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On a Paley-type inequality

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Dedicated to Professor B. Szőkefalvi-Nagy on his 70th birthday

In this paper a new space similar to the dyadic Hardy spaces is investigated. This space is defined by a shift-invariant norm and it is proved that for $1<p<\infty$ this norm is equivalent to the L^p-norm.

1. Introduction

The spaces $L^p=L^p(0,1)$ $(1<p<\infty)$ are considered as real Banach spaces of real-valued functions with the usual norms $\| \|_p$. The "dyadic Hardy spaces" are denoted by $\mathbf{H}^p$. The spaces $\mathbf{H}^p$ $(1\leqq p<\infty)$ coincide with the space of all L^1 functions, quadratic variations of which belong to L^p. The quadratic variation $Q(f)$ of the function $f\in L^1$ is defined by

$$Q(f):=\left(\sum_{n=0}^{\infty}|\Delta_n(f)|^2\right)^{1/2} \tag{1}$$

where $\Delta_n(f)=E_n(f)-E_{n-1}(f)$ $(n=0,1,\ldots)$, $E_{-1}f=0$ and $E_n(f)$ denotes the 2^n-th partial sum of the Walsh—Fourier series of f. The operator E_n is equal to the conditional expectation with respect to the σ-algebra generated by the intervals $[k2^{-n}, (k+1)2^{-n})$ $(k=0,1,\ldots,2^n-1)$. The dyadic $\mathbf{H}^p$-norm of the function f is

$$\|f\|_{\mathbf{H}^p}:=\|Q(f)\|_p \quad (1\leqq p<\infty). \tag{2}$$

It was proved by R. E. A. C. Paley [1] that for $1<p<\infty$ there exist constants c_p and c_p' depending only on p such that

$$c_p'\|f\|_p\leqq\|Q(f)\|_p\leqq c_p\|f\|_p \quad (1<p<\infty), \tag{3}$$

i.e., for $1<p<\infty$ the L^p-norm and the $\mathbf{H}^p$-norm are equivalent. In the case $p=1$ the inequality (3) is not true. B. Davis [2] has proved (in a more general form) that

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the $\mathbf{H}^1$-norm of f is equivalent to the L^1-norm of the dyadic maximal function $E^*(f)$ of f: $\|Q(f)\|_1 \sim \|E^*(f)\|_1$ where $E^*(f)=\sup_n |E_n(f)|$. Furthermore, it is known that

$$\|E^*(f)\|_p \sim \|Q(f)\|_p \sim \int_0^1 \|T(f;\, x)\|_p\, dx \quad (1 \leqq p < \infty) \tag{4}$$

where

$$T(f;\, x) := \sum_{n=0}^{\infty} r_n(x) \Delta_n(f)$$

and $r=(r_n,\ n\in\mathbf{N})$ $(\mathbf{N}:=\{0, 1, 2, \ldots\})$ denotes the Rademacher system. A special case of (3) is the well-known Khintchine inequality:

$$\left(\sum_{n=0}^{\infty} a_n^2\right)^{1/2} \sim \left\|\sum_{n=0}^{\infty} a_n r_n\right\|_p \quad (1 < p < \infty).$$

The L^p-norms $(1<p<\infty)$ are invariant with respect to the dyadic shift operators $s_n(f):=f\Psi_n$ $(n\in\mathbf{N})$, where the Ψ_n-s are the Walsh—Paley functions, i.e., $\|f\|_p = \|f\Psi_n\|_p$ $(1<p<\infty,\ n\in\mathbf{N})$. The $\mathbf{H}^1$-norm has not this property. An easy computation shows that for the functions

$$D_{2^n}(x) = \begin{cases} 2^n, & \text{if } 0 \leqq x < 2^{-n}, \\ 0, & \text{if } 2^{-n} \leqq x < 1 \end{cases} \quad (n\in\mathbf{N})$$

we have

$$\|Q(D_{2^n})\|_1 > 3^{-1/2} n, \quad \|Q(\Psi_{2^n} D_{2^n})\|_1 = 1. \tag{5}$$

