\colon G \to \mathbf{C}$, $u_{-1} \equiv 0$ is called an eigenfunction of order -1 of the operator L with the eigenvalue λ. As it is usual, a function $u_i\colon G \to \mathbf{C}$, $u_i \not\equiv 0$ $(i = 0, 1, \ldots)$ is said to be an eigenfunction of order i of the operator L with the eigenvalue λ if u_i, together with its first $n-1$

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derivatives is absolute continuous on every compact subinterval of G and if for almost all $x \in G$ the equation

$$(Lu_i)(x) = \lambda u_i(x) + u_{i-1}(x)$$

holds, where u_{i-1} is an eigenfunction of order $i-1$ with the eigenvalue λ.

We shall prove the following result:

Theorem 1. *Given any pair of integers $m \geqq 0$, $n \geqq 2$ there exist entire functions f, f_{jik}, h_j $(j \geqq 0, 0 \leqq i < n, 1 \leqq k \leqq N \equiv (m+1)n)$ with $f(z) \neq 0$ for $|z| < \pi^n$ such that the following formulas are valid:*

Given any eigenfunction u_m of order $\leqq m$ of the operator (1) *with the eigenvalue $\lambda \in \mathbf{C}$, introducing for $j < m$ the functions*

$$u_j\colon G \to \mathbf{C}, \quad u_j = Lu_{j+1} - \lambda u_{j+1}, \tag{2}$$

we have that

$$\begin{aligned} f(\lambda t^n) t^{jn+i} u_{m-j}^{(i)}(x) &= \sum_{k=1}^{N} f_{jik}(\lambda t^n) u_m(x+kt) + \\ &+ \sum_{k=1}^{N} f_{jik}(\lambda t^n) \sum_{r=0}^{m} \sum_{s=1}^{n} \int_x^{x+kt} (x+kt-\tau)^{n(r+1)-1} h_r\big(\lambda(x+kt-\tau)^n\big) p_s(\tau) u_{m-r}^{(n-s)}(\tau)\, d\tau \end{aligned} \tag{3}$$

for all $j \geqq 0$, $0 \leqq i < n$, and for all $x \in G$ with $x + Nt \in G$.

The functions f_{00k} are multiples of f and therefore if $j = i = 0$, this formula can be simplified by f.

Consider now the special case

$$Lu = u^{(n)} + p_2 u^{(n-2)} + \ldots + p_n u, \quad G \subset \mathbf{R} \text{ is a bounded open interval.} \tag{4}$$

It is well-known that the eigenfunctions of the operator (4) can be extended to absolute continuous functions on $\bar{G}$ (see [1]). Using Theorem 1 we shall prove the following estimates:

Theorem 2. *There exist constants*

$$\mathcal{K}_m = \mathcal{K}_m(n, |G|, \|p_2\|_1, \ldots, \|p_n\|_1), \quad m = 0, 1, \ldots,$$

($|G|$ denotes the length of G) *such that given any eigenfunction u_m of order m of the operator* (4) *with the eigenvalue $\lambda \in \mathbf{C}$, we have*

$$\|u_{m-j}^{(i)}\|_\infty \leqq \mathcal{K}_m\left(1 + \left|\sqrt[n]{\lambda}\right|\right)^{jn+i+(1/r)} \|u_m\|_r \tag{5}$$

for all $0 \leqq j \leqq m$, $0 \leqq i < n$ and $1 \leqq r \leqq \infty$.

Moreover, if $p_2, \ldots, p_n \in L^p(G)$ for some $1 < p \leqq \infty$, then there exist constants

$$\mathcal{K}_m^p = \mathcal{K}_m^p(n, |G|, \|p_2\|_p, \ldots, \|p_n\|_p), \quad m = 0, 1, \ldots,$$

such that, putting $q=(1-1/p)^{-1}$,

$$\|u_{m-j}^{(i)}\|_q \leqq \mathcal{K}_m^p(1+|\sqrt[n]{\lambda}|)^{jn+i}\|u_m\|_q \tag{6}$$

for all $0\leqq j\leqq m$ *and* $0\leqq i<n$.

In the first section of this paper we prove Theorem 1 for the case $G=\mathbf{R}$, $p_1=p_2=\ldots=p_n\equiv 0$. In its full generality Theorem 1 is proved in Section 2. Finally, Theorem 2 will be proved in Section 3.

