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Non-horizontal geodesics of a Riemannian submersion

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Dedicated to Professor B. Szőkefalvi-Nagy on his 70th birthday

1. Introduction. For Riemannian manifolds M and P, a submersion $\pi: P \to M$ is a smooth mapping of P onto M which has maximal rank and preserves the length of horizontal vectors. A tangent vector to P at x is called *horizontal* if it is orthogonal to the fiber $\pi^{-1} \circ \pi(x)$ through x, *vertical* if it is tangent to the fiber. The fundamental concepts of a Riemannian submersion were introduced by B. O'Neill [2]. The horizontal geodesics of P were studied in [3].

Our aim here is to investigate the non-horizontal geodesics of P and to characterize them with their "projections" on the basic manifold M and on the fibers through their points. As an application we shall get a stability property of some fibers with respect to the geodesic flow of a class of Riemannian submersions.

We use the method of moving frame; for the notation and the basic relations of the invariants of a submersion we refer to [1].

The paper is organised as follows. Section 2 is devoted to the basic concepts of a Riemannian submersion. In Section 3 we discuss the translation of fibers along a curve of M defined by the horizontal subspaces and the relation of this translation to the Riemannian parallel translation. In Section 4 we treat the equation of geodesics as we need. In Section 5 we apply our result in a special class of Riemannian submersions where the translation of fibers is homothetic transformation. Finally in Section 6 we investigate the stability of fibers with respect to the geodesic flow in the above discussed class of submersions.

Throughout this paper the indices $i, j, k, \ldots, a, b, c, \ldots$ and $\alpha, \beta, \gamma, \ldots$ will run from 1 to $n+k$, form 1 to n and from $n+1$ to $n+k$, respectively, where $n = \dim M$, $n+k = \dim P$. The summation convention will be adopted.

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2. Adapted frames. Let $\{L(P), p, P\}$ and $\{L(M), q, M\}$ denote the principal fiber bundle of linear frames of P and M, respectively. The bundle of adapted frames $\{L_M(P), p, P\}$ over P of the submersion $\pi: P\to M$ is defined as a subbundle of $\{L(P), p, P\}$ consisting of frames $\{x; e_1, \ldots, e_{n+k}\}\in L_x(P)$ such that the vectors $e_1, \ldots, e_n$ are horizontal and the vectors $e_{n+1}, \ldots, e_{n+k}$ are vertical. The structure group of the adapted frame bundle is isomorphic to the group $GL(n)\times\times GL(k)\subset GL(n+k)$.

Let ω and φ denote the $\mathbf{R}^{n+k}$-valued canonical form and the $\mathfrak{gl}(n+k)$-valued Riemannian commection form on $L(P)$. ω and φ satisfy the structure equation

$$d\omega = -\varphi\wedge\omega \quad \text{or} \quad d\omega^i = -\varphi^i_k\wedge\omega^k$$

where ω^i and φ^i_k are the components of the forms ω and φ with respect to the canonical bases of $\mathbf{R}^{n+k}$ and $\mathfrak{gl}(n+k)$.

The fundamental tensors of the submersion are of the form

$$A = A_{\beta}{}^{a}{}_{c}e_a\otimes\omega^{\beta}\otimes\omega^c, \quad T = T_{\beta}{}^{a}{}_{\gamma}e_a\otimes\omega^{\beta}\otimes\omega^{\gamma}$$

where $\{e_1, \ldots, e_{n+k}\}$ is an adapted frame and $\{\omega^1, \ldots, \omega^{n+k}\}$ is its dual coframe. The Riemannian metric tensor of P can be written as

$$g = g_{ab}\omega^a\otimes\omega^b + g_{\alpha\beta}\omega^{\alpha}\otimes\omega^{\beta}$$

on the adapted frame bundle. The metric tensor of the basic manifold is $\tilde{g}=g_{ab}\tilde{\omega}^a\otimes\tilde{\omega}^b$, where $\omega^a=\pi^*\tilde{\omega}^a$. The Riemannian connection form ψ of M defines a form on $L_M(P)$ in a natural way whose components are denoted also by ψ^a_b.

