Acta Sci. Math., 45 (1983), 293—304

On the strong and extra strong approximation of orthogonal series

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In honour of Professor Béla Szőkefalvi-Nagy on his seventieth birthday

1. Let $\{\varphi_n(x)\}$ be an orthonormal system on the finite interval (a, b). We consider the orthogonal series

$$\sum_{n=1}^{\infty} c_n \varphi_n(x) \quad \text{with} \quad \sum_{n=1}^{\infty} c_n^2 < \infty. \tag{1}$$

By the Riesz—Fischer theorem the series (1) converges in L^2 to a square-integrable function f. Let us denote the partial sums of (1) by $s_n(x)$.

In [1] the first author proved that *if* $0<\gamma<1$ *and*

$$\sum_{n=1}^{\infty} c_n^2 n^{2\gamma} < \infty \tag{2}$$

then

$$\frac{1}{n} \sum_{k=1}^{n} \bigl(s_k(x)-f(x)\bigr) = o_x(n^{-\gamma})$$

almost everywhere (a.e.) in (a, b).

G. Sunouchi [8] generalized this results to strong approximation, and his result was generalized by one of us ([2]) to very strong approximation as follows:

Theorem A. *Suppose that* $\alpha>0$, $0<\gamma<1$, $0<p<\gamma^{-1}$, *and that* (2) *is satisfied. Then*

$$C_n(f, \alpha, p, \{m_k\};\ x) := \left\{\frac{1}{A_n^\alpha} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} |s_{m_k}(x)-f(x)|^p\right\}^{1/p} = o_x(n^{-\gamma}) \tag{3}$$

holds a.e. for any increasing sequence $\{m_k\}$, *where* $A_n^\alpha = \binom{n+\alpha}{n}$.

This theorem with $m_k=k$ reduces to that of Sunouchi.

Received October 29, 1981.

Recently the first author [3] showed that in the special case $\alpha=1$ the restriction $\gamma<1$ can be omitted, i.e. *if* $\gamma>0$ *and* $0<p<\gamma^{-1}$, *then* (2) *implies that*

$$\left\{\frac{1}{n+1}\sum_{k=0}^{n}|s_{m_k}(x)-f(x)|^p\right\}^{1/p} = o_x(n^{-\gamma}) \tag{4}$$

holds a.e. in (a, b) *for any increasing sequence* $\{m_k\}$.

In the present work, among others, we prove that the restriction $\gamma<1$ from the assumptions of Theorem A can be omitted for any $\alpha>0$ and not only for $\alpha=1$ alone.

Namely we have

Theorem 1. *If* α *and* γ *are positive numbers and* $0<p\gamma<1$ *then condition* (2) *implies that* (3) *holds a.e. in* (a, b) *for any increasing sequence* $\{m_k\}$.

We mention that Theorem 3 of [6] made a moderate step towards this result, namely it states that (3) holds for any positive γ if $\alpha>p\gamma$.

Two further generalizations of (4) were given in the papers [4] and [5], from them we can unify the following

Theorem B. *Suppose that* $\gamma>0$, $0<p\gamma<\beta$, *and that* (2) *holds. Moreover if*

(i) $\beta\leqq 2$ *or* $\beta>2$ *but at least either* $\gamma<1$ *or* $p\leqq 2$;

(ii) $p\geqq 2$ *and* $\sum_{n=1}^{\infty} c_n^2 n^{2\gamma+1-2/p}<\infty$;

then

$$h_n(f, \beta, p, \{m_k\};\ x) := \left\{(n+1)^{-\beta}\sum_{k=0}^{n}(k+1)^{\beta-1}|s_{m_k}(x)-f(x)|^p\right\}^{1/p} = o_x(n^{-\gamma}) \tag{5}$$

holds a.e. in (a, b) *for any increasing sequence* $\{m_k\}$.

To help the lucidity of fulfilment of the assumptions we define certain ranges of the positive parameters p and γ. Let us denote by $A(\beta)$ the range of the positive parameters p and γ determined by the condition $p\gamma<\beta$, i.e.

$$A(\beta) := \{p, \gamma \mid p>0,\quad \gamma>0 \quad \text{and} \quad p\gamma<\beta\},$$

moreover let

$$B(\beta) := \{p, \gamma \mid p>2,\quad \gamma\geqq 1 \quad \text{and} \quad p\gamma<\beta\}.$$

Theorem B shows that if $(p, \gamma)\in A(\beta)\setminus B(\beta)$ then (2) implies (5), but if $(p, \gamma)\in B(\beta)$ then we can only prove (5) under an additional condition.

