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Necessary and sufficient condition for the maximal inequality of convex Young functions

J. MOGYORÓDI and T. F. MÓRI

Dedicated to Professor B. Szőkefalvi-Nagy on his 70th birthday

1. Young functions

Let $\varphi(t)$ be a non-decreasing and left-continuous function defined on $[0, +\infty)$ such that $\varphi(0)=0$ and $\lim_{t\to+\infty}\varphi(t)=+\infty$. For $x\geqq 0$ define

$$\Phi(x) = \int_0^x \varphi(t)\,dt.$$

Then Φ is non-decreasing, continuous and convex. Φ is called a Young function.

The conjugate Young function is defined as follows: for $t>0$ put $\psi(t)=\sup\{x>0: \varphi(x)<t\}$ and let $\psi(0)=0$. One can show that ψ satisfies all the properties imposed on φ. Further, we trivially have

$$\psi(\varphi(x)) \leqq x \leqq \psi(\varphi(x)+0). \tag{1}$$

The Young function

$$\Psi(x) = \int_0^x \psi(t)\,dt$$

is said to be conjugate to Φ.

The pair (Φ, Ψ) of mutually conjugate Young functions satisfies the following inequality of Young:

$$xy \leqq \Phi(x)+\Psi(y) \quad \text{for arbitrary} \quad x\geqq 0,\ y\geqq 0.$$

Equality holds if and only if $y\in[\varphi(x), \varphi(x+0)]$ or $x\in[\psi(y), \psi(y+0)]$.

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We say that Φ satisfies the moderated growth condition if one of the following three equivalent conditions is met:

$$\limsup_{x\to+\infty} \frac{\varphi(c_1 x)}{\varphi(x)} < +\infty \quad \text{for some constant} \quad c_1 > 1, \tag{2}$$

$$\limsup_{x\to+\infty} \frac{\Phi(c_2 x)}{\Phi(x)} < +\infty \quad \text{for some constant} \quad c_2 > 1, \tag{3}$$

$$p = \limsup_{x\to+\infty} \frac{x\varphi(x)}{\Phi(x)} < +\infty. \tag{4}$$

In this note the quantity p is referred to as the power of Φ. The power q of the conjugate Young function Ψ is defined similarly. One can easily prove that

$$\liminf_{x\to+\infty} \frac{x\varphi(x)}{\Phi(x)} = \frac{q}{q-1}. \tag{5}$$

(Here and in the sequel let $\frac{1}{0} = +\infty$, $\frac{+\infty}{+\infty} = 1$, $\frac{1}{+\infty} = 0$ by definition.) Further, for arbitrary constant $c > 1$ we have

$$c^{\frac{q}{q-1}} \leqq \liminf_{x\to+\infty} \frac{\Phi(cx)}{\Phi(x)} \leqq \limsup_{x\to+\infty} \frac{\Phi(cx)}{\Phi(x)} \leqq c^p. \tag{6}$$

The above assertions and further information about the theory of Young functions can be found, e.g., in [4] and in [8].

We prove the following

Lemma. *Let (Φ, Ψ) be a pair of conjugate Young functions. In order that the power q of Ψ be finite it is necessary and sufficient that the condition*

$$\limsup_{x\to+\infty} \frac{1}{\varphi(x)} \int_1^x \frac{\varphi(t)}{t}\, dt = \beta < +\infty \tag{7}$$

be satisfied.

Proof. Integrating by parts yields

$$\frac{1}{\varphi(x)} \int_1^x \frac{\varphi(t)}{t}\, dt = \frac{\Phi(x)}{x\varphi(x)} - \frac{\Phi(1)}{\varphi(x)} + \frac{1}{\varphi(x)} \int_0^x \frac{\Phi(t)}{t\varphi(t)} \frac{\varphi(t)}{t}\, dt. \tag{8}$$

Combining this with (5) we obtain that for arbitrary $\varepsilon > 0$

$$\frac{1}{\varphi(x)} \int_1^x \frac{\varphi(t)}{t}\, dt \leqq \left(\frac{q-1}{q} + \varepsilon\right)\left(1 + \frac{1}{\varphi(x)} \int_1^x \frac{\varphi(t)}{t}\, dt\right) + O\left(\frac{1}{\varphi(x)}\right)$$

holds, hence $\beta \leqq q-1$. Thus the growth condition implies (7).

