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Extension of Banach's principle for multiple sequences of operators

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Dedicated to Professor Béla Sz.-Nagy on his 70th birthday

1. Introduction

Let $(X, \mathscr{F})$ be a measurable space with a positive finite measure μ. Denote by $S=S(X, \mathscr{F})$ the set of the a.e. finite real-valued functions on X measurable with respect to $\mathscr{F}$. As is well-known, X endowed with the distance notion

$$d(\varphi, \psi)=\int_X \frac{|\varphi(x)-\psi(x)|}{1+|\varphi(x)-\psi(x)|}\, d\mu(x) \quad (\varphi, \psi\in S)$$

is a complete metric space (a so-called Frèchet space), and the convergence notion induced by d is equivalent with the convergence in measure.

Let B be a Banach space and let $T\colon B\to S$ be an operator. As usual, T is said to be *subadditive* if

(i) $|T(f+g)(x)|\leqq|Tf(x)|+|Tg(x)|$ a.e. on X for every $f, g\in B$,

and *positive homogeneous* if

(ii) $|T(\alpha f)(x)|=|\alpha Tf(x)|$ a.e. on X for every $\alpha\geqq 0$ and $f\in B$.

We shall deal only with subadditive and positive homogeneous operators T on B (sometimes these operators are said to be *convex*, too) for which the following condition is also satisfied:

(iii) T is *continuous in measure*, i.e. if $f_n, f\in B$ and $\|f_n-f\|\to 0$ as $n\to\infty$, then for every $\varepsilon>0$ we have

$$\mu\{x\colon |Tf_n(x)-Tf(x)|>\varepsilon\}\to 0 \quad \text{as} \quad n\to\infty.$$

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In certain cases we shall need a further property of the operators T, namely

(iv) $|Tf(x)-Tg(x)| \leqq \varkappa\{|T(f-g)(x)|+|T(g-f)(x)|\}$ a.e. on X for every $f, g \in B$,

where $\varkappa$ is a positive constant.

It is clear that if T is a linear operator, then (iv) is satisfied with $\varkappa=1/2$. Another example is the following: If T is an operator with properties (i) and

(v) T is *positive*, i.e. $Tf(x) \geqq 0$ a.e. on X for every $f \in B$,

then T possesses property (iv). In fact, now

$$Tf(x) = T(f-g+g)(x) \leqq T(f-g)(x)+Tg(x)$$

and similarly

$$Tg(x) \leqq T(g-f)(x)+Tf(x),$$

whence (iv) follows with $\varkappa=1$.

We note that if we replace property (ii) by

$\overline{\text{(ii)}}$ $|T(\alpha f)(x)|=|\alpha Tf(x)|$ a.e. on X for every real number α and $f \in B$,

then we can replace property (iv) by

$\overline{\text{(iv)}}$ $|Tf(x)-Tg(x)| \leqq 2\varkappa|T(f-g)(x)|$ a.e. on X for every $f, g \in B$.

Now, it is not hard to check that $\overline{\text{(iv)}}$ in the special case $2\varkappa=1$ implies property (i). So, if $\overline{\text{(ii)}}$ and (v) are satisfied, then properties (i) and $\overline{\text{(iv)}}$ with $2\varkappa=1$ are equivalent to each other.

2. Banach's principle for single series

Given an ordinary sequence $\{T_n : n=1, 2, \ldots\}$ of operators, we shall put, for every $f \in B$,

$$T^*f(x) = \sup_{n \geqq 1} |T_n f(x)|.$$

It is obvious that if the sequence $\{T_n f(x)\}$ is convergent a.e. on X for every $f \in B$, then a fortiori we also have that

$$T^*f(x) < \infty \text{ a.e. on } X \text{ for every } f \in B. \tag{1}$$

The following results are well-known (see [1] and also [2, pp. 1—4], where the operators T_n are supposed to be linear, but the proofs apply, after some simple modifications, to the more general operators indicated in Section 1).

Theorem 0. *Let the operators T_n possess properties* (i)—(iv). *If condition* (1) *is satisfied, then the set of those $f \in B$ for which the sequence $\{T_n f(x)\}$ is a.e. convergent is closed.*

This immediately yields

Corollary. *Let the operators T_n possess properties* (i)—(iv). *If condition* (1) *is satisfied and the sequence $\{T_n f(x)\}$ is a.e. convergent for a set of $f \in B$ which is dense in B, then $\{T_n f(x)\}$ is a.e. convergent for every $f \in B$.*

The next lemma plays a decisive role in the proof of Theorem 0 and sometimes is called Banach's principle in a strict sense.

