Solutions to three problems concerning the overconvergence of complex interpolating polynomials

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To Professor B. Szőkefalvi-Nagy on his seventieth birthday

The aim of this note is to solve the problems raised in [1] by J. SZABADOS and R. S. VARGA. We keep the notations of [1].

The answer to the first problem is positive: $\hat{G}_l(z, \varrho) = G_l(z, \varrho)$. By the definition of $\hat{G}_l(z, \varrho)$ it is sufficient to show that for fixed z, $G_l(z, \varrho)$ is a monotonically decreasing continuous function of ϱ. By Hadamard's three-circle-theorem

$$I_n(\varrho) = \frac{1}{n}\log\left|\max_{|t|=\varrho}(1-t^{-ln})\frac{z^n-1}{t^n-1} - \frac{\omega_n(z, Z)}{\omega_n(t, Z)}\right|$$

is a convex function of $\log\varrho$ on the interval (ϱ', ∞). The proof of [1, Proposition 1] and a trivial estimate yield

$$K_1\log\frac{|z|}{\varrho^{l+1}} \leqq I_n(\varrho) \leqq K_2\log\frac{|z|+\varrho'}{\varrho-\varrho'}, \tag{1}$$

hence

$$\log G_l(z, \varrho) = \limsup_{n\to\infty} I_n(\varrho)$$

is also a convex function of $\log\varrho$ and thus it is continuous in ϱ. Since by (1)

$$\lim_{\varrho\to\infty}\log G_l(z, \varrho) = -\infty,$$

the convexity of $\log G_l(z, \varrho)$ implies its decrease on (ϱ', ∞) as was stated above.

After these the results of [1] imply the formula

$$\Delta_l(z, \varrho, Z) = G_l(z, \varrho) = \max\left(\frac{|z|}{\varrho}g(z, \varrho); \frac{|z|}{\varrho^{l+1}}\right) \quad (|z| > \varrho), \tag{2}$$

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where

$$g(z, \varrho) = \frac{\varrho}{|z|} \limsup_{n\to\infty} \left\{ \max_{|t|=\varrho} \left| \frac{z^n-1}{t^n-1} - \frac{\omega_n(z, Z)}{\omega_n(t, Z)} \right| \right\}^{1/n} =$$

$$= \limsup_{n\to\infty} \left\{ \max_{|t|=\varrho} \left| 1 - \frac{\omega_n(z, Z)}{z^n-1} \frac{t^n-1}{\omega_n(t, Z)} \right| \right\}^{1/n}.$$

Now turning to the second and third problems of [1] we may assume that (geometric) overconvergence takes place at least at one point z_0, $|z_0|>\varrho$, because these problems have interest only from the point of view of the overconvergence. In this case we prove the following rather surprising (see [1]) result.

Theorem. *If* $\Delta_l(z_0, \varrho, Z)<1$ *for some* $|z_0|>\varrho$ *then*

$$\mathfrak{G} = \{z \mid \Delta_l(z, \varrho, Z) = 1\}$$

is a circle with center at the origin, $\Delta_l(z, \varrho, Z)<1$ *inside and* $\Delta_l(z, \varrho, Z)>1$ *outside this circle.*

Proof. Let

$$g_n(z, t) = \left| 1 - \frac{\omega_n(z, Z)}{z^n-1} \frac{t^n-1}{\omega_n(t, Z)} \right|$$

and

$$h(\varrho) = \limsup_{n\to\infty} \left\{ \max_{|t|=\varrho} \left| 1 - \frac{t^n-1}{\omega_n(t, Z)} \right| \right\}^{1/n}.$$

First we prove the equality

$$g(z, \varrho) = h(\varrho) \quad (|z| \geqq \varrho > \varrho') \tag{3}$$

provided either side is less than one.

Suppose $g(z, \varrho)<q<1$. Then for some n_0 and $n \geqq n_0$ we have

$$g_n(z, t) \leqq q^n \quad (|t| = \varrho, \quad n \geqq n_0). \tag{4}$$

This yields

$$1-q^n \leqq \left| \frac{\omega_n(z, Z)}{z^n-1} \frac{t^n-1}{\omega_n(t, Z)} \right| \leqq 1+q^n \quad (|t| = \varrho),$$

and so for any $|t_1|=|t|=\varrho$,

$$1-Kq^n \leqq \left| \frac{t^n-1}{\omega_n(t, Z)} \Big/ \frac{t_1^n-1}{\omega_n(t_1, Z)} \right| \leqq 1+Kq^n. \tag{5}$$

For fixed t the function inside the absolute value marks is homomorphic for $|t_1|>\varrho$ without zeros and with removal singularity at $t_1=\infty$, so we obtain from the maximum modulus principle that (5) holds for all $|t_1|\geqq\varrho$. Letting $t_1\to\infty$ we get

$$1-Kq^n \leqq \left| \frac{t^n-1}{\omega_n(t, Z)} \right| \leqq 1+Kq^n \quad (|t| = \varrho).$$

Again by the maximum modulus principle this is true for every $|t| \geqq \varrho$, so specially

(6) $$1-Kq^n \leqq \left|\frac{z^n-1}{\omega_n(z, Z)}\right| \leqq 1+Kq^n.$$

(4) and (6) yield

(7) $$\left|\frac{t^n-1}{\omega_n(t, Z)}-\frac{t_1^n-1}{\omega_n(t_1, Z)}\right| \leqq Kq^n$$

for every $|t|=|t_1|=\varrho$ and hence also for every $|t|=\varrho \leqq |t_1|$. Letting here $t_1 \to \infty$ it follows

$$\left|1-\frac{t^n-1}{\omega_n(t, Z)}\right| \leqq Kq^n \quad (|t|=\varrho,\ n \geqq n_0)$$

by which $h(\varrho) \leqq q$. Since $g(z, \varrho)<q<1$ was arbitrary, we obtain that $h(\varrho) \leqq g(z, \varrho)$.