Conversely, let y denote $\psi(2x)$. Recalling (1) we can write

$$\frac{1}{\varphi(y)}\int_1^y \frac{\varphi(t)}{t}\,dt \geqq \frac{1}{2x}\int_{\psi(x)}^{\psi(2x)} \frac{\varphi(t)}{t}\,dt \geqq \frac{\varphi(\psi(x)+0)}{2x}\int_{\psi(x)}^{\psi(2x)} \frac{dt}{t} \geqq \frac{1}{2}\log\frac{\psi(2x)}{\psi(x)}.$$

From this it follows that

$$\limsup_{x\to+\infty}\frac{\psi(2x)}{\psi(x)} \leqq e^{2\beta},$$

thus (7) implies the growth condition.

2. The maximal inequality

Definition. We say that for the Young function Φ the maximal inequality is valid with some constants $a, b \geqq 0$ depending only on Φ if for arbitrary non-negative submartingale $(X_n, \mathscr{F}_n)$, $n \geqq 1$, with the maximum $X_n^* = \max_{1\leqq k\leqq n} X_k$ we have

$$E(\Phi(X_n^*)) \leqq a + E(\Phi(bX_n)) \quad n = 1, 2, \ldots. \tag{9}$$

Several papers have been devoted to such type of inequalities, e.g., [1], [3], [7].

The main purpose of the present note is to characterize all the Young functions Φ for which the maximal inequality is valid.

Theorem 1. *Let (Φ, Ψ) be a pair of conjugate Young functions. In order that Φ satisfy the maximal inequality in the above sense it is necessary and sufficient that the power q of Ψ be finite.*

Proof. Although the sufficiency part of the present assertion is already known (cf. [7]), for the sake of completeness we present here a proof to it. Suppose that Ψ obeys the growth condition. Then for arbitrary $b > q$ one can find a constant $a \geqq 0$ to satisfy the inequality $x\psi(x) \leqq a + b\Psi(x)$ for all $x \geqq 0$. We prove that the maximal inequality is valid for Φ with the same constants a and b. To this end we recall the following inequality due to Doob:

$$\lambda P(X_n^* \geqq \lambda) \leqq E(X_n I(X_n^* \geqq \lambda)) \quad \text{for} \quad \lambda \geqq 0.$$

Here $I(\cdot)$ stands for the indicator of the event in the brackets. For any $c > 0$ define $X_k' = \min(X_k, c)$ and set

$$X_n^{**} = \max_{1\leqq k\leqq n} X_k' = \min(X_n^*, c).$$

On the basis of the Doob inequality we have

$$\lambda P(X_n^{**} \geqq \lambda) \leqq E(X_n I(X_n^{**} \geqq \lambda)).$$

Integrating this on $[0, +\infty)$ with respect to the measure generated by $\varphi(\lambda)$ we get

$$\int_0^{+\infty} \lambda E(I(X_n^{**} \geqq \lambda))\, d\varphi(\lambda) \leqq \int_0^{+\infty} E(X_n I(X_n^{**} \geqq \lambda))\, d\varphi(\lambda).$$

Applying the Fubini theorem to both sides we obtain

$$E\Big(\int_0^{X_n^{**}} \lambda\, d\varphi(\lambda)\Big) \leqq E(X_n \varphi(X_n^{**})).$$

By partial integration

$$\int_0^x \lambda\, d\varphi(\lambda) = x\varphi(x) - \int_0^x \varphi(\lambda)\, d\lambda = x\varphi(x) - \Phi(x) = \Psi(\varphi(x)),$$

whence

$$E(\Psi(\varphi(X_n^{**}))) \leqq \frac{1}{b} E(bX_n \varphi(X_n^{**})).$$

Using the Young inequality on the right-hand side yields

$$E(\Psi(\varphi(X_n^{**}))) \leqq \frac{1}{b}\left[E(\Phi(bX_n)) + E(\Psi(\varphi(X_n^{**})))\right].$$

From this it follows that

$$(b-1)E(\Psi(\varphi(X_n^{**}))) \leqq E(\Phi(bX_n)),$$

since X_n^{**} is bounded by c. Now by the assumption

$$\Phi(x) = x\varphi(x) - \Psi(\varphi(x)) \leqq \psi(\varphi(x)+0)\varphi(x) - \Psi(\varphi(x)) \leqq a + (b-1)\Psi(\varphi(x)),$$

from which it follows that

$$E(\Phi(X_n^{**})) \leqq a + E(\Phi(bX_n)).$$

Let c tend to $+\infty$, then $X_n^{**} \to X_n^*$ and the monotone convergence theorem completes the proof of the sufficiency part of our assertion.

