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One-sided convergence conditions for Lagrange interpolation based on the Jacobi roots

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To Professor B. Szőkefalvi-Nagy for his 70th birthday

1. Indroduction. We investigate the Lagrange interpolation for continuous functions on the Jacobi abscissas. By conditions of new type uniform convergence theorem will be established on the whole interval $[-1, 1]$.

2. Notations and preliminary results. Let $\alpha, \beta > -1$, say, $\alpha \geqq \beta$, and let

$$-1 \equiv x_{n+1,n}^{(\alpha,\beta)} < x_{nn}^{(\alpha,\beta)} < x_{n-1,n}^{(\alpha,\beta)} < \ldots < x_{2n}^{(\alpha,\beta)} < x_{1n}^{(\alpha,\beta)} < x_{0n}^{(\alpha,\beta)} \equiv 1 \tag{2.1}$$

be the roots of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ $(n=1, 2, \ldots;$ see e.g. G. SZEGŐ [3]). Let us denote by

$$L_n^{(\alpha,\beta)}(f, x) = \sum_{k=1}^{n} f(x_{kn}^{(\alpha,\beta)}) l_{kn}^{(\alpha,\beta)}(x), \quad n = 1, 2, \ldots, \tag{2.2}$$

the Lagrange interpolatory polynomials of degree $\leqq n-1$ based on the nodes (2.1), i.e., $l_{kn}^{(\alpha,\beta)}(x)$ is the k-th fundamental polynomial of the Lagrange interpolation $(n=1, 2, \ldots)$. If $f \in C$ (f is continuous on $[-1, 1]$) then $L_n^{(\alpha,\beta)}(f, x)$ generally do not tend uniformly to $f(x)$ in $[-1, 1]$ (if $n \to \infty$). However, if we suppose

a) $$\omega(f, t) = \begin{cases} o(|\ln t|^{-1}) & \text{if} \quad -1 < \alpha \leqq -1/2 \\ o(t^{\alpha+1/2}) & \text{if} \quad -1/2 < \alpha < 1/2 \end{cases}$$

when $t \to 0$, or

b) $f \in BC$ if $-1 < \alpha < 1/2$,

then

$$\lim_{n \to \infty} \|L_n^{(\alpha,\beta)}(f, x) - f(x)\| = 0 \tag{2.3}$$

(see [3], 14.4 and P. VÉRTESI [4], respectively). Here $\omega(f, t)$ is the modulus of continuity of f in $[-1, 1]$, $BC = \{f; f \in C$ and is of bounded variation on $[-1, 1]\}$ and $\|g(x)\|_{[a,b]} = \sup_{a \leqq x \leqq b} |g(x)|$; $\|\cdot\|$ stands for $\|\cdot\|_{[-1,1]}$.

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In this note a general convergence criterion is proved, from which, among others, the above mentioned theorems can be deduced.

3. Results. 3.1. We say that $f \in C$ satisfies the one-sided Dini—Lipschitz condition (shortly $f \in sDL$) if

$$f(x+h)-f(x) \leqq \frac{\varepsilon(h)}{|\ln h|}, \quad -1 \leqq x \leqq x+h \leqq 1, \tag{3.1}$$

where $\varepsilon(h) \geqq 0$ $(h \geqq 0)$ and $\lim_{h\to 0} \varepsilon(h)=0$. This definition was introduced by G. P. NEVAI [2]. He proved that for any fixed $\alpha, \beta > -1$ and $[a, b] \subset (-1; 1)$,

$$\lim_{n\to\infty} \|L_n^{(\alpha,\beta)}(f, x)-f(x)\|_{[a,b]} = 0 \quad \text{if} \quad f \in sDL.$$

3.2. According to (3.1) and L. V. ZIZIASVILI [7] we shall define the next δ-modulus for a bounded function defined on $[-1, 1]$ as follows:

$$\delta(f, t) = \sup_{\substack{0 \leqq h \leqq t \\ -1 \leqq x \leqq x+h \leqq 1}} [f(x+h)-f(x)], \quad t \geqq 0. \tag{3.2}$$

It is easy to see the next properties.

1) $0 \leqq \delta(f, t) \leqq \omega(f, t)$,

2) $\delta(f, t) \leqq \delta(f, T)$ if $0 \leqq t \leqq T$,

3) $\lim_{t\to 0} \delta(f, t)=0$ if $f \in C$,

4) $\delta(f, t)=0$ for any $0 \leqq t \leqq t_0$ $(t_0>0)$ if $f \in C$, and is monotone decreasing,

5) $\delta(f, nt) \leqq n\delta(f, t)$, $\delta(f, \lambda t) \leqq (\lambda+1)\delta(f, t)$ where n is a positive integer, λ is a positive real number,

6) $\delta(f_1+f_2, t) \leqq \delta(f_1, t)+\delta(f_2, t)$.

Moreover, by definition

$$f(x)-f(x+h)+\delta(f, t) \geqq 0, \quad 0 \leqq h \leqq t. \tag{3.3}$$

Finally, illustrating 1) let us remark that, e.g., for $g(x)=(x+1)^{1/2}-(x+1)^{1/4}$ we have $c_1 t^{1/2} \leqq \delta(f, t) \leqq c_2 t^{1/2}$ but $c_3 t^{1/4} \leqq \omega(f, t) \leqq c_4 t^{1/4}$.