1. Some properties of the operator $L_0v=v^{(n)}$, $G=\mathbf{R}$. In this section v_m will denote an arbitrarily fixed eigenfunction of order $\leqq m$ of the operator L_0 with the eigenvalue $\lambda=\varrho^n$ and, for $j<m$, we introduce the functions $v_j=v_{j-1}^{(n)}-\lambda v_{j+1}$. We shall also use the notation

$$\varrho_p=\varrho e^{p\frac{2\pi i}{n}}, \quad p=1,2,\ldots,n.$$

The following assertion is obvious:

Lemma 1. v_m *has the form*

$$v_m(x)=\begin{cases}\sum\limits_{r=0}^{m}\sum\limits_{p=1}^{n}a_{rp}(\varrho_p x)^r e^{\varrho_p x} & \text{if } \lambda\neq 0,\\ \sum\limits_{r=0}^{m}\sum\limits_{p=1}^{n}a_{rp}x^{rn+p-1} & \text{if } \lambda=0,\end{cases} \tag{7}$$

with appropriate constants $a_{rp}\in\mathbf{C}$.

For any $t\in\mathbf{R}$, we define the determinant $D(\varrho t)$ of type $N\times N$ in the following way: let the $(rn+p)$-th entry of the k-th row $(1\leqq k\leqq N, 1\leqq p\leqq n, 0\leqq r\leqq m)$ be

$$\frac{(k\varrho_p t)^r}{r!}e^{k\varrho_p t}. \tag{8}$$

One can see easily that D is an entire function with isolated roots. A more thorough investigation shows (cf. [12]) that

$$D(\varrho t)\equiv C(\varrho t)^{n\frac{m(m+1)}{2}}\Big[\prod_{1\leqq p<q\leqq n}(e^{\varrho_p t}-e^{\varrho_q t})\Big]^{(m+1)^2} \tag{9}$$

with some constant $C\neq 0$. Let us denote the subdeterminant of $D(\varrho t)$, corresponding to the element (8) by $D_{kpr}(\varrho t)$, and define formally $D_{kpr}(\varrho t)\equiv 0$ for $r>m$.

Lemma 2. *There exist numbers* $C_{jis}\in\mathbf{C}$, *independent of the choice of* v_m, *such that for all* $j\leqq 0$, $0\leqq i<n$ *and* $x, t\in\mathbf{R}$,

$$v_{m-j}^{(i)}(x)=\sum_{k=1}^{N}\Big\{\sum_{p=1}^{jn+i}C_{jis}\sum_{p=1}^{n}\varrho_p^{jn+i}\frac{D_{kps}(\varrho t)}{D(\varrho t)}\Big\}v_m(x+kt), \tag{10}$$

whenever $D(\varrho t)\neq 0$.

Proof. By (7) and (9), for any $1\leqq k\leqq N$ we have

$$v_m(x+kt)=\sum_{p=1}^{n}\sum_{s=0}^{m}\frac{(k\varrho_p t)^s}{s!}e^{k\varrho_p t}w_{ps}(x), \tag{11}$$

where

$$w_{ps}(x)\equiv\sum_{r=s}^{m}a_{rp}\frac{r!}{(r-s)!}(\varrho_p x)^{r-s}e^{\varrho_p x}.$$

Hence for all $1\leqq p\leqq n$, $0\leqq s\leqq m$,

$$w_{ps}(x)=\sum_{k=1}^{N}\frac{D_{kps}(\varrho t)}{D(\varrho t)}v_m(x+kt). \tag{12}$$

(12) is formally true also for $s>m$ if we put $w_{ps}\equiv 0$. It follows directly from the definition of q_{ps} that for all $1\leqq pn$, $s\geqq 0$, $i\geqq 0$, $x\in\mathbf{R}$,

$$w_{ps}^{(i)}(x)=\varrho_p^i\sum_{q=0}^{i}\binom{i}{q}w_{p,s+q}(x), \tag{13}$$

and hence

$$w_{ps}^{(n)}(x)-\lambda w_{ps}(x)=\lambda\sum_{q=1}^{n}\binom{n}{q}w_{p,s+q}(x). \tag{14}$$