Proposition 1. *The fundamental tensors A, T of the submersion and the Riemannian connection forms of P and M are related by the equations*

(1) $$\varphi^a_b = \psi^a_b + (1/2)A_{\gamma}{}^{a}{}_{b}\omega^{\gamma},$$

(2) $$\varphi^a_{\beta} = (1/2)A_{\beta}{}^{a}{}_{c}\omega^c + (1/2)T_{\beta}{}^{a}{}_{\gamma}\omega^{\gamma},$$

(3) $$\varphi^{\alpha}_b = -(1/2)A^{\alpha}_{\cdot bc}\omega^c - (1/2)T^{\alpha}_{\cdot b\gamma}\omega^{\gamma},$$

where

$$g_{\alpha\beta}A^{\alpha}_{\cdot bc} = g_{ab}A_{\beta}{}^{a}{}_{c}, \quad g_{\alpha\beta}T^{\alpha}_{\cdot b\gamma} = g_{ab}T_{\beta}{}^{a}{}_{\gamma},$$

and the tensors A and T satisfy

(4) $$A_{\gamma ab} + A_{\gamma ba} = 0, \quad T_{\beta}{}^{a}{}_{\gamma} = T_{\gamma}{}^{a}{}_{b},$$

Proof. The more detailed description of the adapted frame bundle and the proof of equations (1), (2), (4) can be found in [1], pp. 155—158 (orthonormed frames are used).

To prove the equation (3) we note that the Riemannian metric tensor satisfies

$$dg_{ij} - \varphi^k_i g_{kj} - \varphi^k_j g_{ik} = 0.$$

Since $g_{a\beta}=0$, we have

$$0 = dg_{a\beta} = g_{ac}\varphi^c_\beta + g_{\beta\gamma}\varphi^\gamma_a.$$

Using (2) we get

$$g_{\beta\gamma}\varphi^\gamma_a = -g_{ac}\varphi^c_\beta = -g_{ac}(1/2)(A_\beta{}^c{}_d\omega^d + T_\beta{}^c{}_\delta\omega^\delta) = -g_{\beta\gamma}(1/2)(A^\gamma_{\cdot ad}\omega^d + T^\gamma_{\cdot a\delta}\omega^\delta).$$

This completes the proof.

3. Translation of fibers. Let be given a curve $y(t)$ of the basic manifold M defined on an open interval $t\in I\subset \mathbf{R}$. If $t_0\in I$, $x_0\in\pi^{-1}(y(t_0))$, there is a unique horizontal curve $x(x_0, t)\,(t\in I)$ of P satisfying $x(x_0, t_0)=x_0$ and $\pi\circ x(x_0, t)=y(t)$. These curves define a 1-parameter family of maps $\tau_{t_0,t}\colon \pi^{-1}(y(t_0))\to\pi^{-1}(y(t))$ along the curve $y(t)$, $t\in I$, such that

$$\tau_{t_0,t}x_0 := x(x_0, t).$$

This map is called translation of fibers along the curve $y(t)$ of M. The derivative map $\tau_{t_0,t*}$ induces a translation of vertical vectors along horizontal curves. A vertical vectorfield Z on P is constant with respect to this translation if and only if the Lie derivative $\mathscr{L}_Y Z=[Y, Z]=0$, where Y is a horizontal vectorfield defined in a neighbourhood of $\pi^{-1}(y(t))\,(t\in I)$ satisfying $Y(x(t))=\dot{x}(t)$ for the horizontal lifts $x(t)$ of $y(t)$. Here the dot denotes the derivation by t.