This phenomenon is curious, and we have had the conjecture (see [4]) that condition (2) implies (5) for any $(p, \gamma)\in A(\beta)$. Now we shall verify this conjecture, namely we prove

Theorem 2. *If* $\gamma>0$ *and* $0<p<\beta$ *then condition* (2) *implies* (5) *a.e. in* $(a; b)$ *for any increasing sequence* $\{m_k\}$.

In connection with the extra strong approximation we shall improve the following theorems given in [3] and [6].

Theorem C. *Suppose that* $\gamma>0, 0<p<\gamma^{-1}$ *and* $p\leqq 2$, *that* $\alpha>p \max(1/2, \gamma)$; *or if* $p=2$ *then* $\alpha\geqq 1$; *moreover that* (2) *holds. Then*

$$C_n(f, \alpha, p, \{\mu_k\}; x) := \left\{\frac{1}{A_n^\alpha}\sum_{k=0}^{n} A_{n-k}^{\alpha-1}|s_{\mu_k}(x)-f(x)|^p\right\}^{1/p} = o_x(n^{-\gamma}) \tag{6}$$

holds a.e. in (a, b) *for any (not necessarily monotone) sequence* $\{\mu_k\}$ *of distinct positive integers.*

Theorem D. *Suppose that* $\gamma>0, p\geqq 2$, *and that* $p\gamma<\min(\alpha, 1)$. *Then*

$$\sum_{n=1}^{\infty} c_n^2 n^{2\gamma+1-(2/p)\min(\alpha,1)} < \infty \tag{7}$$

implies (6) *a.e. in* (a, b) *for any sequence* $\{\mu_k\}$ *of distinct positive integers.*

The next two theorems are certain analogues of Theorems C and D with the means $h_n(f, \beta, p, \{\mu_k\}; x)$.

Theorem C'. *Suppose that* $\gamma>0, 0<p\leqq 2$ *and* $p\gamma<\min(\beta, 1)$, *moreover that* (2) *holds. Then*

$$h_n(f, \beta, p, \{\mu_k\}; x) = o_x(n^{-\gamma}) \tag{8}$$

holds a.e. in (a, b) *for any sequence* $\{\mu_k\}$ *of distinct positive integers.*

We mention that this theorem is a collected form of Theorem 1 and Proposition A of [6].

Theorem D'. *Suppose that* $\gamma>0, p\geqq 2$, *and that* $p\gamma<\min(\beta, 1)$. *Then*

$$\sum_{n=1}^{\infty} c_n^2 n^{2\gamma+1-2/p} < \infty \tag{9}$$

implies (8) *a.e. in* (a, b) *for any sequence* $\{\mu_k\}$ *of distinct positive integers.*

Our two new theorems including these results read as follows:

Theorem 3. *If* $\gamma>0$ *and* $0<p\gamma<\min(\alpha, 1)$ *then* (2) *implies* (6) *a.e. in* (a, b) *for any sequence* $\{\mu_k\}$ *of distinct positive integers.*

Theorem 4. *If* $\gamma>0$ *and* $0<p\gamma<\min(\beta, 1)$ *then* (2) *implies* (8) *a.e. in* (a, b) *for any sequence* $\{\mu_k\}$ *of distinct positive integers.*

2. In order to prove the theorems we require some lemmas.

Lemma 1 ([2], Lemma 5). *Let* $\{\lambda_n\}$ *be a monotone sequence of positive numbers such that*

$$\sum_{n=1}^{m} \lambda_{2^n}^2 \leqq K\lambda_{2^m}^2.^{*)}$$

Then the condition

$$\sum_{n=1}^{\infty} c_n^2 \lambda_n^2 < \infty$$

implies that

$$s_{2^n}(x) - f(x) = o_x(\lambda_{2^n}^{-1})$$

holds a.e. in (a, b).