In the light of (12), our assertion (10) can be written in the form

$$v_{m-j}^{(i)}(x)=\lambda^j\sum_{p=1}^{n}\varrho_p^i\sum_{s=j}^{jn+i}C_{jis}w_{ps}(x). \tag{15}$$

First we prove it for $i=0$, by induction on j. For $j=0$, (15) follows from (11) with $C_{000}=1$. Suppose the formula is true for some $j\geqq 0$; then it is true also for $j+1$. Indeed, we have by (14) and the inductive hypothesis that

$$v_{m-j-1}(x)=\lambda^j\sum_{p=1}^{n}\sum_{s=j}^{jn}C_{j0s}[w_{ps}^{(n)}(x)-\lambda w_{ps}(x)]=\lambda^j\sum_{p=1}^{n}\sum_{s=j}^{jn}C_{j0s}\lambda\sum_{q=1}^{n}\binom{n}{q}w_{p,s+q}(x)=$$

$$=\lambda^{j+1}\sum_{p=1}^{n}\sum_{r=j+1}^{(j+1)n}\left\{\sum_{q=\max(1,r-jn)}^{\min(n,r-j)}\binom{n}{q}C_{j0,r-q}\right\}w_{pr}(x).$$

Thus (15) is true for all $j\geqq 0$, $i=0$. Hence the general case follows by (13):

$$v_{m-j}^{(i)}(x)=\lambda^j\sum_{p=1}^{n}\sum_{s=j}^{jn}C_{j0s}w_{ps}^{(i)}(x)=\lambda^j\sum_{p=1}^{n}\sum_{s=j}^{jn}C_{j0s}\varrho_p^i\sum_{q=0}^{i}\binom{i}{q}w_{p,s+q}(x)=$$

$$=\lambda^j\sum_{p=1}^{n}\varrho_p^i\sum_{r=j}^{jn+i}\left\{\sum_{q=\max(0,r-jn)}^{\min(i,r-j)}\binom{i}{q}C_{j0,r-q}\right\}w_{pr}(x).$$

The lemma is proved.

Now we introduce some special eigenfunctions of the operator L_0 which will also be used in Section 2. Define the functions K_m: $\mathbf{C}\times\mathbf{R}\to\mathbf{C}$ in the following way:

$$K_m(\varrho, x) \equiv 0 \quad \text{if} \quad m < 0;$$

$$K_0(\varrho, x) = \begin{cases} \sum_{p=1}^{n} \frac{\varrho_p}{n\lambda} e^{\varrho_p x} & \text{if} \quad \varrho \neq 0, \\ \frac{x^{n-1}}{(n-1)!} & \text{if} \quad \varrho = 0; \end{cases} \qquad (\lambda \equiv \varrho^n)$$

$$K_m(\varrho, x) \equiv \int_0^x K_0(\varrho, x-t) K_{m-1}(\varrho, t)\, dt \quad \text{if} \quad m > 0.$$

Lemma 3. *For any pair of integers* m *and* $0 \leqq i < n$,

$$D_2^{n+i} K_m(\varrho, x) \equiv \lambda D_2^i K_m(\varrho, x) + D_2^i K_{m-1}(\varrho, x), \tag{16}$$

$$D_2^i K_m(\varrho, 0) = \begin{cases} 1 & \textit{if} \quad m = 0 \quad \textit{and} \quad i = n-1, \\ 0 & \textit{otherwise.} \end{cases} \tag{17}$$

Moreover, there exist entire functions h_m^i *such that* $h_m^i(0) = 1$ *and*

$$D_2^i K_m(\varrho, 2) \equiv \frac{x^{nm+n-1-i}}{(nm+n-1-i)!} h_m^i(\lambda x^n). \tag{18}$$

Consequently, for any $m \geqq 0$, $0 \leqq i < n$ *and* $\varrho \in \mathbf{C}$, $D_2^i K_m(\varrho, \cdot)$ *is an eigenfunction of order* m *of the operator* L_0 *with the eigenvalue* $\lambda = \varrho^n$.