Lemma 0. *Let the operators T_n possess properties* (i)—(iii). *If condition* (1) *is satisfied, then there exists a positive, nonincreasing function $C(\lambda)$, defined for $\lambda > 0$ and tending to zero as $\lambda \to \infty$ such that*

$$\mu\{x\colon T^* f(x) > \lambda \|f\|\} \leqq C(\lambda) \quad \textit{for every} \quad \lambda > 0 \quad \textit{and} \quad f \in B.$$

A simple consequence is the following

Corollary. *Let the operators T_n possess properties* (i)—(iii). *If condition* (1) *is satisfied, then T^* is continuous in measure, even uniformly in f.*

3. Extension to multiple sequences using the convergence notion in Pringsheim's sense

Let $\mathcal{N}^d$ be the set of all d-tuples $\mathbf{k} = (k_1, \ldots, k_d)$ with positive integers for coordinates, where $d \geqq 1$ is a fixed integer. As usual, put

$$\mathbf{k} = (k_1, \ldots, k_d) \leqq (m_1, \ldots, m_d) = \mathbf{m} \quad \text{iff} \quad k_j \leqq m_j \quad (j = 1, \ldots, d),$$

$$\mathbf{k} \pm \mathbf{m} = (k_1 \pm m_1, \ldots, k_d \pm m_d), \ \mathbf{km} = (k_1 m_1, \ldots, k_d m_d), \quad \text{and} \quad \mathbf{1} = (1, \ldots, 1).$$

We recall that a d-multiple sequence $\{t_{\mathbf{m}} \colon \mathbf{m} \in \mathcal{N}^d\}$ of real numbers is said to be *convergent in Pringsheim's sense* if for every $\varepsilon > 0$ there exists an $M = M(\varepsilon)$ so that $|t_{\mathbf{k}} - t_{\mathbf{m}}| < \varepsilon$ whenever

$$\min(k_1, \ldots, k_d) \geqq M \quad \text{and} \quad \min(m_1, \ldots, m_d) \geqq M. \tag{2}$$

We consider a d-multiple sequence $\{T_{\mathbf{k}} \colon \mathbf{k} \in \mathcal{N}^d\}$ of operators having properties (i)—(iii) or (i)—(iv) enumerated in Section 1. It is a simple fact that the sequence $\{T_{\mathbf{k}} f(x)\}$ is convergent a.e. on X in Pringsheim's sense for a given $f \in B$ if and only if

$$\lim_{M \to \infty} \sup_{\text{under}(2)} |T_{\mathbf{k}} f(x) - T_{\mathbf{m}} f(x)| = 0 \quad \text{a.e. on } X,$$

or equivalently, for every $\varepsilon>0$,

$$\mu\{x\colon \sup_{\text{under}(2)} |T_{\mathbf{k}}f(x)-T_{\mathbf{m}}f(x)| > \varepsilon\} \to 0 \quad \text{as} \quad M\to\infty. \tag{3}$$

On the other hand, it is clear that if $\{T_{\mathbf{k}}f(x)\}$ is convergent a.e. on X in Pringsheim's sense for every $f\in B$, then we also have

(4)
$$T_*f(x) = \inf_{M=1,2,\dots} \sup_{\min(k_1,\dots,k_d)\geqq M} |T_{\mathbf{k}}f(x)| < \infty \quad \text{a.e. on } X \quad \text{for every} \quad f\in B.$$

For the sake of brevity, we write

$$T_{*M}f(x) = \sup_{\min(k_1,\dots,k_d)\geqq M} |T_{\mathbf{k}}f(x)| \quad (M=1,2,\dots).$$

The basic fact is again that condition (4) itself already implies the continuity of the operator T_* in measure, uniformly in f. Vice versa, it will be also seen that in certain cases such a continuity property for T_* is all that is needed to establish the a.e. convergence of the d-multiple sequence $\{T_{\mathbf{k}}f(x)\}$ in Pringsheim's sense for every $f\in B$.

The following theorem extends Theorem 0.

Theorem 1. *Let the operators $T_{\mathbf{k}}$, $\mathbf{k}\in\mathcal{N}^d$, possess properties* (i)—(iv). *If condition* (4) *is satisfied, then the set of those $f\in B$ for which the d-multiple sequence $\{T_{\mathbf{k}}f(x)\}$ is a.e. convergent in Pringsheim's sense is closed.*

This implies the next

Corollary 1. *Let the operators $T_{\mathbf{k}}$ possess properties* (i)—(iv). *If condition* (4) *is satisfied and the d-multiple sequence $\{T_{\mathbf{k}}f(x)\}$ is a.e. convergent in Pringsheim's sense for a set of $f\in B$ which is dense in B, then $\{T_{\mathbf{k}}f(x)\}$ is a.e. convergent in Pringsheim's sense for every $f\in B$.*

The continuity property of T_* mentioned above is expressed in the following

Lemma 1. *Let the operators $T_{\mathbf{k}}$, $\mathbf{k}\in\mathcal{N}^d$, possess properties* (i)—(iii). *If condition* (4) *is satisfied, then there exists a positive, nonincreasing function $C(\lambda)$, defined for $\lambda>0$ and tending to zero as $\lambda\to\infty$ such that*

(5)
$$\mu\{x\colon \sup_{\min(k_1,\dots,k_d)\geqq\lambda} |T_{\mathbf{k}}f(x)| > \lambda\|f\|\} \leqq C(\lambda) \quad \textit{for every} \quad \lambda>0 \quad \textit{and} \quad f\in B.$$

This immediately yields $\mu\{x\colon T_*f(x)>\lambda\|f\|\}\leqq C(\lambda)$, which can be reformulated as follows:

Corollary 2. *Let the operators $T_{\mathbf{k}}$ possess properties* (i)—(iii). *If condition* (4) *is satisfied, then T_* is continuous in measure, even uniformly in f.*

Proof of Lemma 1. It is modelled upon the proof of Lemma 0 (see in [2, pp. 2—3]).