Lemma 2 ([7], Lemma 2). *If* $\sum_{n=0}^{\infty} c_n^2 < \infty$ *then for any positive* α *and* p

$$\int_a^b \left\{ \sup_{1\leqq n<\infty} \left(\frac{1}{A_n^\alpha} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} |s_k(x) - \sigma_k(x)|^p \right)^{1/p} \right\}^2 dx \leqq A(\alpha, p) \sum_{n=0}^{\infty} c_n^2,$$

where $\sigma_k(x) := (k+1)^{-1} \sum_{i=0}^{k} s_i(x)$.

Lemma 3 ([5], Lemma 3). *Let* $\varkappa>0$ *and* $\{\lambda_n\}$ *be an arbitrary sequence of positive numbers. Assuming that the condition*

$$\sum_{n=1}^{\infty} \lambda_n \left\{ \sum_{k=n}^{\infty} c_k^2 \right\}^{\varkappa} < \infty \tag{2.1}$$

implies a "certain property $T = T(\{s_n(x)\})$" *of the partial sums* $s_n(x)$ *of* (1) *for any orthonormal system, then* (2.1) *implies that the partial sums* $s_{m_k}(x)$ *of* (1) *also have the same property* T *for any increasing sequence* $\{m_k\}$, *i.e.*

if (2.1)$\Rightarrow T(\{s_n(x)\})$ *then* (2.1)$\Rightarrow T(\{s_{m_k}(x)\})$ *for any increasing sequence* $\{m_k\}$.

Lemma 4. *We have for any positive* p *and* $m \geqq 1$

$$\int_a^b \left\{ \frac{1}{2^m} \sum_{k=2^m}^{2^{m+1}-1} |s_k(x) - s_{2^m}(x) - \sigma_k^*(x)|^p \right\}^{2/p} dx \leqq K(p) \sum_{n=2^m+1}^{2^{m+1}} c_n^2, \tag{2.2}$$

$^{*)}$ $K, K_1, K_2, \ldots$ denote positive constants not necessarily the same at each occurrence.

where

$$\sigma_n^*(x) = \begin{cases} c_0\varphi_0(x) & \text{if } n=0, \\ \dfrac{1}{n-2^{m-1}} \sum\limits_{k=2^m}^{n} \big(s_k(x)-s_{2^m}(x)\big) & \text{if } 2^m \leqq n < 2^{m+1};\ m=0,1,\dots . \end{cases}$$

Proof. Using Lemma 2 with $\alpha=1$ for the following partial sums and $(C, 1)$-means

$$s_n'(x) := \begin{cases} 0 & \text{if } 0 \leqq n \leqq 2^{m-1}, \\ s_{n+2^{m-1}}(x)-s_{2^m}(x) & \text{if } 2^{m-1} < n < 2^{m+1}-2^{m-1}; \end{cases} \tag{2.3}$$

and

$$\sigma_n'(x) := \frac{1}{n}\sum_{k=1}^{n} s_k'(x) = \begin{cases} 0 & \text{if } 0 \leqq n \leqq 2^{m-1}, \\ \sigma_{n+2^{m-1}}^*(x) & \text{if } 2^{m-1} < n < 2^{m+1}-2^{m-1}, \end{cases} \tag{2.4}$$

where m is an arbitrary fixed natural number, we obtain (2.2) immediately, which completes the proof.

Lemma 5. *Let $\gamma>0$, and $p\geqq 2$. Then under condition* (2) *we have that the sum*

$$\tau_1(x) := \sum_{m=1}^{\infty}\sum_{k=2^m}^{2^{m+1}-1} (k+1)^{p\gamma-1}|s_k(x)-s_{2^m}(x)-\sigma_k^*(x)|^p$$

is finite a.e. in (a, b).

Proof. By $p\geqq 2$ and Lemma 4 we have that

$$\int_a^b (\tau_1(x))^{2/p}\,dx \leqq K_1 \int_a^b \sum_{m=0}^{\infty} 2^{m2\gamma}\Big\{2^{-m}\sum_{k=2^m}^{2^{m+1}-1} |s_k(x)-s_{2^m}(x)-\sigma_k^*(x)|^p\Big\}^{2/p} dx \leqq$$

$$\leqq K_2 \sum_{m=0}^{\infty} 2^{2m\gamma}\cdot \sum_{k=2^m+1}^{2^{m+1}} c_k^2 \leqq K_2 \sum_{n=1}^{\infty} c_n^2 n^{2\gamma} < \infty,$$

whence by B. Levi's theorem the statement of Lemma 5 follows.