Proof. For $m=0$, (16)—(18) can be shown by easy computation, using the identity

$$\sum_{p=1}^{n} \varrho_p^i = \begin{cases} n\lambda & \text{if} \quad i = n, \\ 0 & \text{if} \quad 0 \leqq i < n; \end{cases}$$

for $m<0$ they are obvious. Suppose they are true for some $m \geqq 0$, and we shall conclude from this their validity also for $m+1$. It suffices to show (16) and (18) for $i=0$. In fact, the cases $i>0$ of (16) and (18) hence follow by repeated derivation and (17) is a consequence of (18). Using the definition of K_{m+1} and the inductive hypothesis,

$$D_2^n K_{m+1}(\varrho, x) = \frac{d^n}{dx^n} \int_0^x K_0(\varrho, x-t) K_m(\varrho, t)\, dt =$$

$$= \sum_{j=0}^{n-1} D_2^j K_0(\varrho, 0) D_2^{n-1-j} K_m(\varrho, x) + \int_0^x D_2^n K_0(\varrho, x-t) K_m(\varrho, t)\, dt =$$

$$= K_m(\varrho, x) + \int_0^x \lambda K_0(\varrho, x-t) K_m(\varrho, t)\, dt = \lambda K_{m+1}(\varrho, x) + K_m(\varrho, x),$$

and (16) is proved. To show (18), we use the explicit forms

$$(19)\qquad h_j^0(z) \equiv \sum_{k=0}^{\infty} a_k^j z^k, \quad \lim_{k\to\infty} \sqrt[k]{|a_k^j|} = 0, \quad a_0^j = 1, \quad j = 0, 1, \ldots, m.$$

We can write

$$(20)\quad K_{m+1}(\varrho, x) = \int_0^x \frac{(x-t)^{n-1}}{(n-1)!} \sum_{k=0}^{\infty} a_k^0 (\varrho(x-t))^{nk} \frac{t^{nm+n-1}}{(nm+n-1)!} \sum_{r=0}^{\infty} a_r^m (\varrho t)^{nr}\, dt =$$

$$= x^{n(m+1)+n-1} \sum_{k=0}^{\infty} \sum_{r=0}^{\infty} (\varrho x)^{(k+r)n} a_k^0 a_r^m \int_0^1 \frac{(1-\xi)^{n-1+kn}}{(n-1)!} \frac{\xi^{nm+n-1+rn}}{(nm+n-1)!}\, d\xi =$$

$$= \frac{x^{n(m+1)+n-1}}{(n(m+1)+n-1)!} \sum_{s=0}^{\infty} a_s^{m+1} (\varrho x)^{ns},$$

where

$$(21)\quad a_s^{m+1} = (n(m+1)+n-1)! \sum_{k=0}^{s} a_k^0 a_{s-k}^m \int_0^1 \frac{(1-\xi)^{n-1+nk}}{(n-1)!} \frac{\xi^{nm+n-1+(s-k)n}}{(nm+n-1)!}\, d\xi;$$

hence, in view of (19), we easily obtain

$$(22)\qquad a_0^{m+1} = 1 \quad \text{and} \quad \lim_{k\to\infty} \sqrt[k]{|a_k^{m+1}|} = 0$$

(to deduce the first equality, we integrate by parts $n-1$ times). (22) shows the legality of the demonstration of (20). Finally, (20) and (22) yield (18).

Lemma 4. *Given any eigenfunction v_m of order $\leqq m$ with some eigenvalue λ, there exists a sequence $v_{k,m}$ such that $v_{k,m}$ is an eigenfunction of order $\leqq m$ with the eigenvalue $\lambda_k \neq \lambda$, $\lambda_k \to \lambda$, and for all $0 \leqq j \leqq m$, $0 \leqq i < n$ and $x \in \mathbf{R}$, we have $v_{k,m-j}^{(i)}(x) \to v_{m-j}^{(i)}(x)$ $(k\to\infty)$.*

Proof. For $\lambda \neq 0$ this is a direct consequence of Lemma 1. For $\lambda = 0$, it follows from Lemmas 1 and 3 (see (18)).