We introduce the following shift-invariant norm: for $1 \leqq p < \infty$ let

$$\|f\|_{\mathbf{H}_p^*} := \left\|\sup_n Q(f\Psi_n)\right\|_p,$$

and denote by $\mathbf{H}_p^*$ the set of L^1 functions f, for which $\|f\|_{\mathbf{H}_p^*} < \infty$. Obviously, $\mathbf{H}_p^* \subseteqq \mathbf{H}_p$. By means of (5) a function f_0 can be constructed such that $\|f_0\|_{\mathbf{H}^1} < \infty$ and $\|f_0\|_{\mathbf{H}_1^*} = \infty$. In [3] it was proved that the sublinear operator

$$Q^*(f) = \sup_n Q(f\Psi_n) \quad (f\in L^1)$$

has weak type (2, 2), i.e., there exists a constant C independent of f such that for every $y>0$,

$$\text{mes}\,\{x\in[0, 1):\ Q^*(f)(x) > y\} < C\|f\|_2^2/y^2.$$

In this paper we give the following generalization of the above result.

Theorem. 1. *For $1<p<\infty$ the $\mathbf{H}_p^*$-norm is equivalent to the L^p-norm:*

$$\|Q^*(f)\|_p \sim \|f\|_p \quad (1 < p < \infty). \tag{6}$$

2. *There exists a function in* $\mathbf{H}_1$ *with infinite* $\mathbf{H}_1^*$-*norm.*

The first part of Theorem is a consequence of the following

Lemma 1. *The operators*

$$Q_N^*(f) = \sup_{m<2^N} \left(\sum_{n=1}^{N-1} |\Delta_n(f\Psi_m)|^2 \right)^{1/2} \quad (N \in \mathbf{N}) \tag{7}$$

are of restricted weak type (p, p) *for every* $1<p<\infty$, *i.e., for every measurable set* $H \in [0, 1)$,

$$\text{mes}\{x\colon\ Q_N^*(\chi_H)(x) > y\} < C_p \|\chi_H\|_p^p / y^p \quad (y > 0), \tag{8}$$

where χ_H *is the characteristic function of the set* H *and* C_p *is a constant depending only on* p.

It is easy to see that for every $f \in L^1$ there exists a linear operator $L_f\colon L^1 \to L^1$ such that

$$\text{i)}\ L_f(f) = Q_N^*(f), \quad \text{ii)}\ |L_f(g)| \leqq Q_N^*(g) \quad (g \in L^1) \tag{9}$$

hold. Indeed, for $x \in [0, 1)$ let $0 \leqq M(x) < 2^N$ be such a number for which

$$Q_N^*(f)(x) = \left(\sum_{n=1}^{N-1} |\Delta_n(f\Psi_{M(x)})(x)|^2 \right)^{1/2}.$$

Furthermore, let

$$L_f(g)(x) = \sum_{n=1}^{N-1} \varepsilon_n(x) \Delta_n(g\Psi_{M(x)})(x),$$

where

$$\varepsilon_m(x) = \text{sign}\, \Delta_m(f\Psi_{M(x)})(x) \Big/ \left(\sum_{n=0}^{N-1} |\Delta_n(f\Psi_{M(x)})|^2 \right)^{1/2} \quad (1 \leqq m \leqq N).$$

It is obvious that for the linear operator L_f (9) is satisfied, and by (9) ii) it is also of restricted weak type (p, p) for $1<p<\infty$. Applying the Stein—Weiss interpolation theorem (see, e.g., [5], p. 191) we get that the operator $L_f\colon L^p \to L^p$ $(1<p<\infty)$ and consequently on the basis of (9) i) the operators $Q_N^*\colon L^p \to L^p$ $(1<p<\infty)$ are also uniformly bounded.