Necessity. Suppose that the maximal inequality is valid for Φ with some constants a, b. We can set $b \geqq 1$. Let us define a sequence $\{x_n\}$ of numbers with the following properties:

$$x_1 = 1, \quad x_n < x_{n+1} < 2x_n \quad \text{for} \quad n = 1, 2, \ldots, \quad \lim_{n\to\infty} x_n = +\infty$$

and

$$\text{(10)} \qquad \limsup_{n\to\infty} \frac{1}{\varphi(bx_n)} \int_1^{bx_n} \frac{\varphi(t)}{t}\, dt = \limsup_{x\to+\infty} \frac{1}{\varphi(x)} \int_1^x \frac{\varphi(t)}{t}\, dt.$$

Let Ω be the set of the positive integers and let $\mathscr{A}$ be the σ-field of all subsets of Ω. On the measurable space $(\Omega, \mathscr{A})$ we define the probability P by the formula

$$P(\{n\}) = \frac{1}{x_n} - \frac{1}{x_{n+1}}, \quad n = 1, 2, \ldots.$$

Let $\mathscr{F}_n$ be the σ-field generated by the partition

$$(\{1\}, \{2\}, \dots, \{n-1\}, \{n, n+1, \dots\}).$$

Clearly we have $\mathscr{F}_1 \subset \mathscr{F}_2 \subset \dots$. Further, for $n=1, 2, \dots$ define the random variable X_n by

$$X_n(\omega) = x_n I(\omega \geqq x_n), \quad \omega \in \Omega.$$

It is easy to see that $(X_n, \mathscr{F}_n)$ is a nonnegative martingale and that

$$X_n^*(\omega) = \begin{cases} x_\omega, & \text{if} \quad \omega < n \\ x_n, & \text{if} \quad \omega \geqq n. \end{cases}$$

In virtue of the maximal inequality we have

$$\sum_{k=1}^{n-1} \Phi(x_k)\left(\frac{1}{x_k} - \frac{1}{x_{k+1}}\right) + \frac{1}{x_n}\Phi(x_n) \leqq a + \frac{1}{x_n}\Phi(bx_n). \tag{11}$$

The sum of the left hand side of (11) can be estimated as follows:

$$\sum_{k=1}^{n-1} \Phi(x_k)\left(\frac{1}{x_k} - \frac{1}{x_{k+1}}\right) = \sum_{k=1}^{n-1} \Phi(x_k)\frac{1}{2}\int_{x_k/2}^{x_{k+1}/2} \frac{1}{t^2}\,dt \geqq \frac{1}{2}\int_1^{x_n/2} \frac{\Phi(t)}{t^2}\,dt.$$

Integrating by parts we obtain

$$\frac{1}{2}\int_1^{x_n/2} \frac{\Phi(t)}{t^2}\,dt = \frac{1}{2}\left[\Phi(1) - \frac{\Phi(x_n/2)}{x_n/2} + \int_1^{x_n/2} \frac{\varphi(t)}{t}\,dt\right],$$

hence (11) implies

$$\frac{1}{2}\int_1^{x_n/2} \frac{\varphi(t)}{t}\,dt \leqq a + \frac{1}{x_n}\Phi(bx_n).$$

On the other hand,

$$\frac{1}{2}\int_{x_n/2}^{bx_n} \frac{\varphi(t)}{t}\,dt \leqq \frac{1}{2}\varphi(bx_n)\log 2b,$$

consequently

$$\frac{1}{\varphi(bx_n)}\int_1^{bx_n} \frac{\varphi(t)}{t}\,dt \leqq \frac{2a}{\varphi(bx_n)} + 2b\frac{\Phi(bx_n)}{bx_n\varphi(bx_n)} + \log 2b.$$

Keeping in mind the property (10) of the sequence $\{x_n\}$ we conclude

$$\limsup_{x \to +\infty} \frac{1}{\varphi(x)}\int_1^x \frac{\varphi(t)}{t}\,dt \leqq 2b + \log 2b,$$

thus by our Lemma Ψ fulfils the growth condition.

3. Estimates for the best constants in the maximal inequality

Denote by b^* the infimum of the constants b the maximal inequality is valid with. b^* appears to measure somehow the rate of growth of the Young function Φ: the faster Φ grows, the smaller b^* is. Hence it would be of interest either to find the connection between b^* and the quantities introduced while formulating the growth condition, or to give some estimates at least. The assertion proved below may be regarded as the first step in this direction.

Theorem 2. *Let (Φ, Ψ) be a pair of conjugate Young functions with powers p and q, respectively. Then*

$$\frac{p}{p-1} \leqq b^* \leqq q.$$

Proof. The upper estimate for b^* follows immediately from the proof of the sufficiency part of Theorem 1.

For the lower estimate suppose the maximal inequality is valid for Φ with some constants $a \geqq 0$ and $b < \frac{p}{p-1}$. From this we derive a contradiction. In view of Theorem 1 the case $p=1$ may be left out of consideration.