Now we prove Theorem 1 for $L = L_0$. All the following formulas will be taken for all $j \leqq 0$, $0 \leqq i < n$ and $x, t \in \mathbf{R}$. Introducing the entire functions d_{jik} by the formulas

$$(23)\qquad d_{jik}(\varrho t) \equiv \sum_{s=j}^{jn+i} C_{jis} \sum_{p=1}^{n} (\varrho_p t)^{jn+i} D_{kps}(\delta t), \quad k = 1, \ldots, N$$

(see (10)), we have the identities

$$(24)\qquad D(\varrho t) t^{jn+i} v_{m-j}^{(i)}(x) = \sum_{k=1}^{N} d_{jik}(\varrho t) v_m(x+kt)$$

whenever $D(\varrho t) \neq 0$. Let μ denote the smallest multiplicity of the root 0 in the functions d_{jik}. We claim that μ is greater than or equal to the multiplicity of the root 0 in D. Indeed, in the opposite case, dividing both sides of (24) by $(\varrho t)^{\mu}$ and putting $\varrho\to 0$, $x=0$, $t=1$, we would obtain from Lemma 4 for some j, i that the identity

$$\sum_{k=0}^{N} d_{jik}^{*} v_m(k) = 0$$

holds for all eigenfunctions of order $\leqq m$ with the eigenvalue 0, i.e., for all polynomials of degree $<N$ with some coefficients d_{jik}^{*}, at least one of which differs from zero. But this is impossible because putting $v_m(x)\equiv x^r$, $r=0, 1, \dots, N-1$, the resulting system of linear equations has the only solution $d_{ji1}^{*}=d_{ji2}^{*}=\dots=d_{jiN}^{*}=0$.

Assume $D(\varrho t)\neq 0$. Then taking into account also (9), we can divide (24) by

$$C(\varrho t)^{n\frac{m(m+1)}{2}}\Big[\prod_{1\leqq p<q\leqq n}(\varrho_p t-\varrho_q t)\Big]^{(m+1)^2},$$

and we obtain the identities

$$f^{*}(\varrho t)\, t^{jn+i} v_{m-j}^{(i)}(x) = \sum_{k=1}^{N} f_{jik}^{*}(\varrho t)\, v_m(x+kt) \tag{25}$$

where f^{*}, f_{jik}^{*} are suitable entire functions with the properties

$$f^{*}(0) = 1 \quad\text{and}\quad f^{*}(z)\neq 0 \quad\text{if}\quad |z|<\pi. \tag{26}$$

It follows from the construction of f^{*} and f_{ijk}^{*} that

$$f^{*}\big(\varrho t e^{\frac{2\pi i}{n}}\big)\equiv f^{*}(\varrho t) \quad\text{and}\quad f_{jik}^{*}\big(\varrho t e^{\frac{2\pi i}{n}}\big)\equiv f_{jik}^{*}(\varrho t);$$

therefore there exist entire functions such that

$$f^{*}(\varrho t)\equiv f(\lambda t^{n}) \quad\text{and}\quad f_{jik}^{*}(\varrho t)\equiv f_{jik}(\lambda t^{n}). \tag{27}$$

From (25)—(27) the formulas (3) of Theorem 1 follow whenever $D(\varrho t)\neq 0$. However, this last condition can be eliminated with the aid of Lemma 4. The first part of Theorem 1 is proved. To prove the second part of the theorem, it suffices to show that

$$d_{00k}(\varrho t)\equiv C_{000}\sum_{p=1}^{n} D_{kp0}(\varrho t), \quad k=1, \dots, N$$

is a multiple of

$$(\varrho t)^{n\frac{m(m+1)}{2}}\Big[\prod_{1\leqq p<q\leqq n}(e^{\varrho_q t}-e^{\varrho_p t})\Big]^{(m+1)^2};$$

this can be shown similarly to (9). Theorem 1 for L_0 is proved.

2. Proof of Theorem 1. Using the notations of Theorem 1, introduce the functions

$$M(u_{j-r}, t) \equiv (Lu_{j-r})(t) - u_{j-r}^{(n)}(t) = \sum_{s=1}^{n} p_s(t) u_{j-r}^{(n-s)}(t), \tag{28}$$

$$v_j(x) \equiv u_j(x) + \sum_{r=0}^{j} \int_a^x K_r(\varrho, x-t) M(u_{j-r}, t)\, dt, \tag{29}$$

for $0 \leqq r, j \leqq m$, $t, x \in G$ where a is an arbitrarily fixed point of G. First we show that

$$v_j^{(i)}(x) = u_j^{(i)} x + \sum_{r=0}^{j} \int_a^x D_2^i K_r(\varrho, x-t) M(u_{j-r}, t)\, dt \quad (j \leqq m,\ 0 \leqq i < n), \tag{30}$$

$$v_j = v_{j-1}^{(n)} - \lambda v_{j+1} \quad (j < m), \quad v_{-1} \equiv 0. \tag{31}$$