Lemma 6. *Let $\gamma>0$ and $p\geqq 2$. Then condition* (2) *implies that*

$$\tau_2(x) := \sum_{k=1}^{\infty} k^{p\gamma-1}|\sigma_k^*(x)|^p < \infty$$

a.e. in (a, b).

Proof. An elementary consideration shows that

$$\int_a^b (\tau_2(x))^{2/p}\,dx \leqq K \int_a^b \sum_{m=0}^{\infty} 2^{2m\gamma}\Big\{2^{-m}\sum_{k=2^m}^{2^{m+1}-1} |\sigma_k^*(x)|^p\Big\}^{2/p} dx \leqq \tag{2.5}$$

$$\leqq K \int_a^b \sum_{m=0}^{\infty} 2^{2m\gamma}\{\max_{2^m\leqq k<2^{m+1}} |\sigma_k^*(x)|^2\}\,dx.$$

If $2^m<k<2^{m+1}$ then

$$\sigma_k^*(x)=\frac{1}{k-2^{m-1}}\sum_{i=2^m+1}^{k}(k+1-i)\,c_i\varphi_i(x)=\sum_{i=2^m+1}^{k}\left(1-\frac{i-2^{m-1}-1}{k-2^{m-1}}\right)c_i\varphi_i(x)$$

and $\sigma_{2^m}^*(x)=0$, so using the following simple estimation

$$\max_{2^m\leqq k<2^{m+1}}|\sigma_k^*(x)|^2=\max_{2^m\leqq k<2^{m+1}}|\sigma_k^*(x)-\sigma_{2^m}^*(x)|^2\leqq$$

$$\leqq\left(\sum_{k=2^m+1}^{2^{m+1}-1}|\sigma_k^*(x)-\sigma_{k-1}^*(x)|\right)^2\leqq 2^m\sum_{k=2^m+1}^{2^{m+1}-1}|\sigma_k^*(x)-\sigma_{k-1}^*(x)|^2,$$

we obtain that

$$\text{(2.6)}\qquad \int_a^b\left\{\max_{2^m\leqq k<2^{m+1}}|\sigma_k^*(x)|^2\right\}dx\leqq\frac{2^m}{2^{4(m-1)}}\sum_{k=2^m+1}^{2^{m+1}-1}\sum_{i=2^m+1}^{k}(i-2^{m-1}-1)^2c_i^2\leqq$$

$$\leqq 2^6\sum_{i=2^m+1}^{2^{m+1}-1}c_i^2.$$

Hence, by (2.5), we get that

$$\int_a^b(\tau_2(x))^{2/p}\,dx\leqq K_1\sum_{m=0}^{\infty}2^{2m\gamma}\sum_{n=2^m+1}^{2^{m+1}}c_n^2\leqq K_1\sum_{n=1}^{\infty}c_n^2n^{2\gamma}<\infty,$$

and this proves Lemma 6.

Lemma 7. *Condition* (2) *with any positive* γ *implies that*

$$\text{(2.7)}\qquad \sigma_n^*(x)=o_x(n^{-\gamma})$$

holds a.e. in (a,b).

Proof. Using estimation (2.6) we immediately obtain that

$$\int_a^b\sum_{m=0}^{\infty}(2^{m\gamma}\max_{2^m<n<2^{m+1}}|\sigma_n^*(x)|)^2\,dx\leqq K_2\sum_{n=1}^{\infty}c_n^2n^{2\gamma}<\infty,$$

whence (2.7) follows, which ends the proof.

Lemma 8. *Let* $\gamma>0,\ p\geqq 2$ *and* $p\gamma<1$. *For a given sequence* $\{\mu_k\}$ *of distinct positive integers we define another sequence* $\{m_k\}$ *as follows*: $m_k=2^m$ *if* $2^m\leqq\mu_k<2^{m+1}$. *Then* (2) *implies that the sum*

$$\mu_1(x):=\sum_{k=0}^{\infty}(k+1)^{p\gamma-1}|s_{\mu_k}(x)-s_{m_k}(x)-\sigma_{\mu_k}^*(x)|^p$$

is finite a.e. in (a,b).