Since

$$Q^*(f) \leqq \sup_m |E_0(f\Psi_m)| + \sup_m \left(\sum_{n=1}^{\infty} |\Delta_n(f\Psi_m)|^2 \right)^{1/2} = \sup_m |E_0(f\Psi_m)| + \lim_{N\to\infty} Q_N^*(f),$$

we have

$$\|Q^*(f)\|_p \leqq C_p^* \|f\|_p \quad (1 < p < \infty),$$

and by the Paley-inequality,

$$c_p' \|f\|_p < \|Q(f)\|_p \leqq \|Q^*(f)\|_p.$$

This proves (6).

Let us introduce another shift-invariant norm by means of the maximal function

$$E^{**}(f) := \sup_{m,n\in\mathbf{N}} |E_n(f\Psi_m)|$$

as follows: let

$$\|f\|_p^* := \|E^{**}(f)\|_p \quad (1 \leqq p < \infty).$$

Since $E^*(f) \leqq E^{**}(f) \leqq E^*(|f|)$, the Doob-inequality (see [4]), implies that $\|f\|_p^* \sim \|f\|_p$ $(1<p<\infty)$, i.e., for $1<p<\infty$ the $\mathbf{H}_p$-norm is equivalent to the $\|\ \|_p^*$-norm. We do not know whether the $\mathbf{H}_1$-norm and the $\|\ \|_1^*$-norm are equivalent or not.

2. Two lemmas

Let

$$\mathscr{I}_N := \{[k2^n, (k+1)2^n) \colon 0 \leqq n < N, \quad (k+1)2^n < 2^N, \ k, n\in\mathbf{N}^n\},$$

and for an interval $I=[k2^n, (k+1)2^n)$ we set $m(I)=k2^n$, $|I|=2^n$ and

$$E_I(f) = \sum_{n\in I}\left(\int_0^1 f\Psi_n\, dx\right)\Psi_n.$$

Then, $E_n(f)=E_{[0,2^n)}(f)$ and for all $j\in I=[k2^n, (k+1)2^n)$ we have $E_I(f)=E_n(f\Psi_j)\Psi_j$.

By means of the intervals of $\mathscr{I}_N$ the function $Q_N^*(f)$ can be written in the form

$$Q_N^*(f) = \sup_{j<2^N}(\sum_{j\in I}|\Delta_I(f)|^2)^{1/2},$$

where $\Delta_I(f)=E_{I_+}(f)-E_I(f)$ and I_+ denotes the interval for which $I\subset I_+$ and $|I_+|=2|I|$ hold.

To estimate $Q_N^*(f)$ we use an elementary observation with respect to series, in which the indices of the terms are the elements of $\mathscr{I}_N$. We need the following

Lemma 2. *Let $g_I\colon [0,1)\to\mathbf{R}$ $(I\in\mathscr{I}_N)$ be a sequence of functions and $B_I\subset[0,1)$ $(I\in\mathscr{I}_N)$ a sequence of increasing sets* (i.e., $I\subseteqq J$ *implies* $B_I\subseteqq B_J$). *Further let $A_I=B_I\setminus\bigcap_{J\subset I} B_J$. Then*

(10) $$\sup\{|\sum_{I\subseteqq J\subset K}\chi_{B_J}g_J| \colon I\subset K,\ I, K\in\mathscr{I}_N\} \leqq G := 2\sup_{I\in\mathscr{I}_N}\chi_{A_I}\sup_{I\subset K}|\sum_{I\subseteqq J\subset K} g_J|.$$ [1)]

Proof. To prove (10), let $x\in[0,1)$ and $S_{IK}=|\sum_{I\subseteqq J\subset K}\chi_{B_J}g_J|$. We show that $S_{IK}(x)\leqq G(x)$.