Proposition 2. *If $Y=Y^a e_a$ and $Z=Z^\alpha e_\alpha$ are horizontal and vertical vectorfields on P then the expression $\nabla_Y Z-\mathscr{L}_Y Z$ is a $(1,1)$-type tensorfield satisfying*

$$\nabla_Y Z-\mathscr{L}_Y Z = (1/2)A_\gamma{}^a{}_c Y^c Z^\gamma e_a - (1/2)T^\alpha_{\cdot c\gamma}Y^c Z^\gamma e_\alpha.$$

Proof. We fix a point $x_0\in P$. Let $U\subset M$ be a neighbourhood of $y_0=\pi(x_0)\in M$, and $t\in I\mapsto y(t)\in M$ is a curve in U such that $y(t_0)=y_0$ $(t_0\in I)$ and $\dot{y}(t_0)=\pi_* Y(x_0)$. Let $\bar{Y}$ be a horizontal vectorfield defined on a neighbourhood of x_0 such that $\bar{Y}(x(t))=\dot{x}(t)$ for horizontial lifts $x(t)$ of $y(t)$. Let be given a frame field $\{\tilde{e}_1(y), \dots, \tilde{e}_n(y)\}$ on U and an adapted frame field $\{e_1(x), \dots, e_{n+k}(x)\}$ on a neighbourhood of x_0 such that $\pi_* e_a(x)=\tilde{e}_a(\pi(x))$. For a smooth function f on P we denote the components of its differential with respect to the adapted coframe $\{\omega^i\}$ dual to $\{e_i\}$ by $\partial_i f$, i.e., $df=(\partial_i f)\omega^i$. We can write for $\bar{Y}=\bar{Y}^a e_a$

$$\nabla_Y Z-\mathscr{L}_Y Z = \nabla_Z \bar{Y} = ((\partial_\gamma \bar{Y}^a)Z^\gamma + \varphi^a_c(Z)\bar{Y}^c)e_a + \varphi^\alpha_c(Z)\bar{Y}^c e_\alpha.$$

Since the components $\bar{Y}^a$ are constant on the fiber we have $\partial_\gamma \bar{Y}^a=0$. By Proposition 1 we get

$$\nabla_Y Z-\mathscr{L}_Y Z = \nabla_Z\bar{Y} = \varphi^a_c(Z)\bar{Y}^c e_a + \varphi^\alpha_c(Z)\bar{Y}^c e_\alpha = (1/2)A_\gamma{}^a{}_c Z^\gamma \bar{Y}^c e_a - (1/2)T^\alpha_{\cdot c\gamma}Z^\gamma\bar{Y}^c e_\alpha,$$

since the forms ψ^a_c are lifted from a form on $L(M)$ and therefore $\psi^a_c(Z)=0$. At the point x_0; $\bar{Y}(x_0)=Y(x_0)$, and the proof is complete.

4. Equation of geodesics. Let $x(s)$ $(s\in I)$ be an arc-length parametrized curve in P, $y(s)=\pi\circ x(s)$ is its projection curve in the basic manifold M and $z(s)=\tau_{s,s_0}x(s)$ is its development in the fiber $\pi^{-1}(y(s_0))$, $s_0\in I$. Comma denotes the derivation by s. The tangent vector $x'(s)$ can be written as $x'(s)=Y(s)+Z(s)$, where $Y(s)$ is its horizontal part and $Z(s)$ is its vertical part. The curve $x(s)$ is a geodesic if and only if the equations

$$Y^{a\prime}+\varphi_c^a(x')Y^c+\varphi_\gamma^a(x')Z^\gamma=0,\quad Z^{\alpha\prime}+\varphi_c^\alpha(x')Y^c+\varphi_\gamma^\alpha(x')Z^\gamma=0$$

are satisfied, where $Y=Y^a e_a$, $Z=Z^\alpha e_\alpha$ for an adapted frame field $\{e_i(s)\}$ along $x(s)$. By Proposition 1 these equations can be written in the form

$$Y^{a\prime}+\psi_c^a(Y)Y^c+A_\beta{}^a{}_cZ^\beta Y^c+(1/2)T_\beta{}^a{}_\gamma Z^\beta Z^\gamma=0, \tag{5}$$

$$Z^{\alpha\prime}+\varphi_\gamma^\alpha(x')Z^\gamma-(1/2)A^\alpha_{\cdot bc}Y^bY^c-(1/2)T^\alpha_{\cdot c\beta}Y^cZ^\beta=0. \tag{6}$$