Proof. Choosing q such that $1<q<(1-p\gamma)^{-1}$ and applying Hölder's inequality with this q and $q'=q/(q-1)$ we obtain that

$$\mu_1(x)=\sum_{m=0}^{\infty}\sum_{2^m\leqq\mu_k<2^{m+1}}(k+1)^{p\gamma-1}|s_{\mu_k}(x)-s_{m_k}(x)-\sigma^*_{\mu_k}(x)|^p\leqq$$

$$\leqq\sum_{m=0}^{\infty}\Big\{\sum_{2^m\leqq\mu_k<2^{m+1}}(k+1)^{(p\gamma-1)q}\Big\}^{1/q}\Big\{\sum_{2^m\leqq\mu_k<2^{m+1}}|s_{\mu_k}(x)-s_{m_k}(x)-\sigma^*_{\mu_k}(x)|^{pq'}\Big\}^{1/q'}\leqq$$

$$\leqq K\sum_{m=0}^{\infty}\Big\{\sum_{k=1}^{2^m}k^{(p\gamma-1)q}\Big\}^{1/q}\Big\{\sum_{i=2^m}^{2^{m+1}-1}|s_i(x)-s_{2^m}(x)-\sigma_i^*(x)|^{pq'}\Big\}^{1/q'}\leqq$$

$$\leqq K_1\sum_{m=0}^{\infty}2^{m(p\gamma-1/q')}\Big\{\sum_{i=2^m}^{2^{m+1}-1}|s_i(x)-s_{2^m}(x)-\sigma_i^*(x)|^{pq'}\Big\}^{1/q'}.$$

Hence, by Lemma 4 and $p\geqq 2$, we get that

$$\int_a^b(\mu_1(x))^{2/p}\,dx\leqq K_2\sum_{m=0}^{\infty}2^{m2\gamma}\sum_{n=2^m+1}^{2^{m+1}}c_n^2<\infty,$$

which proves Lemma 8.

Lemma 9. *Let $\gamma>0$, $p\geqq 2$ and $p\gamma<1$. Then, for any given sequence $\{\mu_k\}$ of distinct positive integers, the sum*

$$\mu_2(x):=\sum_{k=0}^{\infty}(k+1)^{p\gamma+1}|\sigma^*_{\mu_k}(x)|^p$$

is finite a.e. in (a,b) if (2) holds.

Proof. In a similar way as in the proof of Lemma 8 we obtain with Hölder's inequality $(1<q<(1-p\gamma)^{-1}$ and $1/q+1/q'=1)$ that

$$\int_a^b(\mu_2(x))^{2/p}\,dx\leqq\int_a^b\sum_{m=0}^{\infty}\Big\{\sum_{2^m\leqq\mu_k<2^{m+1}}(k+1)^{p\gamma-1}|\sigma^*_{\mu_k}(x)|^p\Big\}^{2/p}dx\leqq$$

$$\leqq\int_a^b\sum_{m=0}^{\infty}2^{m(p\gamma-1/q')(2/p)}\Big\{\sum_{i=2^m}^{2^{m+1}-1}|\sigma_i^*(x)|^{pq'}\Big\}^{2/pq'}dx\leqq$$

$$\leqq\sum_{m=0}^{\infty}2^{2m\gamma}\int_a^b\Big\{\frac{1}{2^m}\sum_{k=2^m}^{2^{m+1}-1}|\sigma_k^*(x)|^{pq'}\Big\}^{2/pq'}dx.$$

From this step we can continue the proof as in Lemma 6, and so we obtain the conclusion.

3. Proof of Theorem 1. Putting

$$C_n(x):=C_n(f,\alpha,p,\{k\};x),$$

and if $2^m \leqq n < 2^{m+1}$ $(m \geqq 2)$ holds, then

$$C_n(x) \leqq K\left(\left\{\frac{1}{A_n^\alpha}\sum_{k=0}^{2^m-1} A_{n-k}^{\alpha-1}|s_k(x)-f(x)|^p\right\}^{1/p}+\right. \tag{3.1}$$

$$\left.+\left\{\frac{1}{A_n^\alpha}\sum_{k=2^{m-1}+1}^{n} A_{n-k}^{\alpha-1}|s_k(x)-f(x)|^p\right\}^{1/p}\right) =: K\left(C_n^{(1)}(x)+C_n^{(2)}(x)\right).$$

Here the first term $C_n^{(1)}(x)$, by (4), has the order $o_x(n^{-\gamma})$, namely it is known that for any $\beta>-1$, $0<K_1<\frac{A_n^\beta}{n^\beta}<K_2$.

