Acta Sci. Math., 45 (1983), 381—388

On symplectic actions of compact Lie groups with isotropy subgroups of maximal rank

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Dedicated to Professor B. Szőkefalvi-Nagy on his 70th birthday

Let (M, ω) be a symplectic manifold, G a connected Lie group and $\Phi: G\times M\to M$ a symplectic action. If the action Φ has a momentum map $\mu: M\to\mathfrak{g}^*$, where $\mathfrak{g}^*$ is the dual space of the Lie algebra $\mathfrak{g}$ of G, then there is an action $\Psi: G\times\mathfrak{g}^*\to\mathfrak{g}^*$ such that μ is equivariant with respect to the actions Φ, Ψ. In this setting are derived the results of A. A. Kirillov, B. Kostant, J. M. Souriau and of others concerning Hamiltonian systems with symmetries ([1], pp. 276—311). Restriction to the case where G is compact offers a situation with peculiar features, a subject which seems to deserve special concern. A result pertaining to the above case is presented below. In fact, it is shown that if G is compact and the isotropy subgroups of Φ are of maximal rank then all the orbits of Φ are equivariantly isomorphic.

The concepts and results applied subsequently are in conformity with those laid down in the work of R. ABRAHAM and J. E. MARSDEN [1], however, in the notations there are some deviations.

The following lemma presents a simple but for the subsequent results essential observation.

Lemma 1. *Let (M, ω) be a symplectic manifold, G a connected Lie group and $\Phi: G\times M\to M$ a symplectic action with a momentum map $\mu: M\to\mathfrak{g}^*$. Then the kernel of the tangent linear map $T_z\mu: T_zM\to T_{\mu(z)}\mathfrak{g}^*$ is given by*

$$\operatorname{Ker} T_z\mu = (T_z G(z))^{\perp}$$

where the orthogonal complement is taken with respect to the symplectic form ω.

Proof. Let $Z\in T_zM$ and $\varphi: I\to M$ a curve with $\varphi(0)=z, \varphi'(0)=Z$. Then the following holds for any fixed $X\in\mathfrak{g}$ according to the definition of the momentum

Received June 1, 1982.

map:

$$\frac{d}{d\tau}\langle \mu(\varphi(\tau)), X\rangle\big|_{\tau=0} = Z\langle \mu(x), X\rangle = d(\langle \mu(x), X\rangle)(Z) = (i(\bar{X})\omega)(Z) = \omega(\bar{X}, Z)$$

where $\bar{X}$ is the infinitesimal generator of the action Φ corresponding to the element X of the Lie algebra $\mathfrak{g}$ of G. On the other hand, the following is obviously valid:

$$\frac{d}{d\tau}\langle \mu(\varphi(\tau)), X\rangle\big|_{\tau=0} = \langle \iota_\xi \circ T_z\mu Z, X\rangle$$

where $\iota_\zeta\colon T_\zeta\mathfrak{g}^* \to \mathfrak{g}^*$ is the canonical isomorphism at $\zeta = \mu(z)$. Consequently, the following is obtained:

$$\langle \iota_\xi \circ T_z\mu Z, X\rangle = \omega(\bar{X}, Z) \quad \text{for} \quad X \in \mathfrak{g}.$$

Therefore $Z \in \operatorname{Ker} T_z\mu$ if and only if $Z \in (T_zG(z))^\perp$ holds, since $T_zG(z)$ is spanned by the values of $\bar{X}$ at z as X runs through $\mathfrak{g}$.

Let now (M, ω) be a symplectic manifold and $\langle\, ,\rangle$ a Riemannian metric on M. Then there is a unique tensor field A of type $(1; 1)$ on M such that $\omega(X, Y) = \langle AX, Y\rangle$ holds for any vector fields $X, Y \in \mathcal{T}(M)$. Moreover, since ω is non-degenerate, $A_z\colon T_zM \to T_zM$, the value of A at the point $z \in M$ is an automorphism of the tangent space. Consider now with respect to the inner product $\langle\, ,\rangle_z$ the polar decomposition $A_z = S_z \circ J_z$ of A_z, then the symmetric tensor S_z and the orthogonal tensor J_z are uniquely defined since A_z is injective ([2], pp. 169—170). Thus, tensor fields S, J are obtained on M. Moreover, the tensor field J is an almost complex structure on M and $\langle X, Y\rangle = \omega(JX, Y)$ holds for arbitrary vector fields $X, Y \in \mathcal{T}(M)$ according to a basic result ([1], pp. 172—174).