Indeed, using (29) and (17), we get that for any $j \leqq m$, $0 \leqq i \leqq n$,

$$v_j^{(i)}(x) = u_j^{(i)}(x) + \sum_{r=0}^{j} \frac{d^i}{dx^i} \int_a^x K_r(\varrho, x-t) M(u_{j-r}, t)\, dt =$$

$$= u_j^{(i)}(x) + D_2^{i-1} K_0(\varrho, 0) M(u_j, x) + \sum_{r=0}^{j} \int_a^x D_2^i K_r(\varrho, x-t) M(u_{j-r}, t)\, dt.$$

For $0 \leqq i < n$ this implies (30) in view of (17). Now let $i = n$. Using also (29), (2), (28) and (16), we conclude that

$$v_j^{(n)}(x) - \lambda v_j(x) =$$

$$= u_j^{(n)}(x) - \lambda u_j(x) + M(u_j, x) + \sum_{r=0}^{j} \int_a^x [D_2^n K_r(\varrho, x-t) - \lambda K_r(\varrho, x-t)] M(u_{j-r}, t)\, dt =$$

$$= u_{j-1}(x) + \sum_{r=0}^{j-1} \int_a^x K_r(\varrho, x-t) M(u_{j-1-r}, t)\, dt,$$

whence the first part of (31) follows. $v_{-1} \equiv 0$ is obvious by (29). v_m being the restriction of an eigenfunction of order $\leqq m$ of the operator L_0 with the eigenvalue λ (by (31)); we can apply Theorem 1 for v_m. Using also (30), we obtain the identities

$$f(\lambda t^n) t^{jn+i} \left(u_{m-j}^{(i)}(x) + \sum_{r=j}^{m} \int_a^x D_2^i K_{r-j}(\varrho, x-\tau) M(u_{m-r}, \tau)\, d\tau \right) =$$

$$= \sum_{k=1}^{N} f_{jik}(\lambda t^n) \left(u_m(x+kt) + \sum_{r=0}^{m} \int_a^{x+kt} K_r(\varrho, x+kt-\tau) M(u_{m-r}, \tau)\, d\tau \right)$$

for all $j \geqq 0$, $0 \leqq i < n$, $x \in G$ and $x + Nt \in G$. By (18) and (28) this identity would coincide with (3) if we could replace the lower bound a in all the integrals by x.

But this is allowed by the following remark: $K_r(\varrho, \cdot)$ is an eigenfunction of order r of L_0 with the eigenvalue λ (Lemma 3), and therefore we have

$$f(\lambda t^n) t^{jn+i} D_2^i K_{r-j}(\varrho, x-\tau) = \sum_{k=1}^{N} f_{jik}(\lambda t^n) K_r(\varrho, x-\tau+kt)$$

for any $j \geqq 0$, $0 \leqq i < n$ and $x, t, \tau \in \mathbf{R}$. Furthermore, $D_2^i K_{r-j} \equiv 0$ for any $j > r$, $0 \leqq i < n$. Thus Theorem 1 is proved.

3. Proof of Theorem 2. Using the notations of Theorem 1, let us fix a constant C such that for all $0 \leqq j \leqq m$, $0 \leqq i < n$ and $1 \leqq k \leqq N$,

$$|f_{jik}(z)| \leqq C|f(z)| \quad \text{if} \quad |z| \leqq 1 \tag{32}$$

and

$$|h_j(z)| \leqq C \quad \text{if} \quad |z| \leqq N^N. \tag{33}$$

Assume $p_2, \ldots, p_n \in L^p(G)$ $(1 \leqq p \leqq \infty)$, and define the numbers ε, R, M_q (where $p^{-1}+q^{-1}=1$) as follows:

$$\varepsilon \equiv (4N)^{-1}(b-a)^{-1/q} \quad (G \equiv (a, b)), \tag{34}$$

$$R \equiv \min\left\{\frac{1}{|\sqrt[n]{\lambda}|}, \frac{b-a}{2N}, \min\left\{\sqrt[s-1]{\frac{\varepsilon}{C^2 N^{N+1}\|p_s\|_p}} : 2 \leqq s \leqq n\right\}\right\}, \tag{35}$$

$$M_q \equiv \max\{R^{jn+i}\|u_{m-j}^{(i)}\|_q \colon 0 \leqq j \leqq m,\ 0 \leqq i < n\}. \tag{36}$$