If $S_{IK}(x)\neq 0$, then the (linearly ordered) set $\{J\in\mathscr{I}_N\colon I\subseteqq J\subset K, x\in B_J\}$ is not empty. Denote by $\bar{I}$ the minimum element (with respect to the ordering $\subseteqq$) of

[1)] $J\subset K$ means that $J\subseteqq K$ and $J\neq K$.

this set. If $I \subset \bar{I}$, then by the definition of $\bar{I}$ we have that for $I \subseteqq J \subset \bar{I}$, $x \notin B_J$. Let I^* be such an element of the set $\tilde{\mathcal{I}} = \{J \in \mathcal{I}_N : J \subset \bar{I},\ x \in B_J\}$ $(\neq \emptyset)$, for which $|I^*| = \min\{|J| : J \in \tilde{\mathcal{I}}\}$. From the definition of I^* it follows that for every $J \subset I^*$ we have $J \notin \tilde{\mathcal{I}}$. Thus, for such J's, $x \notin B_J$ and consequently $x \in A_{I^*}$. From these we get

$$|S_{IK}(x)| = |S_{IK}(x)| = |S_{I^*K}(x) - S_{I^*I}(x)| \leqq$$

$$\leqq \chi_{A_{I^*}}(x) \Big|\sum_{I^* \subseteqq J \subset K} g_J(x)\Big| + \chi_{A_{I^*}}(x) \Big|\sum_{I^* \subseteqq J \subset K} g_J(x)\Big| \leqq G(x),$$

and (10) is proved.

Let

$$F_I f = \sup\{|E_I(f)| : J \subset I,\ 2|J| = |I|\} \quad (I \in \mathcal{I}_N,\ |I| \geqq 2),$$

$$F_I f = |E_I(f)| \quad (I \in \mathcal{I}_N,\ |I| = 1), \tag{11}$$

$$F_I^* f = \sup\{F_J f : J \subseteqq I\}, \quad F^* f = \sup\{F_I^* f : I \in \mathcal{I}_N\}.$$

The σ-algebra generated by the intervals $[k2^{-n}, (k+1)2^{-n})$ $(k=0, 1, \ldots, 2^n-1)$ will be denoted by $\mathcal{A}_n$ $(n \in \mathrm{N})$ and for $I \in \mathcal{I}_N$, $|I| = 2^n$, set $\mathcal{A}_I = \mathcal{A}_n$. The sequence $(E_I(f), I \in \mathcal{I}_N)$ is predictable. Indeed, since $E_I(f) = E_{I'}(f) + E_{I''}(f)$ $(I = I' \cup I''$, $I' \cap I'' = \emptyset)$, $F_I^* f$ is $\mathcal{A}_{n-1}$-measurable and $|E_I(f)| < 2F_I^* f$.

For $y > 0$ let

$$B_I^y = \{x \in [0, 1) : (F_I^* f)(x) > y\}, \quad A_I^y = B_I^y \setminus \bigcup_{J \subset I} B_J^y,$$

$$C_I^y = \{x \in [0, 1) : (F_I^* f)(y) \leqq ey\}. \tag{12}$$

Then the following statement is true.

Lemma 3. *For every* $y > 0$,

$$\sum_{I \in \mathcal{I}_N} \operatorname{mes} A_I^y < \frac{1}{y^2} \int_{\{F^* f > y\}} |f|^2\, dx. \tag{13}$$

Proof. On the basis of the definition of A_I^y and B_I^y it is obvious that $(F_I f)(x) > y$ if $x \in A_I^y$. Let

$$D_{I'}^y = \{x \in A_I^y : |E_{I'}(f)(x)| > y\}, \quad D_{I''}^y = A_I^y \setminus D_{I'}^y,$$

where $I' \subset I$, $I'' = I \setminus I'$ and $2|I'| = |I|$. We set

$$P_I = \chi_{D_{I'}} E_{I'} + \chi_{D_{I''}} E_{I''}.$$

Since $E_I E_J \doteq 0$ if $I \cap J = \emptyset$, and $\chi_{A_I^y} \chi_{A_J^y} = 0$, if $I \subset J$, on the basis of the $\mathcal{A}_I$-homogeneity of E_I (which means $E_I(\lambda f) = \lambda E_I f$, if λ is $\mathcal{A}_I$-measurable) we get that the P_I's are orthogonal projections, i.e., $P_I P_J = \delta_{IJ} P_I$ $(I, J \in \mathcal{I}_N)$. Thus

$$\|\chi_{\{F^* f > y\}} f\|_2^2 \geqq \Big\| \sum_{I \in \mathcal{I}_N} P_I f \Big\|_2^2 = \sum_{I \in \mathcal{I}_N} \|P_I f\|_2^2 =$$

$$= \sum_{I \in \mathcal{I}_N} \int_{D_{I'}} |E_{I'} f|^2 \, dx + \int_{D_{I''}} |E_{I''} f|^2 \, dx \geqq y^2 \sum_{I \in \mathcal{I}_N} \operatorname{mes} A_I^y,$$

and Lemma 3 is proved.