Now let us suppose conversely that $h(\varrho)<q<1$. Then for some n_0 we have

$$\left|1-\frac{t^n-1}{\omega_n(t, Z)}\right| < q^n \quad (|t|=\varrho,\ n \geqq n_0)$$

Applying the maximum modulus principle once more we obtain that

$$\frac{z^n-1}{\omega_n(z, Z)}=1+\varepsilon_n(z), \quad |\varepsilon_n(z)| \leqq q^n, \quad |z| \geqq \varrho,$$

by which

$$\frac{\omega_n(z, Z)}{z^n-1}=1+\eta_n(z), \quad |\eta_n(z)| \leqq Kq^n.$$

Multiplying this by

$$\frac{t^n-1}{\omega_n(t, Z)}=1+\varepsilon_n(t), \quad |\varepsilon_n(t)| \leqq q^n, \quad |t|=\varrho$$

it follows readily that

$$g_n(z, t) \leqq Kq^n \quad (|t|=\varrho,\ |z| \geqq \varrho, \quad n \geqq n_0),$$

and the inequality $g(z, \varrho) \leqq h(\varrho)$ can be deduced as the opposite inequality above.

Let

$$\varphi(\varrho)=\max\left(\frac{h(\varrho)}{\varrho};\ \frac{1}{\varrho^{l+1}}\right).$$

So far we have proved (see (2) and (3)) the formula

(8) $$\Delta_l(z; \varrho, Z)=|z|\varphi(\varrho) \quad (|z|>\varrho)$$

under the assumption $\min(\varphi(\varrho), \Delta_l(z)/|z|)<1/\varrho$, and by the first part of our paper here $\varphi(\varrho)$ $(\varrho>\varrho')$ is a monotonically decreasing convex function of ϱ with bounds

$$\frac{1}{\varrho^{l+1}} \leqq \varphi(\varrho) \leqq \frac{1}{\varrho-\varrho'}.$$

The Theorem follows immediately from (8).

Remarks. 1. We have proved somewhat more, namely for a $\varrho>\varrho'$, (geometric) overconvergence occurs if and only if

$$\limsup_{n\to\infty}\max_{|t|=\varrho}\left|1-\frac{t^n-1}{\omega_n(t,Z)}\right|^{1/n}<1,$$

and in this case the "overconvergence radius" is

$$\frac{1}{\varphi(\varrho)}=\min\left(\varrho^{l+1};\ \frac{\varrho}{\limsup\limits_{n\to\infty}\max\limits_{|t|=\varrho}\left|1-\dfrac{t^n-1}{\omega_n(t,Z)}\right|^{1/n}}\right).$$

2. The Theorem does not hold without the assumption "$\Delta_l(z_0,\varrho,Z)<1$ for some $|z_0|>\varrho$". Indeed, if the points $z_{k,n}\,(1\leqq k\leqq n)$ are "very near to $(-1)^n$" then the interior of the set

$$\{z\,|\,\Delta_l(z,\varrho,Z)<1\}$$

is the common part of the discs $|z-1|<\varrho-1$ and $|z+1|<\varrho-1$ and $\mathfrak{G}$ is on its boundary.

3. The formula (8) yields very easily the following result of J. Szabados and R. S. Varga (see [2, Theorem 2, 3]): If

$$\limsup_{n\to\infty}\sqrt[n]{\max_{1\leqq k\leqq n}|z_{k,n}-\exp 2\pi ik/n|}\leqq\delta<1$$

then

$$\Delta_l(z,\varrho,Z)\leqq\frac{|z|}{\varrho}\max\left(\frac{1}{\varrho^l},\delta\right).$$

Indeed, for any $\varepsilon>0$ we have for large n

$$\left|1-\frac{t^n-1}{\omega_n(t,Z)}\right|^{1/n}=\left|1-\prod_{k=1}^{n}\frac{t-\exp 2\pi ik/n}{t-z_{k,n}}\right|^{1/n}\leqq$$

$$\leqq\left|\prod_{k=1}^{n}\left(1+\frac{(\delta+\varepsilon)^n}{\varrho-\varrho'}\right)-1\right|^{1/n}\leqq(Kn)^{1/n}(\delta+\varepsilon),$$

and so $h(\varrho)\leqq\delta$.

References

[1] J. Szabados and R. S. Varga, On the overconvergence of complex interpolating polynomials. II. Domain of geometric convergence to zero, *Acta Sci. Math.*, **45** (1983), 377—380.

[2] J. Szabados and R. S. Varga, On the overconvergence of complex interpolating polynomials, *J. Approx. Theory*, to appear.

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