The tensor field J is called the *almost complex structure defined by the symplectic form ω and by the Riemannian metric $\langle\, ,\rangle$.*

The following corollary is a consequence of the preceding lemma and of the above mentioned facts.

Corollary. *Let (M, ω) be a symplectic manifold, G a connected Lie group and $\Phi\colon G \times M \to M$ a symplectic action with a momentum map $\mu\colon M \to \mathfrak{g}^*$. Moreover, let there be a Riemannian metric $\langle\, ,\rangle$ on M which is left invariant by the action Φ and let J be the almost complex structure defined by ω and $\langle\, ,\rangle$. Then the kernel of the tangent linear map $T_z\mu$ is given at any point $z \in M$ by*

$$\operatorname{Ker} T_z\mu = J_z(N_zG(z))$$

where $N_zG(z)$ is the orthogonal complement of the tangent space $T_zG(z)$ with respect to the inner product $\langle\, ,\rangle_z$.

Proof. According to the above mentioned relation of ω, $\langle\ ,\ \rangle$, and J the following inclusions obviously hold:

$$J(N_z G(z)) \subset (T_z G(z))^{\perp}, \quad J^{-1}((T_z G(z)^{\perp}) \subset N_z G(z)$$

for $z \in M$. Consequently, the Corollary follows directly from the preceding lemma.

The following lemma which is again a consequence of the relation of ω, $\langle\ ,\ \rangle$, and J, is essential for the subsequent results.

The lemma concerns the induced action on the tangent bundle. In fact, if an action $\Phi: G \times M \to M$ is given, then by $\Phi_g(z) = \Phi(g, z)$, $z \in M$, a diffeomorphism $\Phi_g: M \to M$ is defined for any $g \in G$. Consequently the tangent linear map $T\Phi_g: TM \to TM$ is a transformation of TM for $g \in G$. Thus an action of G on TM is obtained which is called the *induced action* of G on TM.

Lemma 2. *Let (M, ω) be a symplectic manifold, G a connected Lie group, $\Phi: G \times M \to M$ a symplectic action, and $\langle\ ,\ \rangle$ a Riemannian metric on M which is left invariant by the action Φ. Then the almost complex structure J defined by ω and $\langle\ ,\ \rangle$ is equivariant for the induced action of G on TM; in other words, $T\Phi_g \circ J = J \circ T\Phi_g$ is valid for any element g of G.*

Proof. Let A be the tensor field defined by ω and $\langle\ ,\ \rangle$ on M and S, J those obtained by the polar decomposition of A. The invariance of ω and $\langle\ ,\ \rangle$ yields that the following is valid for arbitrary vector fields $X, Y \in \mathscr{T}(M)$ and $g \in G$:

$$\langle T\Phi_g^{-1} \circ A \circ T\Phi_g X, Y \rangle = \langle A \circ T\Phi_g X, T\Phi_g Y \rangle =$$
$$= \omega(T\Phi_g X, T\Phi_g Y) = \omega(X, Y) = \langle AX, Y \rangle.$$

But then $A = T\Phi_g^{-1} \circ A \circ T\Phi_g$ holds for $g \in G$. Consequently, the following is valid, too:

$$A = S \circ J = (T\Phi_g^{-1} \circ S \circ T\Phi_g) \circ (T\Phi_g^{-1} \circ J \circ T\Phi_g), \quad g \in G.$$

But, then $T\Phi_g^{-1} \circ S \circ T\Phi_g$, $T\Phi_g^{-1} \circ J \circ T\Phi_g$ yields a polar decomposition of A for $g \in G$, since the Riemannian metric $\langle\ ,\ \rangle$ is left invariant by the action Φ. Since A_z, $z \in M$, is injective, its polar decomposition is unique, as mentioned before. Consequently, the validity of

$$J = T\Phi_g^{-1} \circ J \circ T\Phi_g, \quad g \in G$$

is obtained which yields the assertion of the lemma.