By (ii), we need only establish (5) for $\|f\|=1$. Let an $\varepsilon>0$ be given. Owing to (4) for every $f\in B$ there exists an M, possibly depending on ε and f, such that

$$\mu\{x\colon\ T_{*M}f(x) > M\} \leqq \varepsilon.$$

In other words, this means that

$$B = \bigcap_{M=1}^{\infty} \{f\colon\ \mu\{x\colon\ T_{*M}f(x) > M\} \leqq \varepsilon\}.$$

We shall show that each set on the right of the last equality is closed. To this effect, observe that for each M,

$$\{f\colon\ \mu\{x\colon\ T_{*M}f(x) > M\} \leqq \varepsilon\} = \bigcap_{N=M}^{\infty} \{f\colon\ \mu\{x\colon\ T_{*MN}f(x) > M\} \leqq \varepsilon\}, \tag{6}$$

where

$$T_{*MN}f(x) = \max_{\substack{M \leqq \min(k_1,\ldots,k_d) \leqq \\ \leqq \max(k_1,\ldots,k_d) \leqq N}} |T_{\mathbf{k}}f(x)| \quad (M,\ N = 1, 2, \ldots;\ M \leqq N).$$

By (i), for every f and g in B we have

$$|T_{*MN}f(x) - T_{*MN}g(x)| \leqq T_{*MN}(f-g)(x) + T_{*MN}(g-f)(x).$$

Consequently, for every $\delta>0$,

$$\mu\{x\colon\ |T_{*MN}f(x) - T_{*MN}g(x)| > \delta\} \leqq$$

$$\leqq \sum_{k_1=M}^{N} \cdots \sum_{k_d=M}^{N} \left[\mu\left\{x\colon\ |T_{\mathbf{k}}(f-g)(x)| > \frac{\delta}{2}\right\} + \mu\left\{x\colon\ |T_{\mathbf{k}}(g-f)(x)| > \frac{\delta}{2}\right\}\right].$$

Since each operator $T_{\mathbf{k}}$ is continuous in measure (property (iii)), hence it follows that the operators T_{*MN} are also continuous in measure. Therefore, each of the sets

$$\{f\colon\ \mu\{x\colon\ T_{*MN}f(x) > M\} \leqq \varepsilon\}$$

is closed, and thus so is the set in (6).

Now we apply the Baire category theorem and conclude that one of the sets in (6) contains a closed ball, say with some center $f_0\in B$ and radius $\varrho>0$. This means that if $f\in B$ and $\|f-f_0\| \leqq \varrho$, then

$$\mu\{x\colon\ T_{*M}f(x) > M\} \leqq \varepsilon.$$

In other words, if $g\in B$ and $\|g\| \leqq 1$, then

$$\mu\{x\colon\ T_{*M}(f_0+\varrho g)(x) > M\} \leqq \varepsilon.$$

This yields

(7)
$$\mu\left\{x\colon T_{*M}g(x) > \frac{2M}{\varrho}\right\} \leqq \mu\{x\colon T_{*M}(f_0+\varrho g)(x) > M\} + \mu\{x\colon T_{*M}f_0(x) > M\} \leqq 2\varepsilon$$
$$\text{for every} \quad g\in B,\ \|g\| \leqq 1.$$

It is not hard to verify that (7) already implies (5) to be proved. In fact, put
$$C(\lambda) = \sup_{\|g\|\leqq 1} \mu\{x\colon T_{*[\lambda]}f(x) > \lambda\},$$
where by $[\lambda]$ we denote the integral part of $\lambda>0$. Inequality (7) shows that $C(\lambda)\leqq 2\varepsilon$ if $\lambda\geqq\max(M, 2M/\varrho)$. Thus we have

(8)
$$\lim_{\lambda\to\infty} C(\lambda) = 0$$

and our assertion is proved.

Proof of Theorem 1. Denote by $\mathscr{C}$ the set of $f\in B$ for which the d-multiple sequence $\{T_{\mathbf{k}}f(x)\}$ is a.e. convergent in Pringsheim's sense. We are to show that if for a given $f\in B$ it is true that for every $\varepsilon>0$ there is a $g\in\mathscr{C}$ such that $\|f-g\|<\varepsilon$, then $f\in\mathscr{C}$ as well.

By (iv),
$$|T_{\mathbf{k}}f(x)-T_{\mathbf{m}}f(x)| \leqq |T_{\mathbf{k}}f(x)-T_{\mathbf{k}}g(x)|+|T_{\mathbf{k}}g(x)-T_{\mathbf{m}}g(x)|+|T_{\mathbf{m}}g(x)-T_{\mathbf{m}}f(x)| \leqq$$
$$\leqq \varkappa[|T_{\mathbf{k}}(f-g)(x)|+|T_{\mathbf{k}}(g-f)(x)|+|T_{\mathbf{m}}(g-f)(x)|+|T_{\mathbf{m}}(f-g)(x)|]+|T_{\mathbf{k}}g(x)-T_{\mathbf{m}}g(x)|.$$
Thus, for every $\lambda>0$ and $M\geqq 1$,

(9)
$$\mu\{x\colon \sup_{\text{under}(2)} |T_{\mathbf{k}}f(x)-T_{\mathbf{m}}f(x)| > \lambda\|f-g\|\} \leqq$$
$$\leqq \mu\left\{x\colon T_{*M}(f-g)(x) > \frac{\lambda}{5\varkappa}\|f-g\|\right\}+\mu\left\{x\colon T_{*M}(g-f)(x) > \frac{\lambda}{5\varkappa}\|f-g\|\right\}+$$
$$+\mu\left\{x\colon \sup_{\text{under}(2)} |T_{\mathbf{k}}g(x)-T_{\mathbf{m}}g(x)| > \frac{\lambda}{5}\|f-g\|\right\}.$$