Next we estimate $C_n^{(2)}(x)$ as follows:

$$C_n^{(2)}(x) \leqq K\left(\left\{\frac{1}{A_n^\alpha}\sum_{k=2^{m-1}+1}^{2^m-1} A_{n-k}^{\alpha-1}|s_k(x)-s_{2^{m-1}}(x)-\sigma_k^*(x)|^p\right\}^{1/p}+\right. \tag{3.2}$$

$$+\left\{\frac{1}{A_n^\alpha}\sum_{k=2^{m-1}+1}^{2^m-1} A_{n-k}^{\alpha-1}|s_{2^{m-1}}(x)-f(x)|^p\right\}^{1/p}+\left\{\frac{1}{A_n^\alpha}\sum_{k=2^m}^{n} A_{n-k}^{\alpha-1}|s_k(x)-s_{2^m}(x)-\sigma_k^*(x)|^p\right\}^{1/p}+$$

$$\left.+\left\{\frac{1}{A_n^\alpha}\sum_{k=2^m}^{n} A_{n-k}^{\alpha-1}|s_{2^m}(x)-f(x)|^p\right\}^{1/p}+\left\{\frac{1}{A_n^\alpha}\sum_{k=2^{m-1}+1}^{n} A_{n-k}^{\alpha-1}|\sigma_k^*(x)|^p\right\}^{1/p}\right) =: K\sum_{i=1}^{5} D_n^{(i)}(x).$$

An easy consideration shows in view of Lemma 1 and Lemma 7 that

$$D_n^{(2)}(x)+D_n^{(4)}(x)+D_n^{(5)}(x) = o_x(n^{-\gamma}). \tag{3.3}$$

To estimate $D_n^{(1)}(x)$ and $D_n^{(3)}(x)$ we use again Hölder's inequality with such a q to be chosen so that $q>1$ and $(\alpha-1)q>-1$. Then

$$D_n^{(1)}(x) \leqq \frac{1}{(A_n^\alpha)^{1/p}}\left\{\sum_{k=2^{m-1}+1}^{2^m-1}(A_{n-k}^{\alpha-1})^q\right\}^{1/pq}\left\{\sum_{k=2^{m-1}+1}^{2^m-1}|s_k-s_{2^{m-1}}-\sigma_k^*|^{pq'}\right\}^{1/pq'} \leqq$$

$$\leqq K\left\{\frac{1}{2^m}\sum_{k=2^{m-1}}^{2^m-1}|s_k(x)-s_{2^{m-1}}(x)-\sigma_k^*(x)|^{pq'}\right\}^{1/pq'} =: D_m^*(x),$$

whence by Lemma 4 we obtain that

$$\sum_{m=1}^{\infty}\int_a^b (2^{m\gamma}D_m^*(x))^2\,dx \leqq K_1\sum_{m=1}^{\infty}2^{2m\gamma}\sum_{n=2^{m-1}+1}^{2^m} c_n^2 < \infty, \tag{3.4}$$

which implies that

$$D_n^{(1)}(x) = o_x(n^{-\gamma}) \tag{3.5}$$

also holds a.e. in (a, b).

Similarly

$$D_n^{(3)}(x) \leqq \frac{1}{(A_n^{\alpha})^{1/p}} \left\{ \sum_{k=2^m}^{n} (A_{n-k}^{\alpha-1})^q \right\}^{1/pq} \left\{ \sum_{k=2^m}^{n} |s_k(x) - s_{2^m}(x) - \sigma_k^*(x)|^{pq'} \right\}^{1/pq'} \leqq$$

$$\leqq K \left\{ \frac{1}{2^{m+1}} \sum_{k=2^m}^{2^{m+1}-1} |s_k(x) - s_{2^m}(x) - \sigma_k^*(x)|^{pq'} \right\}^{1/pq'} = D_{m+1}^*(x),$$

and so

(3.6) $$D_n^{(3)}(x) = o_x(n^{-\gamma})$$

also holds a.e. in (a, b) by (3.4).