3. Proof of Lemma 1

Let

$$\varepsilon_I^y = \frac{1}{y} \chi_{\{(1/e) F_{I_+}^* f \leqq y < F_{I_+}^* f\}} = \frac{1}{y} \chi_{B_{I_+}^y} \chi_{C_{I_+}^y} \quad (y > 0). \tag{14}$$

Then ε_I^y is $\mathcal{A}_I$-measurable and

$$\Big(\int_0^{+\infty} \varepsilon_I^y \, dy \Big) \Delta_I f = \Delta_I f.$$

Using this, the quadratic variation can be estimated as follows:

$$Q_n(f) = \Big(\sum_{n \in I \in \mathcal{I}_N} |\Delta_I f|^2 \Big)^{1/2} = \Bigg(\sum_{n \in I \in \mathcal{I}_N} \bigg| \int_0^{+\infty} \varepsilon_I^y \Delta_I f \, dy \bigg|^2 \Bigg)^{1/2} \leqq$$

$$\leqq \int_0^{+\infty} \Big(\sum_{n \in I \in \mathcal{I}_N} |\varepsilon_I^y \Delta_I f|^2 \Big)^{1/2} dy,$$

and by Lemma 2 we have

$$Q_N^*(f) < \int_0^{+\infty} \sup_{I \in \mathcal{I}_N} R_I^y f \, dy,$$

where

$$R_I^y f = 2 \chi_{A_I^y} \Big(\sum_{I \subseteqq J \in \mathcal{I}_N} |\varepsilon_I^y \Delta_I f|^2 \Big)^{1/2},$$

and consequently

$$\chi_{\{F^* f < \lambda\}} Q_N^*(f) \leqq \int_0^{\lambda} \sup_{I \in \mathcal{I}_N} R_I^y \, dy. \tag{15}$$

Using Abel's transformation, an easy computation shows that

$$\Big| \sum_{I \subseteqq J \in \mathcal{I}_N} \varepsilon_I^y \Delta_I f \Big| \leqq 4e,$$

thus by the Paley-inequality we get

$$\begin{aligned} \|\chi_{A_I^y} R_I^y\|_p &\leqq C_p \Big\| \sum_{I \subseteqq J \in \mathcal{I}_N} \varepsilon_I^y \Delta_I (f \chi_{A_I^y}) \Big\|_p = \\ &= C_p \Big\| \chi_{A_I^y} \sum_{I \subseteqq J \subseteqq \mathcal{I}_N} \varepsilon_I^y \Delta_I f \Big\|_p < 4e C_p \|\chi_{A_I^y}\|_p . \end{aligned} \tag{16}$$

Let first $p>2$. Then by (13) and (15),

$$\|\chi_{\{F^*f\leqq\lambda\}}Q_N^*(f)\|_{2p} \leqq \int_0^{\lambda}\Big(\sum_{I\in\mathcal{I}_N}\|R_I^y f\|_{2p}^{2p}\Big)^{1/2p}dy \leqq$$

$$\leqq 2(4eC_{2p})^{2p}\int_0^{\lambda}\Big(\sum_{I\in\mathcal{I}_N}\text{mes }A_I^y\Big)^{1/2p}dy \leqq C_p'\int_0^{\lambda}\Big(\int_{\{F^*f>y\}}|f|^2/y^2\,dx\Big)^{1/2p}dy \leqq$$

$$\leqq C_p'\Big(\int_0^{\lambda}y^{-1/2}\,dy\Big)\Big(\int_0^1(F^*f)^{p-2}|f|^2\,dx\Big)^{1/2p} \leqq 2C_p'\lambda^{1/2}\Big(\int_0^1|F^*f|^p\Big)^{1/2p}.$$