In order to state a corollary of the preceding lemma the introduction of a concept is convenient. In fact, let $\Phi: G \times M \to M$ be a smooth action of a connected Lie group G. Then

$$R_z = \{X \mid T_z \Phi_g X = X \quad \text{for} \quad g \in G_z \quad \text{where} \quad X \in T_z M\}$$

is a subspace of the tangent space $T_z M$ at any point z of the manifold M.

Corollary. *Let* (M, ω) *be a symplectic manifold,* G *a connected Lie group,* $\Phi: G\times M\to M$ *a symplectic action and* $\langle\,,\rangle$ *a Riemannian metric on* M *which is left invariant by the action* Φ. *If* J *is the almost complex structure defined by* ω *and* $\langle\,,\rangle$ *then* $J_z(R_z)=R_z$ *holds at any point* $z\in M$.

Proof. If $X\in R_z$ then $T_z\Phi_g(J_z(X))=J_z(T_z\Phi_z(X))=J_z(X)$ holds for $g\in G_z$, and this implies the assertion of the corollary.

In case of a smooth action of a compact connected Lie group there is a standard classification of the orbits of the action and accordingly principal, exceptional and singular orbits are distinguished. As a result of R. Palais [5] shows the above classification can be introduced in case of isometric actions of connected Lie groups so that almost all the fundamental results concerning smooth actions of compact Lie groups remain valid. Therefore, if $\Phi: G\times M\to M$ is an isometric action of a connected Lie group on a Riemannian manifold M then there are points $z\in M$ such that $G(z)$ is a principal orbit; moreover, if in this case

$$T_z M = T_z G(z)\oplus N_z G(z)$$

is the orthogonal decomposition with respect to the Riemannian metric $\langle\,,\rangle$, then $N_zG(z)\subset R_z$ holds.

For the formulation of the next lemma the introduction of the following concept is convenient. Consider a smooth action $\Phi: G\times M\to M$ of a connected Lie group G on a differentiable manifold M and a non-zero tangent vector $X\in T_zG(z), z\in M$; it is said that X is an *isotropy fixed tangent vector* for the action Φ provided that the following is valid:

$$X = T_z\Phi_g X \quad \text{for} \quad g\in G_z.$$

Some results concerning basic properties of the above concept will be given elsewhere.

Lemma 3. *Let* (M, ω) *be a symplectic manifold,* G *a connected Lie group,* $\Phi: G\times M\to M$ *a symplectic action with a momentum map* $\mu: M\to \mathfrak{g}^*$ *and* $\langle\,,\rangle$ *a Riemannian metric on* M *which is left invariant by the action* Φ. *If the action* Φ *has no isotropy fixed tangent vectors then*

$$\operatorname{Ker} T_z\mu = N_z G(z)$$

holds at any point $z\in M$ *such that* $G(z)$ *is a principal orbit of the action.*

Proof. Let $z\in M$ be such that $G(z)$ is a principal orbit and consider the orthogonal decomposition

$$T_z M = T_z G(z)\oplus N_z G(z)$$

with respect to the Riemannian metric $\langle\,,\rangle$. Let $X\in R$ and

$$X = X'+X'', \quad X'\in T_z G(z),\ X''\in N_z G(z)$$

its corresponding decomposition. Then both X' and X'' are left fixed by the action $T_z\Phi_g\colon T_zM\to T_zM$, $g\in G_z$. Thus, the assumption that Φ has no isotropy fixed tangent vectors implies that $X'=0$ holds. Consequently, $R_z\subset N_zG(z)$ is valid. On the other hand, the assumption that $G(z)$ is a principal orbit implies that $N_zG(z)\subset R_z$ holds. Thus, $N_zG(z)=R_z$. Now, the corollaries to Lemma 1 and to Lemma 2 as well as the preceding assertion yield that

$$\operatorname{Ker} T_z\mu = J_z(N_zG(z)) = J_z(R_z) = R_z = N_zG(z)$$

holds. Thus, the assertion of the lemma is proved.