We know $A^\alpha_{\cdot bc}Y^bY^c=0$ by the skew symmetry of A. The tangent vector y' of the projection curve $y(s)$ has the components Y^a with respect to the frame $\tilde e_a=\pi_* e_a$ hence the equations (5) are equivalent to

$$\tilde\nabla_s y'=-(A_\beta{}^a{}_cZ^\beta Y^c+(1/2)T_\beta{}^a{}_\gamma Z^\beta Z^\gamma)\tilde e_a. \tag{7}$$

From the equations (6) and (7) it follows immediately that the horizontal lifts of geodesics of M are geodesics of P (in this case $Z^\alpha\equiv 0$).

It is also clear that if a geodesic $x(s)$ of P is horizontal at $s_0\in I$ then it is horizontal for all $s\in I$.

We investigate here the non-horizontal geodesics of P. If $x(s)$, $s\in I$, is a non-horizontal curve then $(\tau_{s,s_0}x(s))'\neq 0$ for all $s\in I$. Let us denote by $\bar x_\sigma(s)$ the horizontal lift of $y(s)=\pi\circ x(s)$ through $x(\sigma)$ that is $\bar x_\sigma(\sigma)=x(\sigma)$ $(\sigma\in I)$. Let $Y(x)$ be a horizontal vectorfield defined in a neighbourhood of the curve $x(s)$ such that $Y(\bar x_\sigma(s))=\bar x_\sigma'(s)$ for all $\sigma, s\in I$. Let $Z(x)$ be a vertical vectorfield defined in a neighbourhood of the curve $x(s)$ such that $Z(\bar x_\sigma(s_1))=\tau_{s_2,s_1*}Z(\bar x_\sigma(s_2))$ is satisfied for all $s_1, s_2\in I$, where τ_{s_1,s_2} is the translation $\pi^{-1}(y(s_1))\to\pi^{-1}(y(s_2))$ along $y(s)$. From the definition of the vectorfields Y and Z we get $\mathcal{L}_YZ=0$. It follows from (6) that

$$0=(\partial_bZ^\alpha)Y^b+\varphi_\gamma^\alpha(Y)Z^\gamma+(\partial_\beta Z^\alpha)Z^\beta+\varphi_\gamma^\alpha(Z)Z^\gamma-(1/2)T^\alpha_{\cdot c\beta}Z^\beta Y^c.$$

Since

$$((\partial_bZ^\alpha)Y^b+\varphi_\gamma^\alpha(Y)Z^\gamma)e_\alpha=\nabla_YZ-\varphi_\gamma^a(Y)Z^\gamma e_a,$$

and by Proposition 2,

$$\nabla_YZ=\mathcal{L}_YZ+(1/2)A_\gamma{}^a{}_cY^cZ^\gamma e_a-(1/2)T^\alpha_{\cdot c\gamma}Y^cZ^\gamma e_\alpha,$$

which together with $\mathcal{L}_YZ=0$ imply

$$((\partial_\beta Z^\alpha)Z^\beta+\varphi_\gamma^\alpha(Z)Z^\gamma)e_\alpha-\varphi_\gamma^a(Y)Z^\gamma e_a+(1/2)A_\gamma{}^a{}_cY^cZ^\gamma e_a-T^\alpha_{\cdot c\gamma}Y^cZ^\gamma e_\alpha=0.$$