Let us fix a $\delta>0$ and an $\varepsilon>0$. In virtue of (5) and (8) we get
$$\mu\{x\colon T_{*M}(f-g)(x) > M\|f-g\|\} \leqq C(M) \leqq \frac{\delta}{3}$$
if M is large enough, say $M\geqq M_1$, independently of $g\in\mathscr{C}$. Taking $\lambda=5\varkappa M_1$, hence it follows

(10)
$$\mu\left\{x\colon T_{*M}(f-g)(x) > \frac{\lambda}{5\varkappa}\|f-g\|\right\}+$$
$$+\mu\left\{x\colon T_{*M}(g-f)(x) > \frac{\lambda}{5\varkappa}\|f-g\|\right\} \leqq \frac{2\delta}{3} \quad \text{for} \quad M\geqq M_1.$$

Now let us choose $g\in\mathscr{C}$ in such a way that $\lambda\|f-g\|\leqq\varepsilon$. Due to (3), there exists an M_2 such that

$$\text{(11)}\qquad \mu\left\{x\colon \sup_{\text{under}(2)} |T_{\mathbf{k}}g(x)-T_{\mathbf{m}}g(x)| > \frac{\lambda}{5}\|f-g\|\right\} \leqq \frac{\delta}{3} \quad \text{for} \quad M \geqq M_2.$$

Collecting together (9)—(11), we can infer

$$\mu\{x\colon \sup_{\text{under}(2)} |T_{\mathbf{k}}f(x)-T_{\mathbf{m}}f(x)| > \varepsilon\} \leqq \delta \quad \text{for} \quad M \geqq \max(M_1, M_2).$$

Since δ and ε are arbitrary, we obtain relation (3). But this is equivalent to the a.e. convergence of the d-multiple sequence $\{T_{\mathbf{k}}f(x)\}$ in Pringsheim's sense.

4. Extension to multiple sequences using the notion of regular convergence

Following HARDY [3] (cf. [5], where this kind of convergence was rediscovered and called "convergence in a restricted sense") we say that a d-multiple series

$$\sum_{\mathbf{k}\in\mathscr{N}^d} b_{\mathbf{k}} = \sum_{k_1=1}^{\infty} \cdots \sum_{k_d=1}^{\infty} b_{k_1,\dots,k_d}$$

of real numbers is *regularly convergent* if for every $\varepsilon>0$ there exists an $M=M(\varepsilon)$ so that

$$\text{(12)}\qquad \Big|\sum_{\mathbf{m}\leqq\mathbf{k}\leqq\mathbf{n}} b_{\mathbf{k}}\Big| = \left|\sum_{k_1=m_1}^{n_1} \cdots \sum_{k_d=m_d}^{n_d} b_{k_1,\dots,k_d}\right| < \varepsilon$$

whenever

$$\text{(13)}\qquad \max(m_1, \dots, m_d) \geqq M \quad \text{and} \quad \mathbf{n} \geqq \mathbf{m}.$$

It is a trivial fact that the regular convergence of series (12) implies the convergence of the rectangular partial sums

$$s_{\mathbf{m}} = \sum_{\mathbf{1}\leqq\mathbf{k}\leqq\mathbf{m}} b_{\mathbf{k}} \qquad (\mathbf{m}\in\mathscr{N}^d)$$

in Pringsheim's sense.

Given a d-multiple sequence $\{t_{\mathbf{m}}\colon \mathbf{m}\in\mathscr{N}^d\}$ of real numbers, first we define the "total" finite differences $\Delta t_{\mathbf{m}}$ as follows

$$\Delta t_{\mathbf{m}} = \sum_{\eta_1=0}^{1} \cdots \sum_{\eta_d=0}^{1} (-1)^{d-\eta_1-\dots-\eta_d} t_{m_1-1+\eta_1,\dots,m_d-1+\eta_d}$$

with the agreement that $t_{k_1,\dots,k_d}$ is taken to equal 0 if $k_j=0$ for at least one j, $1\leqq j\leqq d$. Then we consider the d-multiple series

$$\text{(14)}\qquad \sum_{\mathbf{m}\in\mathscr{N}^d} \Delta t_{\mathbf{m}},$$

whose rectangular partial sums coincide with the t_m. Now we say that the d-multiple sequence $\{t_\mathbf{m}\}$ is regularly convergent if series (14) is regularly convergent. In other words, this requires that for every $\varepsilon>0$ there exists an $M=M(\varepsilon)$ so that

$$\left|\sum_{\eta_1=0}^{1} \cdots \sum_{\eta_d=0}^{1} (-1)^{\eta_1+\cdots+\eta_d} t_{\boldsymbol{\eta}\mathbf{m}+(\mathbf{1}-\boldsymbol{\eta})\mathbf{n}}\right| < \varepsilon, \quad \boldsymbol{\eta} = (\eta_1, \ldots, \eta_d),$$

whenever (13) is satisfied. For brevity, denote by $\Delta_{\mathbf{m},\mathbf{n}} t_\mathbf{k}$ the expression between the absolute signs.