The following theorem presents the result already mentioned at the beginning. The rank of compact Lie groups occurring here is taken in the usual sense given in terms of the maximal tori or of the Cartan subalgebras.

Theorem. *Let (M,ω) be a symplectic manifold, G a compact connected Lie group and $\Phi\colon G\times M\to M$ a symplectic action with a momentum map. If the isotropy subgroups of Φ are of maximal rank then all the orbits of Φ are equivariantly isomorphic.*

Proof. Since the group G is compact, there is a Riemannian metric $\langle\,,\rangle$ on M which is left invariant by the action Φ. The assumption that the isotropy subgroups of Φ are of maximal rank implies that Φ has no isotropy fixed tangent vectors. In fact, assume that there is a $V\in T_zG(z)$ for some $z\in M$ which is an isotropy fixed tangent vector of Φ. Consider now the maps

$$\pi_z\colon G\to G/G_z,\quad \varepsilon_z\colon G/G_z\to G(z),$$

which are the canonical projection and the canonical equivariant isomorphism, and fix a reductive decomposition $\mathfrak{g}=\mathfrak{g}\oplus\mathfrak{m}_z$ where $\mathfrak{g}_z\subset\mathfrak{g}$ is the subalgebra of the Lie algebra $\mathfrak{g}$ corresponding to the isotropy subgroup G_z. Then with the usual identifications $T_eG=\mathfrak{g}$, $T_eG_z=\mathfrak{g}_z$, a restricted map

$$T_o\varepsilon_z\circ T_e\pi_z\colon \mathfrak{m}_z\to T_zG(z)$$

is obtained where $o=\pi_z(e)$, and this restricted map is a vector space isomorphism which is equivariant for the following actions:

$$\operatorname{Ad}(g)\colon \mathfrak{m}_z\to\mathfrak{m}_z,\quad T_z\Phi_g\colon T_zG(z)\to T_zG(z),\quad g\in G_z.$$

Now, the existence of the vector $V\in T_zG(z)$ yields an element X of the Lie algebra $\mathfrak{g}$ such that

$$X\in\mathfrak{m}-\{0\}\quad\text{and}\quad [\mathfrak{g}_z, X]=0$$

are valid. Since $G_z\subset G$ is of maximal rank, there is a Cartan subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ included in $\mathfrak{g}_z$. But then $[\mathfrak{k},X]=0$ holds, and therefore X is in the normalizer of $\mathfrak{k}$. Since $X\notin\mathfrak{k}$ is valid, a contradiction is obtained with the definition of Cartan subalgebras. Therefore, the action Φ has no isotropy fixed tangent vectors. Let now $\mu: M \to g^*$ be a momentum map of the action Φ. Then the preceding lemma applies and yields that

$$\operatorname{Ker} T_z \mu = N_z G(z)$$

is valid for $z \in M$ provided that $G(z)$ is a principal orbit of the action Φ.

Fix now a $z \in M$ such that $G(z)$ is a principal orbit of Φ and consider the component F_z containing the point z of the following set

$$\{x \mid \Phi(g, x) = x \quad \text{for} \quad g \in G_z, \text{ where } x \in M\}.$$

Then, F_z is a totally geodesic submanifold of the Riemannian manifold M according to a fundamental result ([4], pp. 59—61) and F_z intersects every orbit of the action Φ [6]. Let $F_z' \subset F_z$ be the set of points $x \in F_z$ such that $G(x)$ is a principal orbit. Then, F_z' is an open, everywhere dense subset of F_z in consequence of the Principal Isotropy Type Theorem. Moreover, observations made in the proof of the preceding lemma imply that

$$T_x F_z = N_x G(x)$$

holds for $x \in F_z'$. Therefore, the assertion of Lemma 3 yields that

$$\operatorname{Ker} T_z \mu = T_x F_z$$

is valid for $x \in F_z'$. But then $\mu(F_z')$ is a single point and consequently $\mu(F_z)$ is a single point too. Consider now the action $\Psi: G \times g^* \to g^*$ on the dual space g^*, which is associated with the action Φ ([1], pp. 276—294). The image of μ is a single orbit of the action Ψ, owing to the facts that μ is equivariant for Φ and Ψ, that F_z intersects every orbit of Φ, and that $\mu(F_z)$ is a single point.