By Proposition 1 $\varphi^\alpha_\gamma(Y)Z^\gamma=(1/2)A_\gamma{}^a{}_c Y^c Z^\gamma$, we find

$$(\partial_\beta Z^\alpha)Z^\beta+\varphi^\alpha_\gamma(Z)Z^\gamma = T^\alpha_{\cdot c\gamma}Y^c Z^\gamma. \tag{8}$$

Now, we fix a parameter $s_0\in I$. If we map the fibers $\pi^{-1}(y(s))$ onto $\pi^{-1}(y(s_0))$ using the translation τ_{s,s_0} along $y(s)$, we get the curve $z(s):=\tau_{s,s_0}x(s)$ on $\pi^{-1}(y(s_0))$. The vertical vectors $\{e_{n+1}(s),\dots,e_{n+k}(s)\}$ of the adapted frame field $\{e_i(s)\}$ are mapped into the vertical frame field $\overset{*}{e}_\alpha(s):=\tau_{s,s_0*}e_\alpha(s)$ along $z(s)$; the vertical vectors $Z(x(s))$ and $Z(z(s))$ have the same components with respect to the corresponding frames. Thus the left hand side of the equation (8) is $(\overset{*}{\nabla}_s z')(s_0)$; where $\overset{*}{\nabla}_s$ is the induced covariant derivation along $z(s)$ on the submanifold $\pi^{-1}(y(s_0))$ defined by the induced Riemannian connection form $\varphi^\alpha_\gamma(Z)$.

We can summarize the obtained results.

Theorem 1. *Let $x(s)$, $s\in I$, be an arc-length parametrized curve in P. It is a geodesic of P if and only if*

(i) *the projection curve $y(s)=\pi\circ x(s)$ satisfies*

$$\tilde{\nabla}_s y' = -\pi_*[A(x')x'+(1/2)T(x',x')] = -[A_\beta{}^a{}_c Z^\beta Y^c+(1/2)T_\beta{}^a{}_\gamma Z^\beta Z^\gamma]\tilde{e}_a$$

where $\tilde{\nabla}_s$ is the covariant derivative in the basic manifold M;

(ii) *for all $s_0\in I$ the development $z(s)=\tau_{s,s_0}x(s)$ of $x(s)$ in the fiber $\pi^{-1}(y(s_0))$ satisfies*

$$\overset{*}{\nabla}_s z' = T^\alpha_{\cdot c\beta}Y^c Z^\beta e_\alpha \quad at \quad s=s_0,$$

where $\overset{*}{\nabla}_x$ is the induced covariant derivative in the fiber $\pi^{-1}(y(s_0))$.

Proof. We have proved already that the conditions (i) and (ii) are necessary for a geodesic $x(s)$ of P. The sufficiency follows from the fact that the conditions (i) and (ii) give a second order differential equation for $x(s)$, it has for all initial points and tangent vectors a unique solution which has to be the same curve as the geodesic with this initial point and tangent vector.

5. Homothetic fibers. Here we give a more detailed discussion of the case which can be obtained from a Riemannian submersion whose fibers are totally geodesic submanifolds by a bundle-like homothetic deformation with a positive smooth function defined on the basic manifold (cf. [1]).

If $g_{ab}\omega^a\otimes\omega^b+g_{\alpha\beta}\omega^\alpha\otimes\omega^\beta$ is the metric tensor of the submersion $\{P,\pi,M\}$ with totally geodesic fibers (i.e., $T_\beta{}^a{}_\gamma\equiv 0$ and the translation of the fibers is isometry), the submersion with the metric tensor

$$g_{ab}\,\omega^a\otimes\omega^b+\exp(-\varrho)\,g_{\alpha\beta}\,\omega^\alpha\otimes\omega^\beta,$$

where $\varrho\colon M\to\mathbf{R}$ is a smooth function, is a Riemannian submersion such that the translation of the fibers is homothetic map. In this case the second fundamental tensor of the fibers is of the form

$$T_{\beta\alpha\gamma} = \varrho_\alpha \exp(-\varrho)\, g_{\beta\gamma}, \quad \text{where} \quad d\varrho = \varrho_a \omega^a. \tag{9}$$

(cf. Corollary of Theorem 2 in [1], p. 161.)