After these preliminaries, consider again a d-multiple sequence $\{T_\mathbf{k}\colon \mathbf{k}\in\mathcal{N}^d\}$ of operators possessing properties (i)—(iv). The a.e. regular convergence can be characterized as follows. The d-multiple sequence $\{T_\mathbf{k} f(x)\}$ is regularly convergent a.e. on X for an $f\in B$ if and only if

$$\lim_{M\to\infty} \sup_{\text{under}(13)} |\Delta_{\mathbf{m},\mathbf{n}} T_\mathbf{k} f(x)| = 0 \quad \text{a.e. on } X,$$

or equivalently, for every $\varepsilon>0$,

$$\mu\Big\{x\colon \sup_{\text{under}(13)} |\Delta_{\mathbf{m},\mathbf{n}} T_\mathbf{k} f(x)| > \varepsilon\Big\} \to 0 \quad \text{as} \quad M \to \infty. \tag{15}$$

It is obvious that if $\{T_\mathbf{k} f(x)\}$ is regularly convergent a.e. on X for every $f\in B$, then a fortiori we also have that

$$T^* f(x) = \sup_{\mathbf{k}\in\mathcal{N}^d} |T_\mathbf{k} f(x)| < \infty \quad \text{a.e. on } X \text{ for every } f\in B. \tag{16}$$

The fundamental fact is again that condition (16) itself already implies that the operator T^* is continuous in measure, uniformly in f. Indeed, both Lemma 0 and its Corollary are plainly true for the set $\{T_\mathbf{k}\colon \mathbf{k}\in\mathcal{N}^d\}$ of operators under properties (i)—(iii) and condition (16).

The extension of Theorem 0 reads as follows.

Theorem 2. *Let the operators* $T_\mathbf{k}, \mathbf{k}\in\mathcal{N}^d$, *possess properties* (i)—(iv). *If condition* (16) *is satisfied, then the set* $\mathcal{C}$ *of those* $f\in B$ *for which the d-multiple sequence* $\{T_\mathbf{k} f(x)\}$ *is a.e. regularly convergent is closed.*

An immediate consequence is that if the a.e. regular convergence of $\{T_\mathbf{k} f(x)\}$ is established when f belongs to some special class which is dense in B, then the a.e. regular convergence of $\{T_\mathbf{k} f(x)\}$ for every $f\in B$ is completely equivalent to the fulfilment of inequality (16).

Proof of Theorem 2. We have to prove that if $f\in B$ is such that for every $\varepsilon>0$ there is a $g\in\mathcal{C}$ for which $\|f-g\|<\varepsilon$, then $f\in\mathcal{C}$ as well. To this end, we prove (15).

A simple estimation shows that

$$\mu\{x\colon \sup_{\text{under}(13)} |\Delta_{\mathbf{m},\mathbf{n}} T_{\mathbf{k}} f(x)| > \lambda \|f-g\|\} \leqq$$

$$\leqq \mu\left\{x\colon \sup_{\text{under}(13)} |\Delta_{\mathbf{m},\mathbf{n}} T_{\mathbf{k}} f(x) - \Delta_{\mathbf{m},\mathbf{n}} T_{\mathbf{k}} g(x)| > \frac{\lambda}{2}\|f-g\|\right\} +$$

$$+ \mu\left\{x\colon \sup_{\text{under}(13)} |\Delta_{\mathbf{m},\mathbf{n}} T_{\mathbf{k}} g(x)| > \frac{\lambda}{2}\|f-g\|\right\}.$$

As to the first term on the right, we illuminate the situation in the particular case $d=2$:

$$|\Delta_{\mathbf{m},\mathbf{n}} T_{\mathbf{k}} f(x) - \Delta_{\mathbf{m},\mathbf{n}} T_{\mathbf{k}} g(x)| \leqq |T_{n_1 n_2} f(x) - T_{n_1 n_2} g(x)| + |T_{m_1 n_2} f(x) - T_{m_1 n_2} g(x)| +$$
$$+ |T_{n_1 m_2} f(x) - T_{n_1 m_2} g(x)| + |T_{m_1 m_2} f(x) - T_{m_1 m_2} g(x)| \leqq$$
$$\leqq \varkappa[|T_{n_1 n_2}(f-g)(x)| + |T_{n_1 n_2}(g-f)(x)| + \ldots] \leqq 4\varkappa[T^*(f-g)(x) + T^*(g-f)(x)].$$

So, it can be easily seen that

$$\mu\{x\colon \sup_{\text{under}(13)} |\Delta_{\mathbf{m},\mathbf{n}} T_{\mathbf{k}} f(x)| > \lambda\|f-g\|\} \leqq \tag{17}$$

$$\leqq \mu\left\{x\colon T^*(f-g)(x) > \frac{\lambda}{2^{d+2}\varkappa}\|f-g\|\right\} + \mu\left\{x\colon T^*(g-f)(x) > \frac{\lambda}{2^{d+2}\varkappa}\|f-g\|\right\} +$$

$$+ \mu\left\{x\colon \sup_{\text{under}(13)} |\Delta_{\mathbf{m},\mathbf{n}} T_{\mathbf{k}} g(x)| > \frac{\lambda}{2}\|f-g\|\right\}.$$