Define $\Omega = \{1, 2, ..., n\}$, let $\mathscr{A}$ be the σ-field of all subsets of Ω and let $P(\{\omega\}) = \frac{1}{n}$; $\omega \in \Omega$. On the probability space $(\Omega, \mathscr{A}, P)$ define the nonnegative martingale $(X_k; \mathscr{F}_k)$, $k=1, ..., n$, as follows: let $\mathscr{F}_{n+1-k}$ be the σ-field generated by the partition

$$(\{1, 2, ..., k\}, \quad \{k+1\}, ..., \{n\})$$

and let

$$X_{n+1-k}(\omega) = \begin{cases} c\omega^{-1/p}, & \text{if} \quad \omega > k \\ \frac{c}{k} \sum_{i=1}^{k} i^{-1/p}, & \text{if} \quad \omega \leqq k, \end{cases}$$

where $c = \Phi^{-1}(n)/b$.

Clearly, $(X_k, \mathscr{F}_k)$ has the martingale property. One can easily see that

$$X_n^*(\omega) = \frac{c}{\omega} \sum_{i=1}^{\omega} i^{-1/p} \geqq \frac{c}{\omega} \frac{p}{p-1} (\omega^{1-1/p} - 1),$$

from which we have

$$X_n^*(\omega) > b(1+\varepsilon) X_n(\omega) \quad \text{for} \quad \omega \geqq k_0,$$

where $\varepsilon > 0$ satisfies $b(1+\varepsilon) < \frac{p}{p-1}$ and the threshold k_0 does not depend on n.

Hence

$$E(\Phi(X_n^*)) \geqq \frac{1}{n} \sum_{\omega=k_0}^{n} E(\Phi(b(1+\varepsilon)X_n(\omega))) \geqq (1+\varepsilon)\frac{1}{n} \sum_{\omega=k_0}^{n} E(\Phi(bX_n(\omega))) \geqq$$

$$\geqq (1+\varepsilon)\left[E(\Phi(bX_n)) - \frac{k_0}{n}\Phi(bc)\right] = (1+\varepsilon)\left[E(\Phi(bX_n)) - k_0\right].$$

Applying the maximal inequality to the martingale $(X_k, \mathcal{F}_k)$ we obtain

(12) $$a + E(\Phi(bX_n)) \geqq (1+\varepsilon)[E(\Phi(bX_n)) - k_0].$$

Now let n tend to infinity. Then from (6) it follows that

$$\liminf_{n\to\infty} \frac{1}{n}\Phi(bX_n(\omega)) = \liminf_{n\to\infty} \frac{\Phi(bc\omega^{-1/p})}{\Phi(bc)} \geqq \frac{1}{\omega}$$

for arbitrary fixed positive integer ω. Consequently,

$$\lim_{n\to\infty} E(\Phi(bX_n)) = +\infty$$

which contradicts (12).

4. Remarks

(i) *Convexity inequality.* We say that for the Young function Ψ the convexity inequality is valid with some constants $a, b \geqq 0$, if for arbitrary sequence $\{Z_n\}$ of nonnegative random variables and increasing sequence $\{\mathcal{F}_n\}$ of σ-fields

$$E\left[\Psi\left(\sum_{i=1}^{n} E(Z_i|\mathcal{F}_i)\right)\right] \leqq a + E\left[\Psi\left(b\sum_{i=1}^{n} Z_i\right)\right], \quad n = 1, 2, \ldots$$

holds. By the duality theorem of [6] the maximal inequality is valid for a Young function Φ if and only if the convexity inequality holds for Ψ, the conjugate to Φ. So Theorem 1 of the present note affords also a necessary and sufficient condition for a Young function Ψ to satisfy the convexity inequality, namely, that Ψ should meet the growth condition.

(ii) *An open problem.* Denote

$$\liminf_{x\to+\infty} \frac{1}{\varphi(x)} \int_0^x \frac{\varphi(t)}{t}\,dt$$

by α. Returning to (8) we can see that $\alpha \geqq \frac{1}{p}(1+\alpha)$, thus $\alpha \geqq \frac{1}{p-1}$. If we rewrite this into the form

$$\frac{p}{p-1} \leqq \alpha + 1 \leqq \beta + 1 \leqq q,$$

the following problem arises. Is it true that

$$\alpha+1 \leqq b^* \leqq \beta+1 \tag{13}$$

holds for every Young function Φ the conjugate of which has a finite power? If Φ itself also satisfies the growth condition, another proof of the maximal inequality shows that $b^* \leqq p\beta$ (see [5]). Since $\beta+1 \leqq p\beta$ always holds, the upper bound in (13) seems to be rather sharp if not false.

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DEPARTMENT OF PROBABILITY THEORY
EÖTVÖS LORÁND UNIVERSITY
MÚZEUM KRT. 6—8
1088 BUDAPEST, HUNGARY