Theorem 2. *Let $\{P, \pi, M\}$ be a Riemannian submersion satisfying* (9). *The curve $x(s)$ $(s\in I)$ is an arc-length parametrized geodesic if and only if*

(i) *the projection curve $y(s)=\pi\circ x(s)$ satisfies*

$$\tilde{\nabla}_s y' = -\pi_*[A(x')x'] - (c/2)\,\mathrm{grad}\,(\exp\varrho(y(s)))$$

for a positive constant c;

(ii) *the development $z(s)=\tau_{s,s_0}x(s)$ in the fiber $\pi^{-1}(y(s_0))$ is a geodesic parametrized with the speed $\|z'\|=\sqrt{c}\cdot\exp\varrho(y(s))$, where the constant satisfies in an initial point $s_0\in I$,*

$$\|y'(s_0)\|^2 + c\cdot\exp 2\varrho(y(s_0)) = 1.$$

Proof. The conditions (i) and (ii) of Theorem 1 give the equations

$$\tilde{\nabla}_s y' = -\pi_* A(x')x' - (1/2)(\mathrm{grad}\,\varrho)\langle z', z'\rangle, \tag{10a}$$

$$\overset{*}{\nabla}_s z' = \varrho' z'. \tag{10b}$$

The equation (10a) can be obtained immediately by substitution of (9). For the proof of (10b) we note that in our case the translation of fibers τ_{s,s_0} is homothetic and consequently affine map, thus for all $s_1, s_2\in I$ the developments $z_1(s)$ and $z_2(s)$ in the fibers $\pi^{-1}(y(s_1))$ and $\pi^{-1}(y(s_2))$, respectively, satisfy $\tau_{s_1,s_2*}(\overset{*}{\nabla}_s z_1') = \overset{*}{\nabla}_s z_2'$. It follows that the condition (ii) of Theorem 1 can be considered in a fixed fiber $\pi^{-1}(y(s_0))$ $(s_0\in I)$ for all $s\in I$. The right hand side of the equation (10b) can be obtained by (9)

$$\overset{*}{\nabla}_s z' = T^{\alpha}_{\cdot c\beta} Y^c Z^\beta \overset{*}{e}_\alpha = z'(\varrho_c Y^c) = z'\varrho'.$$

The equation (10b) means that the curve $z(s)$ is a geodesic in $\pi^{-1}(y(s_0))$, its speed can be computed as follows:

$$\overset{*}{\nabla}_s(\exp(-\varrho)z') = -\varrho'\exp(-\varrho)z' + \exp(-\varrho)\overset{*}{\nabla}_s z' = 0,$$

that is

$$\langle z', z'\rangle = \exp(-\varrho)\cdot c\exp 2\varrho = c\cdot\exp\varrho, \quad c = \text{constant}.$$

We substitute this in equation (10a):

$$\tilde{\nabla}_s y' = -\pi_* A(x')x' - (1/2)(\mathrm{grad}\,\varrho)c\cdot\exp\varrho = -\pi_* A(x')x' - (c/2)\,\mathrm{grad}\,(\exp\varrho),$$

and the necessity of the conditions of Theorem 2 is proved. But they are sufficient since for all initial points and tangent vectors these two second order equations have the same unique solution.