Using (3), (32), (33), (35), and (36), for any $0 \leqq j \leqq m$, $0 \leqq i < n$, $a \leqq x \leqq \dfrac{a+b}{2}$ and $0 \leqq t \leqq R$ we can write

$$t^{jn+i}|u_{m-j}^{(i)}(x)| \leqq C\sum_{k=1}^{N}|u_m(x+kt)| + NC^2\sum_{r=0}^{m}\sum_{s=2}^{n}(NR)^{n(r+1)-1}\|p_s\|_p\|u_{m-r}^{(n-s)}\|_q =$$

$$= C\sum_{k=1}^{N}|u_m(x+kt)| + N^N C^2\sum_{r=0}^{m}\sum_{s=2}^{n}(R^{s-1}\|p_s\|_p)(R^{rn+n-s}\|u_{m-r}^{(n-s)}\|_q) \leqq$$

$$\leqq C\sum_{k=1}^{N}|u_m(x+kt)| + \varepsilon M_q,$$

i.e.,

$$t^{jn+i}|u_{m-j}^{(i)}(x)| \leqq C\sum_{k=1}^{N}|u_m(x+kt)| + \varepsilon M_q. \tag{37}$$

First we prove (5) $(q \equiv \infty)$. Applying the operation

$$NR^{-1}\int_0^R \cdot\, dt$$

to both sides, we obtain

$$R^{jn+i}|u_{m-j}^{(i)}(x)| \leqq NCR^{-1}\sum_{k=1}^{N}\int_0^R |u_m(x+kt)|\,dt + N\varepsilon M_\infty.$$

Using the Hölder inequality, one can easily see that

$$R^{-1}\int_0^R |u_m(x+kt)|\,dt \leqq R^{-1/r}\|u_m\|_r,$$

and therefore

$$R^{jn+i}|u_{m-j}^{(i)}(x)| \leqq N^2CR^{-1/r}\|u_m\|_r + N\varepsilon M_\infty.$$

This is true for all $a \leqq x \leqq \frac{a+b}{2}$, but one can quite similarly prove it for all $\frac{a+b}{2} \leqq x \leqq b$, too. Hence

$$R^{jn+i}\|u_{m-j}^{(i)}\|_\infty \leqq N^2CR^{-1/r}\|u_m\|_r + N\varepsilon M_\infty,$$

and in view of (34) and (36),

$$M_\infty \leqq N^2CR^{-1/r}\|u_m\|_r + \frac{1}{2}M_\infty,$$

$$M_\infty \leqq 2N^2CR^{-1/r}\|u_m\|_r.$$

Hence (5) follows by (34), (35) and (36).

To prove (6), put $t=R$ in (37) and take the $L^q\left(a, \frac{a+b}{2}\right)$ norm of both sides:

$$R^{jn+i}\|u_{m-j}^{(i)}\|_{L^q\left(a,\frac{a+b}{2}\right)} \leqq NC\|u_m\|_q + (b-a)^{1/q}\varepsilon M_q.$$

A similar estimate can be obtained for $\|u_{m-j}^{(j)}\|_{L^q\left(\frac{a+b}{2},b\right)}$, too. Therefore, in view of (36) and (34),

$$M_q \leqq 2NC\|u_m\|_q + \frac{1}{2}M_q,$$

$$M_q \leqq 4NC\|u_m\|_q.$$

Hence (6) follows by (34), (35) and (36).

Remark. For $n=2$ the functions f_{jlk}, h_j in Theorem 1 have some special properties. Using these properties, one can show with the method of the paper [7] the following stronger form of (5):

$$\|u_{m-j}^{(i)}\|_\infty \leqq \mathscr{K}_{2,m}(1+|\sqrt{\lambda}|)^{j+i}(1+|\mathrm{Re}\sqrt{\lambda}|)^{j+1/r}\|u_m\|_r.$$

Just this result was proved in [7].

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DEPARTMENT OF MATH. ANALYSIS, II
LORÁND EÖTVÖS UNIVERSITY
PF. 323
1445 BUDAPEST 8, HUNGARY