Owing to Lemma 0, applied this time to $\{T_{\mathbf{k}}\colon \mathbf{k}\in\mathcal{N}^d\}$, we obtain

$$\mu\left\{x\colon T^*(f-g)(x) > \frac{\lambda}{2^{d+2}\varkappa}\|f-g\|\right\} \leqq C\left(\frac{\lambda}{2^{d+2}\varkappa}\right), \tag{18}$$

independently of $g\in\mathscr{C}$. By choosing $\lambda = 1/\varepsilon\varepsilon_1$ and taking $\|f-g\| \leqq \varepsilon^2\varepsilon_1$, where $\varepsilon_1 > 0$ will be chosen later on, we get from (17) and (18) that

$$\mu\{x\colon \sup_{\text{under}(13)} |\Delta_{\mathbf{m},\mathbf{n}} T_{\mathbf{k}} f(x)| > \varepsilon\} \leqq \tag{19}$$

$$\leqq 2C\left(\frac{1}{2^{d+2}\varkappa\varepsilon\varepsilon_1}\right) + \mu\left\{x\colon \sup_{\text{under}(13)} |\Delta_{\mathbf{m},\mathbf{n}} T_{\mathbf{k}} g(x)| > \frac{\varepsilon}{2}\right\}.$$

By (8), the first term on the right tends to zero as $\varepsilon_1 \to 0$. Given a $\delta > 0$, we can fix $\varepsilon_1 > 0$ so that this term does not exceed $\delta/2$. Then using the fact that $g\in\mathscr{C}$, the second term on the right-hand side of (19) can be made less than $\delta/2$ by choosing M sufficiently large, say $M \geqq M_0$. To sum up, we conclude that

$$\mu\{x\colon \sup_{\text{under}(13)} |\Delta_{\mathbf{m},\mathbf{n}} T_{\mathbf{k}} f(x)| > \varepsilon\} \leqq \delta \quad \text{for} \quad M \geqq M_0.$$

The proof of Theorem 2 is complete.

5. Application to a problem of summability of multiple orthogonal series

Let $\Phi=\{\varphi_{\mathbf{k}}(x); \mathbf{k}\in\mathcal{N}^d\}$ be an orthonormal system (in abbreviation: ONS) on X. We shall consider the d-multiple series

$$(20) \qquad \sum_{\mathbf{k}\in\mathcal{N}^d} c_{\mathbf{k}}\varphi_{\mathbf{k}}(x),$$

where $\{c_{\mathbf{k}}: \mathbf{k}\in\mathcal{N}^d\}$ is a d-multiple sequence of real numbers (coefficients) for which

$$(21) \qquad \sum_{\mathbf{k}\in\mathcal{N}^d} c_{\mathbf{k}}^2 < \infty.$$

By the Riesz—Fischer theorem the sum of series (20) exists in the sense of the mean convergence in $L^2(X)$-metric. In the following we shall be interested in the pointwise summability of series (20).

Let $\mathcal{A}=\{a_{\mathbf{m},\mathbf{k}}: \mathbf{m}, \mathbf{k}\in\mathcal{N}^d\}$ be a given "d-multiple matrix" of real numbers with the following two properties:

$$(22) \qquad a_{\mathbf{m},\mathbf{k}} \to a_{\mathbf{k}} \quad \text{as} \quad \min(m_1, \ldots, m_d) \to \infty \quad \text{for every } \mathbf{k}\in\mathcal{N}^d$$

and this convergence is regular in the sense of Section 4, and

$$(23) \qquad \sum_{\mathbf{k}\in\mathcal{N}^d} a_{\mathbf{m},\mathbf{k}}^2 < \infty \quad \text{for every } \mathbf{m}\in\mathcal{N}^d.$$

The so-called $\mathcal{A}$-means of series (20) are formed as follows

$$t_{\mathbf{m}}(x) = \sum_{\mathbf{k}\in\mathcal{N}^d} a_{\mathbf{m},\mathbf{k}} c_{\mathbf{k}}\varphi_{\mathbf{k}}(x) \quad (\mathbf{m}\in\mathcal{N}^d),$$

which results in a series-sequence transformation. By (21) and (23), the $\mathcal{A}$-means exist in the sense of $L^2(X)$-metric. Now, series (20) is said to be *$\mathcal{A}$-summable* (regularly or in Pringsheim's sense) if $\{t_{\mathbf{m}}(x): \mathbf{m}\in\mathcal{N}^d\}$ as a d-multiple sequence is (regularly or in Pringsheim's sense, respectively) convergent.

We need the *modified Lebesgue functions* $L_M^*(\mathcal{A}, \Phi; x)$ of the system Φ with respect to the summation method $\mathcal{A}$ defined in the following way. We set

$$K_{\mathbf{m}}(\mathcal{A}, \Phi; x, y) = \sum_{\mathbf{k}\in\mathcal{N}^d} a_{\mathbf{m},\mathbf{k}}\varphi_{\mathbf{k}}(x)\varphi_{\mathbf{k}}(y) \quad (\mathbf{m}\in\mathcal{N}^d).$$

Again by (23), the kernel $K_{\mathbf{m}}(\mathcal{A}, \Phi; x, y)$ as a function of y exists in the sense of $L^2(X)$-metric for almost every x. Consequently, the integral

$$L_M^*(\mathcal{A}, \Phi; x) = \int_X \Big(\max_{\max(m_1, \ldots, m_d)\le M} |K_{\mathbf{m}}(\mathcal{A}, \Phi; x, y)|\Big)\, d\mu(y) \quad (M = 1, 2, \ldots)$$

exists for almost every x and even belongs to $L^2(X)$.