Collecting the estimates given under (3.1), (3.2), (3.3), (3.5) and (3.6) we obtain that

$$C_n(f, \alpha, p, \{k\}; x) = o_x(n^{-\gamma})$$

a.e. in (a, b). Hence, using Lemma 3 with $\varkappa = 1$, $\lambda_n = n^{2\gamma-1}$ and $T(\{s_n(x)\}) := := C_n(f, \alpha, p, \{k\}; x) = o(n^{-\gamma})$, the statement of Theorem 1 follows obviously.

The proof is complete.

Proof of Theorem 2. Denote

$$h_n(x) := h_n(f, \beta, p, \{k\}; x).$$

By Theorem B we can assume that $p > 2$, namely otherwise (5) holds. Then with $2^m \leqq n < 2^{m+1}$

(3.7) $$h_n(x) \leqq K \left\{ n^{-\beta} \sum_{\nu=0}^{m} \sum_{k=2^\nu}^{2^{\nu+1}-1} k^{\beta-1} |s_k(x) - f(x)|^p \right\}^{1/p} \leqq$$

$$\leqq K_1 \left(\left\{ n^{-\beta} \sum_{\nu=0}^{m} \sum_{k=2^\nu}^{2^{\nu+1}-1} k^{\beta-1} |s_k(x) - s_{2^\nu}(x) - \sigma_k^*(x)|^p \right\}^{1/p} + \right.$$

$$\left. + \left\{ n^{-\beta} \sum_{\nu=0}^{m} \sum_{k=2^\nu}^{2^{\nu+1}-1} k^{\beta-1} |s_{2^\nu}(x) - f(x)|^p \right\}^{1/p} + \left\{ n^{-\beta} \sum_{k=1}^{2^{m+1}} k^{\beta-1} |\sigma_k^*(x)|^p \right\}^{1/p} \right) =$$

$$=: K_1 \sum_{i=1}^{3} d_n^{(i)}(x).$$

By Lemma 1 and $\beta > p\gamma$ it is easy to show that $d_n^{(2)}(x) = o_x(n^{-\gamma})$, namely

(3.8) $$d_n^{(2)}(x) = \left\{ n^{-\beta} \sum_{\nu=0}^{m} 2^{\nu\beta} o_x(2^{-\nu\gamma p}) \right\}^{1/p} = o_x(n^{-\gamma}).$$

But if we observe that Lemma 5 and Lemma 6 imply that, as $m \to \infty$,

$$2^{m(p\gamma-1)} \sum_{k=2^m}^{2^{m+1}} |s_k(x) - s_{2^m}(x) - \sigma_k^*(x)|^p = o_x(1)$$

and

$$2^{m(p\gamma-1)} \sum_{k=2^m}^{2^{m+1}} |\sigma_k^*(x)|^p = o_x(1)$$

hold a.e. in (a, b), then by the use of these estimates we can easily verify that

$$d_n^{(1)}(x)+d_n^{(3)}(x) = o_x(n^{-\gamma})$$

also holds a.e. in (a, b).

Indeed, by $\beta > p\gamma$, we have that

$$d_n^{(1)}(x) = \left\{n^{-\beta} \sum_{\nu=0}^{m} 2^{\nu(\beta-1)} o_x(2^{\nu(1-p\gamma)})\right\}^{1/p} = o_x(n^{-\gamma})$$

and similarly

$$d_n^{(3)}(x) \leqq \left\{n^{-\beta} \sum_{\nu=0}^{m} \sum_{k=2^\nu}^{2^{\nu+1}} k^{\beta-1} |\sigma_k^*(x)|^p\right\}^{1/p} = o_x(n^{-\gamma}).$$

Summing up our partial estimations we get that

$$h_n(f, \alpha, p, \{k\}; x) = o_x(n^{-\gamma}),$$

whence Lemma 3, as in the proof of Theorem 1, conveys the assertion of Theorem 2.