Corollary 1. *A vertical curve in a fiber* $\pi^{-1}(y_0)$ *is a geodesic of* P *if and only if it is a geodesic in the submanifold* $\pi^{-1}(y_0)$ *and* $(\operatorname{grad} \varrho)(y_0)=0$. *In this case the fiber* $\pi^{-1}(y_0)$ *is a totally geodesic submanifold.*

Corollary 2. *The function* $\Phi(y, y'): TM \to \mathbf{R}$ *defined by* $\Phi(y, y')= =(1/c)\langle y', y'\rangle + \exp \varrho(y)$ *is constant along a projection curve of a geodesic.*

Proof. By Theorem 2 we have for a geodesic $x(s)$,

$$1 = \langle x', x'\rangle = \langle y', y'\rangle + c \cdot \exp \varrho = c \cdot \Phi(y, y').$$

Remark. The constant c along a geodesic $x(s)$ can be expressed by the angle θ between the geodesic $x(s)$ and the fiber:

$$c = \exp(-\varrho)(1-\langle y', y'\rangle) = \exp(-\varrho)\cos^2\theta.$$

Thus the statement that $\sqrt{c}=\exp(-(1/2)\varrho)\cos\theta$ is constant is a generalization of Clairaut's Theorem on surface of revolution.

Corollary 3. *If* $\{P, \pi, M\}$ *is a Riemannian submersion with totally geodesic fibers, the curve* $x(s)$ *is a geodesic of* P *if and only if the following conditions are satisfied:*

(i) *let* σ *denote the arc-length parameter of the projection curve* $y=\pi\circ x$, *the first vector of curvature is*

$$\tilde{\nabla}_\sigma \frac{dy}{d\sigma} = -\pi_* A\left(\frac{dx}{d\sigma}\right)\frac{dx}{d\sigma},$$

(ii) *the development* $z(s)=\tau_{s,s_0}x(s)$ *of* $x(s)$ *in the fiber* $\pi^{-1}(y(s_0))$ *is a constant speed geodesic.*

Proof. Theorem 2 implies in the case ϱ=constant $\|y'\|^2=1-\|z'\|^2$=constant, therefore σ is proportial to s. It follows

$$\tilde{\nabla}_\sigma \frac{dy}{d\sigma} = \|y'\|^{-2}\nabla_s y' = -\|y'\|^{-2}\pi_* A(x')x' = -\pi_* A\left(\frac{dx}{d\sigma}\right)\frac{dx}{d\sigma}.$$

Example. Let us consider the Hopf bundle $\pi: S^{2m+1} \to \mathbf{CP}(m)$ of the unit sphere over the complex projective space equipped with the Fubini—Study metric. It is a Riemannian submersion with totally geodesic fibers (cf. [2], p. 466).

Its tensor A can be expressed in the form $A(Z)Y=\langle Z, N\rangle JY$, where J is the almost complex tensor on $\mathbf{C}^{m+1}$, N is the tangent unit vectorfield of the fibers defined by JM for the unit normal vectorfield M of S^{2m+1}, Z and Y are arbitrary vertical and horizontal tangent vectorfields of S^{2m+1}.

We get that the curve $x(s)$ is a geodesic of S^{2m+1} if and only if
(i) the first vector of curvature of $y(s)$ is expressed by

$$\tilde{\nabla}_{\sigma} \frac{dy}{d\sigma} = -\zeta J \frac{dy}{d\sigma},$$

where σ is the arc-length parameter of $y(s)$ and the vertical part Z of $\frac{dx}{d\sigma}$ is $Z=\zeta N$. We have

$$\tilde{\nabla}_{\sigma}\left(J\frac{dy}{d\sigma}\right) = J\tilde{\nabla}_{\sigma} \frac{dy}{d\sigma} = \zeta \frac{dy}{d\sigma},$$

that is the curve $y(s)$ is a real 2-plane curve in $\mathbf{CP}(m)$ of curvature $-\zeta$ contained in the complex projective line (2-sphere) spanned by y' and Jy'.