Using the maximal inequality $\|F^*f\|_r\leqq(r/(r-1))\|f\|_r$ $(r>1)$ we get

$$\lambda^p\,\text{mes}\,\{Q_N^*(f)>\lambda,\ F^*f\leqq\lambda\}<C_p''\|f\|_p^p,$$

and on the basis of the maximal inequality (8) follows for every $f\in L^p$ $(p\geqq 2)$.

Let now $1<p<2$ and $f=\chi_H$. By a simple integral transformation (15) can be written in the form

$$\chi_{\{F^*f\leqq\lambda^p\}}Q_N^*(f)<\lambda\int_0^{\lambda^{p-1}}\sup_{I\in\mathcal{I}_N}R_I^{\lambda t}f\,dt,$$

and since $\sup_I R_I^{\lambda t}f=\chi_{\{F^*f>\lambda t\}}\sup_I R_I^{\lambda t}f$, by $F^*f\leqq 1$ we have

(17)
$$\chi_{\{F^*f\leqq\lambda^p\}}Q_N^*(f)<\int_0^{\lambda_1}\sup_{I\in\mathcal{I}_N}R_I^{\lambda t}f\,dt,$$

where $\lambda_1=\min(\lambda^{p-1},\lambda^{-1})\leqq 1$. The condition $t\leqq\lambda^{p-1}$ yields $\lambda^{-2}\leqq t^{-(2-p)/(p-1)}\lambda^{-p}$, thus by (13), (16), and (17) with $q=2((2-p)/(p-1)+2)$ we have

$$\|\chi_{\{F^*f<\lambda^p\}}Q_N^*(f)\|_q\leqq\lambda\int_0^{\lambda_1}\Big(\sum_{I\in\mathcal{I}_N}\|R_I^{\lambda t}f\|_q^q\Big)^{1/q}dt\leqq$$

$$\leqq\lambda C_q\int_0^{\lambda_1}\Big(\sum_{I\in\mathcal{I}_N}\text{mes }A_I^{\lambda t}\Big)^{1/q}dt<(\text{mes }H)^{1/q}\lambda C_q\int_0^{\lambda_1}(\lambda t)^{-2/q}\,dt\leqq$$

$$\leqq C_q\lambda^{1-p/q}(\text{mes }H)^{1/q}\int_0^1 t^{-1/2}\,dt=2C_q\lambda^{1-p/q}(\text{mes }H)^{1/q}.$$

From this we obtain

$$\lambda^p\,\text{mes}\,\{Q_N^*(f)>\lambda;\ F^*f\leqq\lambda^p\}\leqq\bar{C}_p\,\text{mes }H.$$

This and the maximal inequality gives (8).

4. Proof of the second part of Theorem

Let

$$f=\sum_{n=0}^{\infty} 2^{-n/2} r_{2^n} D_{2^{2^n}}.$$

Since $\|D_{2^s}\|_1=1$ $(s\in \mathbf{N})$, this series is absolute convergent a.e. and $f\in L^1$. It is obvious that

$$E^*f \leqq \sum_{n=0}^{\infty} 2^{-n/2} D_{2^{2^n}},$$

and consequently $E^*f\in L^1$, i.e., $\|f\|_{\mathbf{H}_1}<\infty$. On the basis of $Q(r_{2^n}f)\geqq 2^{-n/2}Q(D_{2^{2^n}})$ we have

$$\|Q^*(f)\|_1 \geqq \|Q(r_{2^n}f)\|_1 \geqq 2^{-n/2}\|Q(D_{2^{2^n}})\|_1 \geqq 3^{-1/2}2^{n/2} \quad (n\in \mathbf{N}),$$

thus $\|f\|_{\mathbf{H}_1^*}=\infty$.

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