Proof of Theorem 3. At first we prove the special case $\alpha=1$. Then, for $0<p\leqq 2$, Theorem C gives (6), so we assume that $p>2$. Next $\{m_k\}$ denotes the sequence defined in Lemma 8. Using this notation we have

$$\left\{\frac{1}{n+1} \sum_{k=0}^{n} |s_{\mu_k}(x)-f(x)|^p\right\}^{1/p} \leqq K\left(\left\{\frac{1}{n+1} \sum_{k=0}^{n} |s_{\mu_k}(x)-s_{m_k}(x)-\sigma_{\mu_k}^*(x)|^p\right\}^{1/p} +\right.$$

$$\text{(3.9)} \qquad \left. + \left\{\frac{1}{n+1} \sum_{k=0}^{n} |\sigma_{\mu_k}^*(x)|^p\right\}^{1/p} + \left\{\frac{1}{n+1} \sum_{k=0}^{n} |s_{m_k}(x)-f(x)|^p\right\}^{1/p}\right) =$$

$$=: K\big(\mu_n^{(1)}(x)+\mu_n^{(2)}(x)+\mu_n^{(3)}(x)\big).$$

Lemma 8 and Lemma 9 prove that

$$\mu_n^{(1)}(x) = o_x(n^{-\gamma}) \quad \text{and} \quad \mu_n^{(2)}(x) = o_x(n^{-\gamma})$$

a.e. in (a, b).

To prove the same estimation for $\mu_n^{(3)}(x)$ we define a new sequence $\{N_n(m)\}$. Let $N_n(m)$ denote the number of μ_k lying in the interval $[2^m, 2^{m+1})$ and $k \leqq n+1$. It is obvious that

$$N_n(m) \leqq \min(n+1, 2^m) \quad \text{and} \quad \sum_{m=0}^{\infty} N_n(m) = n+1.$$

If $2^{l-1} \leqq n < 2^l$, then we obtain with the aid of Lemma 1 and $p\gamma < 1$ that

$$(\mu_n^{(3)}(x))^p = \frac{1}{n+1} \sum_{k=0}^{n} o_x(m_k^{-\gamma p}) = \frac{1}{n+1} \sum_{m=0}^{\infty} N_n(m) o_x(2^{-mp\gamma}) =$$

$$= \frac{1}{n+1} \left\{ \sum_{m=0}^{l-1} 2^m o_x(2^{-mp\gamma}) + \sum_{m=l}^{\infty} (n+1) o_x(2^{-mp\gamma}) \right\} = o_x(2^{-lp\gamma}) = o_x(n^{-p\gamma}),$$

which proves

$$\mu_n^{(3)}(x) = o_x(n^{-\gamma}),$$

and thus by (3.9)

$$\left\{ \frac{1}{n+1} \sum_{k=0}^{n} |s_{\mu_k}(x) - f(x)|^p \right\}^{1/p} = o_x(n^{-\gamma}) \tag{3.10}$$

holds a.e. in (a, b).

If $\alpha > 1$ then (6) is an immediate consequence of (3.10) because of the relation $A_{n-k}^{\alpha-1}/A_n^{\alpha} = O\left(\frac{1}{n}\right)$ $(0 \leqq k \leqq n)$.

If $0 < \alpha < 1$ we can choose q such that $p\gamma < \frac{1}{q} < \alpha$. Then with $q' = \frac{q}{q-1}$ the inequality $(\alpha-1)q' > -1$ is fulfilled.

Now using Hölder's inequality we obtain that

$$C_n(f, \alpha, p, \{\mu_k\}; x) \leqq \left\{ \frac{1}{(A_n^{\alpha})^{q'}} \sum_{k=0}^{n} (A_{n-k}^{\alpha-1})^{q'} \right\}^{1/pq'} \left\{ \sum_{k=0}^{n} |s_{\mu_k}(x) - f(x)|^{pq} \right\}^{1/pq} \leqq$$

$$\leqq K \left\{ \frac{1}{n+1} \sum_{k=0}^{n} |s_{\mu_k}(x) - f(x)|^{pq} \right\}^{1/pq}.$$

Hence, by $pq\gamma < 1$, using (3.10) we get the assertion of Theorem 3.

Proof of Theorem 4. The case $\beta = 1$ is identical with the special case $\alpha = 1$ of Theorem 3. The cases $\beta > 1$ and $0 < \beta < 1$ may be proved similarly to the cases $\alpha > 1$ and $0 < \alpha < 1$ above, choosing q such that $p\gamma < \frac{1}{q} < \beta$ for $\beta < 1$. We omit the proof.

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