Now we are ready to state

Theorem 3. *Suppose that $\Phi=\{\varphi_{\mathbf{k}}(x): \mathbf{k}\in\mathcal{N}^d\}$ is an ONS on X, $\{c_{\mathbf{k}}: \mathbf{k}\in\mathcal{N}^d\}$ is a sequence of coefficients satisfying condition* (21), *and* $\mathcal{A}=\{a_{\mathbf{m},\mathbf{k}}: \mathbf{m}, \mathbf{k}\in\mathcal{N}^d\}$

is a matrix of real numbers satisfying conditions (22) *and* (23). *If*

$$L := \int_X \{\sup_{M=1,2,\ldots} L_M^*(\mathscr{A}, \Phi; x)\}^2 \, d\mu(x) < \infty, \tag{24}$$

then series (20) *is regularly $\mathscr{A}$-summable a.e. on X.*

This theorem in the special case $d=1$ is due to TANDORI [6].

First we prove the following

Lemma 2. *Under the conditions of Theorem* 3, *except* (22), *we have*

$$\int_X \big(\sup_{\mathbf{m}\in\mathscr{N}^d} |t_{\mathbf{m}}(x)|\big)\, d\mu(x) \leqq \{2L^{1/2} + \big(\sup_{\mathbf{k}\in\mathscr{N}^d} a_{\mathbf{1},\mathbf{k}}^2\big)^{1/2}\}\{\sum_{\mathbf{k}\in\mathscr{N}^d} c_{\mathbf{k}}^2\}^{1/2}. \tag{25}$$

Proof of Lemma 2. It will be done by a modification of the well-known classical method (see, e.g. [4] and also [6]).

For every positive integer M and $x \in X$ define $\mathbf{M}(x) = (M_1(x), \ldots, M_d(x)) \in \mathscr{N}^d$ in a unique way such that $1 \leqq M_j(x) \leqq M$ for each $j = 1, \ldots, d$ and

$$t_{\mathbf{M}(x)}(x) = \max_{\max(m_1,\ldots,m_d)\leqq M} t_{\mathbf{m}}(x) \quad (M = 1, 2, \ldots).$$

Using the representation

$$t_{\mathbf{M}(x)}(x) = \int_X \big(\sum_{\mathbf{k}\in\mathscr{N}^d} c_{\mathbf{k}}\varphi_{\mathbf{k}}(y)\big)\big(\sum_{\mathbf{n}\in\mathscr{N}^d} a_{\mathbf{M}(x),\mathbf{n}}\varphi_{\mathbf{n}}(x)\varphi_{\mathbf{n}}(y)\big)\, d\mu(y),$$

Fubini's theorem and the Schwarz inequality imply that

$$\int_X t_{\mathbf{M}(x)}(x)\, d\mu(x) = \int_X \{\big(\sum_{\mathbf{k}\in\mathscr{N}^d} c_{\mathbf{k}}\varphi_{\mathbf{k}}(y)\big) \int_X \sum_{\mathbf{n}\in\mathscr{N}^d} a_{\mathbf{M}(x),\mathbf{n}}\varphi_{\mathbf{n}}(x)\varphi_{\mathbf{n}}(y)\, d\mu(x)\}\, d\mu(y) \leqq$$

$$\leqq \int_X \{\big|\sum_{\mathbf{k}\in\mathscr{N}^d} c_{\mathbf{k}}\varphi_{\mathbf{k}}(y)\big| \int_X \big(\max_{\max(m_1,\ldots,m_d)\leqq M} \big|\sum_{\mathbf{n}\in\mathscr{N}^d} a_{\mathbf{m},\mathbf{n}}\varphi_{\mathbf{n}}(x)\varphi_{\mathbf{n}}(y)\big|\big)\, d\mu(x)\}\, d\mu(y) =$$

$$= \int_X \big|\sum_{\mathbf{k}\in\mathscr{N}^d} c_{\mathbf{k}}\varphi_{\mathbf{k}}(y)\big| L_M^*(\mathscr{A}, \Phi; y)\, d\mu(y) \leqq \{L \sum_{\mathbf{k}\in\mathscr{N}^d} c_{\mathbf{k}}^2\}^{1/2},$$

the last inequality is by (24). Applying Beppo Levi's theorem, hence it follows that

$$\int_X \{\sup_{\mathbf{m}\in\mathscr{N}^d} t_{\mathbf{m}}(x)\}\, d\mu(x) \leqq \{L \sum_{\mathbf{k}\in\mathscr{N}^d} c_{\mathbf{k}}^2\}^{1/2}.$$

Repeating this argument for $-t_{\mathbf{m}}(x)$, which corresponds to the system $\{-\varphi_{\mathbf{k}}(x) : \mathbf{k} \in \mathscr{N}^d\}$, we obtain

$$\int_X \{\sup_{\mathbf{m}\in\mathscr{N}^d} (-t_{\mathbf{m}}(x))\}\, d\mu(x) \leqq \{L \sum_{\mathbf{k}\in\mathscr{N}^d} c_{\mathbf{k}}^2\}^{1/2}.$$