The restriction of $T_x\mu$ to $T_xG(x)$ is injective provided that $G(x)$ is a principal orbit of Φ, as Lemma 3 implies this. Therefore, the action Φ cannot have singular orbits, since the image of μ is a single orbit of Ψ as observed above. Thus, μ restricted to an orbit of Φ is a covering map. Let now $z \in M$ be such that $G(z)$ is a principal orbit. Then F_z is intersected the same number of times by any principal orbit of Φ. Since in any neighbourhood of an exceptional orbit there are principal ones; F_z is intersected the same number of times by an exceptional orbit of Φ as by the principal ones. Therefore, the existence of exceptional orbits and properties of the momentum map μ imply the existence of different intersecting totally geodesic submanifolds F_z. But the fact that two different ones among such submanifolds intersects entails obviously the existence of singular orbits. Consequently, the action Φ has no exceptional orbits either. Thus, the action Φ has only principal orbits; and this fact implies the assertion of the theorem.

As its following corollary shows, the preceding theorems has consequences concerning the structure of the symplectic manifold as well.

Corollary. *Let* (M, ω) *be a symplectic manifold,* G *a compact connected Lie group and* $\Phi: G\times M\to M$ *a symplectic action with a momentum map. If the isotropy subgroups of the action* Φ *are of maximal rank, then* M *is the total space of a differentiable fibre boundle, where the base manifold is the orbit space of the action* Φ *and the fibers are diffeomorphic to a finite covering of a fixed orbit of the coadjoint action* $\mathrm{Ad}^*: G\times \mathfrak{g}^*\to\mathfrak{g}^*$.

Proof. Since G is compact, there is a momentum map $\tilde{\mu}: M\to\mathfrak{g}^*$ of the action Φ such that the associated action $\tilde{\Psi}: G\times\mathfrak{g}^*\to\mathfrak{g}^*$ is equivalent to the coadjoint action of G. In fact, let $\mu: M\to\mathfrak{g}^*$ be a momentum map of Φ and $\sigma: G\to\mathfrak{g}^*$ the coadjoint cocycle associated to μ; then the associated action Ψ is given as follows:

$$\Psi(g, \xi) = \mathrm{Ad}^*(g^{-1})\xi+\sigma(g) \quad \text{where} \quad (g, \xi)\in G\times\mathfrak{g}^*.$$

Since the group G is compact, the action Ψ must have a fixed point $\zeta\in\mathfrak{g}^*$ and therefore

$$\Psi(g, \zeta) = \mathrm{Ad}^*(g^{-1})\zeta+\sigma(g) = \zeta$$

holds for every $g\in G$. Consequently, the associated action Ψ is given as follows:

$$\Psi(g, \xi) = \mathrm{Ad}^*(g^{-1})(\xi-\zeta)+\zeta \quad \text{where} \quad (g, \xi)\in G\times\mathfrak{g}^*.$$

According to the preceding theorem μ maps to a single orbit of Ψ and is a covering map on each orbit of Φ which are all of the same type. Consequently, the orbits of Φ are diffeomorphic to a finite covering of a single orbit of the coadjoint action. Since the orbits of Φ are all of the same type, the assertion of the corollary follows now by a basic theorem on the union of orbits of the same type of smooth actions of compact Lie groups ([3], pp. 6—9).

Added in proof. The author is indebted to Professor J. E. Marsden for the information that Lemmas 1, 2 and 3 are partially contained in the paper Symmetry and bifurcations of momentum mappings, *Comm. Math. Phys.*, **78** (1981), 445—478 by J. M. Arms, J. E. Marsden and V. Moncrief. Moreover, the author is thankful to Professor J. J. Duistermaat for acquainting him with conjectures that some results of the paper such as the Corollary to the Theorem can be developed further. A detailed account of the above observations together with further related results will be presented in a forthcoming paper.

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