(ii) ζ=constant, that is the fiber curve $z(\sigma)=\tau_{\sigma,\sigma_0}x(\sigma)$ is of the speed $\frac{dz}{d\sigma}=z' : \frac{d\sigma}{ds}=\zeta : \|y'\|=\frac{\zeta}{\sqrt{1-\zeta^2}}$, with respect to the arc-length parameter σ of the basic curve $y(\sigma)$, which is a circle of curvature $-\zeta$ in a complex projective line in $\mathbf{CP}(m)$ $(-1<\zeta<1)$.

6. Stable fibers of the geodesic flow. As we observed in Corollary 1 of Theorem 2 if $y_0 \in M$ is a critical point of the function $\varrho: M \to \mathbf{R}$, $\pi^{-1}(y_0)$ is a total geodesic submanifold of P, or equivalently the tangent bundle $T(\pi^{-1}(y_0))$ is an invariant submanifold of TP with respect to the geodesic flow on TP.

The fiber $\pi^{-1}(y_0)$ is called stable (with respect to the geodesic flow) if for any $\varepsilon>0$ it is possible to find a $\delta=\delta(\varepsilon)>0$ such that if an arc-length parametrized geodesic $x(s)$ satisfies the initial conditions

$$d(\pi \circ x(s_0),\ y_0) < \delta, \quad \|\pi_* x'(s_0)\| < \delta$$

then the inequalities $d(\pi \circ x(s), y_0)<\varepsilon$, $\|\pi_* x'(s)\|<\varepsilon$ hold for any $s \in \mathbf{R}$, where d is the distance on M. (For simplicity we suppose that the manifold P is complete and the geodesics of P are defined for all $s \in \mathbf{R}$.)

Theorem 3. *If the function $\varrho: M \to \mathbf{R}$ has at the point $y_0 \in M$ strict local minimum, the fiber $\pi^{-1}(y_0)$ is stable with respect to the geodesic flow.*

Proof. Since the function $\exp \varrho$ has at y_0 strict local minimum, the strict inequality $\exp \varrho(y) > \exp \varrho(y_0)$ if $y \neq y_0$, is true in a neighbourhood $V \subset M$. If $y' \neq 0$

$$\Phi(y, y') = (1/c)\langle y', y'\rangle + \exp \varrho(y) > \exp \varrho(y) \geqq \exp \varrho(y_0)$$

for $y \in V$, that is the function $\Phi(y, y')$ has a strict local minimum at $\{y_0, 0\} \in TM$. Let be given an $\varepsilon>0$ such that the ε-neighbourhood of y_0 is contained in V. We consider the values of Φ on the boundary of the neighbourhood defined by the inequalities $d(y, y_0)<\varepsilon$, $\|y\|<\varepsilon$. The function Φ reaches its minimum Φ^* on this compact set (if ε is sufficiently small) and $\Phi^*>\exp \varrho(y_0)$. We can find a neighbourhood $d(y, y_0)<\delta$, $\|y'\|<\delta$ in TM such that here $\Phi(y, y')<\Phi^*$. If the initial point and tangent vector of $y(s)=\pi\circ x(s)$ satisfy the inequalities $d(y(s_0), y_0)<\delta$; $\|y'(s_0)\|<\delta$ then $\Phi(y(s_0), y'(s_0))<\Phi^*$. But by Corollary 2 of Theorem 2 Φ is constant along $y(s)=\pi\circ x(s)$ if $x(s)$ is a geodesic of P, consequently $\Phi(y(s), y'(s))<\Phi^*$ for all $s\in\mathbf{R}$. Therefore the curve $\{y(s), y'(s)\}=\{\pi\circ x(s), \pi_* x'\}$ cannot attain the boundary of the ε-neighbourhood of $\{y_0, 0\}$, because there would be $\Phi\geqq\Phi^*$.

This completes the proof.

Corollary. *If $\{P, \pi, M\}$ is a submersion with 1-dimensional fibers, then at the strict minimum point $y_0\in M$ of the function $\varrho\colon M\to\mathbf{R}$ the fiber geodesic $\pi^{-1}(y_0)$ is stable.*

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