Now, the wanted inequality (25) follows from the elementary relation

$$\sup_{\mathbf{m}\in\mathscr{N}^d} |t_{\mathbf{m}}(x)| \leqq \sup_{\mathbf{m}\in\mathscr{N}^d} t_{\mathbf{m}}(x) + \sup_{\mathbf{k}\in\mathscr{N}^d} (-t_{\mathbf{m}}(x)) + |t_{\mathbf{1}}(x)|.$$

Proof of Theorem 3. We recall that the set $l^2(\mathcal{N}^d)$ of those d-multiple sequences $\mathfrak{c}=\{c_{\mathbf{k}}: \mathbf{k}\in\mathcal{N}^d\}$ for which condition (21) is satisfied, endowed with the usual vector operations and Euclidean norm, is a Banach space. The operators

$$\mathfrak{c} \to T_{\mathbf{m}}\mathfrak{c}(x) = t_{\mathbf{m}}(x)\colon l^2(\mathcal{N}^d) \to L^2(X) \quad (\mathbf{m}\in\mathcal{N}^d)$$

are clearly linear and continuous in $L^2(X)$-metric, a fortiori in measure. The continuity in $L^2(X)$-metric is shown by the estimate

$$\int_X t_{\mathbf{m}}^2(x)\,d\mu(x) = \sum_{\mathbf{k}\in\mathcal{N}^d} a_{\mathbf{m},\mathbf{k}}^2 c_{\mathbf{k}}^2 \leqq \big(\max_{\mathbf{k}\in\mathcal{N}^d} a_{\mathbf{m},\mathbf{k}}^2\big) \sum_{\mathbf{k}\in\mathcal{N}^d} c_{\mathbf{k}}^2.$$

Due to Lemma 2, for every $\mathfrak{c}\in l^2(\mathcal{N}^d)$,

$$T^*\mathfrak{c}(x) = \sup_{\mathbf{m}\in\mathcal{N}^d} |t_{\mathbf{m}}(x)| < \infty \quad \text{a.e.} \quad \text{on} \quad X. \tag{26}$$

For every $\mathfrak{c}\in l^2(\mathcal{N}^d)$ and $M=1,2,\ldots$ define $\mathfrak{c}^{(M)}=\{c_{\mathbf{k}}^{(M)}: \mathbf{k}\in\mathcal{N}^d\}$ as follows

$$c_{\mathbf{k}}^{(M)} = \begin{cases} c_{\mathbf{k}} & \text{if} \quad \max(k_1, \ldots, k_d) \leqq M, \\ 0 & \text{otherwise.} \end{cases}$$

It is also clear that these "finite sequences" $\mathfrak{c}^{(M)}$ constitute a dense subset in $l^2(\mathcal{N}^d)$. Furthermore, (22) yields

$$\begin{gathered} T_{\mathbf{m}}\mathfrak{c}^{(M)}(x) = \sum_{k_1=1}^{M} \cdots \sum_{k_d=1}^{M} a_{\mathbf{m},\mathbf{k}} c_{\mathbf{k}} \varphi_{\mathbf{k}}(x) \to \sum_{k_1=1}^{M} \cdots \sum_{k_d=1}^{M} a_{\mathbf{k}} c_{\mathbf{k}} \varphi_{\mathbf{k}}(x) \\ \text{as} \quad \min(m_1, \ldots, m_d) \to \infty \quad \text{for every} \quad M = 1, 2, \ldots \end{gathered} \tag{27}$$

and even this convergence is regular in the sense of Section 4.

On the basis of (26) and (27), Theorem 2 is applicable and results that the d-multiple sequence $T_{\mathbf{m}}\mathfrak{c}(x)=t_{\mathbf{m}}(x)$ is regularly convergent a.e. on X for every $\mathfrak{c}\in l^2(\mathcal{N}^d)$. This finishes the proof of Theorem 3.

On closing, we formulate a slight generalization of Theorem 3. To this effect, let $\Lambda=\{\lambda_{\mathbf{k}}: \mathbf{k}\in\mathcal{N}^d\}$ be a d-multiple sequence of positive numbers, which is nondecreasing in the sense that $\lambda_{\mathbf{k}}\leqq\lambda_{\mathbf{m}}$ whenever $\mathbf{k}\leqq\mathbf{m}$. Denote by $\Phi/\sqrt{\Lambda}$ the system $\{\varphi_{\mathbf{k}}(x)/\sqrt{\lambda_{\mathbf{k}}}: \mathbf{k}\in\mathcal{N}^d\}$. Then

$$L_M^*\left(\mathcal{A}, \frac{\Phi}{\sqrt{\Lambda}};\ x\right) = \int_X \left(\max_{\max(m_1,\ldots,m_d)\leqq M} \left|K_{\mathbf{m}}\left(\mathcal{A}, \frac{\Phi}{\sqrt{\Lambda}};\ x, y\right)\right|\right) d\mu(y) \quad (M = 1, 2, \ldots),$$

where

$$K_{\mathbf{m}}\left(\mathcal{A}, \frac{\Phi}{\sqrt{\Lambda}};\ x, y\right) = \sum_{\mathbf{k}\in\mathcal{N}^d} a_{\mathbf{m},\mathbf{k}} \frac{\varphi_{\mathbf{k}}(x)\varphi_{\mathbf{k}}(y)}{\lambda_{\mathbf{k}}} \qquad (\mathbf{m}\in N^d).$$

The following theorem can be proved analogously to as Theorem 3 is proved.