3.3. By these definitions we can prove

Theorem 3.1. *Let* $-1<\gamma=\max(\alpha, \beta)<1/2$ *be fixed. If* $f \in C$ *and*

$$\delta(f, t) = \begin{cases} o(|\ln t|^{-1}) & \text{if} \quad -1 < \gamma \leqq -1/2 \\ o(t^{\gamma+1/2}) & \text{if} \quad -1/2 < \gamma < 1/2 \end{cases} \tag{3.4}$$

then

$$\lim_{n\to\infty} \|L_n^{(\alpha,\beta)}(f, x)-f(x)\| = 0. \tag{3.5}$$

3.4. To obtain (3.5) from 2a) we remark that from 2a) (by 3.2.1)) we get (3.4); moreover if $f \in BC$ then $f=f_1-f_2$ where f_1 and f_2 are monotone decreasing, i.e., $\delta(f_1, t)=\delta(f_2, t)\equiv 0$. Hence by Theorem 3.1 we have (3.5). These mean, Theorem 3.1 includes the statements of 2, indeed.

3.5. If $\gamma\geqq 1/2$, (3.4) generally does not involve (3.5) even if $\delta(f, t)=0$. More exactly if $\mu=\min(\alpha, \beta)$ then we have

Theorem 3.2. *If $\gamma>1/2$ or, if $\gamma\geqq 1/2$ and $\mu=1/2$, then there exists a continuous monotone decreasing function g for which* (3.5) *does not hold.*

3.6. Remarks. a) Theorems corresponding to Theorems 3.1 and 3.2 can be obtained for continuous monotone increasing functions g considering that now $(-g)$ is monotone decreasing.

b) It is worthwhile to state the next

Corollary 3.3. *If $-1<\gamma<1/2$ then for any $f\in BC$ we have* (3.5). *On the other hand, if $\gamma>1/2$ or, if $\gamma\geqq 1/2$ and $\mu=1/2$, then* (3.5) *does not hold for a certain $g\in BC$.*

c) It is easy to prove Theorem 3.1 if $-1\leqq\beta\leqq\alpha<1/2$. Further, we can prove the corresponding theorems if we consider the Lagrange polynomials based on $\{x_{kn}^{(\alpha,\beta)}\}_{k=0}^{n}$, $\{x_{kn}^{(\alpha,\beta)}\}_{k=1}^{n+1}$ or $\{x_{kn}^{(\alpha,\beta)}\}_{k=0}^{n+1}$, respectively. Omitting the details we refer to [4] and [6].

4. Proofs. 4.1. Proof of Theorem 3.1. By $f=f(x), f_i=f(x_i)$ and $l_i=l_i(x)$,

$$(4.1)\quad \begin{aligned} 2[f(x)-L_n^{(\alpha,\beta)}(f, x)] &= 2\sum_{k=1}^{n}[f(x)-f(x_k)]\,l_k(x) = \\ &= (f-f_1)l_1+\sum_{k=1}^{n-1}(f-f_k)(l_k+l_{k+1})+\sum_{k=1}^{n-1}(f_k-f_{k+1})l_{k+1}+(f-f_n)l_n. \end{aligned}$$

To estimate the sums first we prove

Lemma 4.1. *Let $-1<\alpha, \beta$ and $\varepsilon, \eta>0$ be fixed. If $k\geqq M_1$, $\vartheta_k\leqq\pi-\varepsilon$; then for any $x\in[-1+\eta, 1]$ we have*

$$(4.2)\qquad |l_{kn}^{(\alpha,\beta)}(x)+l_{k+1,n}^{(\alpha,\beta)}(x)| = O(1)|\bar{l}_{kn}^{(\alpha,\beta)}(x)|\left[\frac{1}{k}+\frac{k}{(k+j)(|k-j|+1)}\right]$$

uniformly in x and k. Here $|\bar{l}_k(x)|=\max(|l_k(x)|, |l_{k+1}(x)|)$, and M_1 depends on α and β.

To obtain (4.2) we shall use the next relations. If $x_{kn}=x_{kn}^{(\alpha,\beta)}=\cos\vartheta_{kn}^{(\alpha,\beta)}$, $0\leqq k\leqq n+1$ (with $x_0\equiv 1, x_{n+1}\equiv -1$), then

$$(4.3)\qquad \vartheta_{k+1,n}^{(\alpha,\beta)}-\vartheta_{kn}^{(\alpha,\beta)}\sim\frac{1}{n},\quad k=0, 1, \dots, n;$$

$$(4.4)\qquad |P_n^{(\alpha,\beta)}(x)|\sim|\vartheta-\vartheta_j|\vartheta_j^{-\alpha-1/2}n^{1/2}\sim|x-x_j|\vartheta_j^{-\alpha-3/2}n^{1/2}$$

uniformly in $x\in[-1+\eta, 1]$; moreover, with $P_n^{(\alpha,\beta)}(x)=P_n(x)$,

$$P_n'(x_k)=(-1)^{k-1}\sqrt{\frac{n}{\pi}}\,\frac{1+O(k^{-1})}{2\left(\sin\frac{\vartheta_k}{2}\right)^{\alpha+3/2}\left(\cos\frac{\vartheta_k}{2}\right)^{\beta+3/2}} \tag{4.5}$$

uniformly in k if $k\geqq M_0$ and $0<\vartheta_k\leqq\pi-\varepsilon$,

$$|P_n'(x_k)|\sim k^{-\alpha-3/2}n^{\alpha+2},\quad 0\leqq\vartheta_k\leqq\pi-\varepsilon, \tag{4.6}$$

$$P_n^{(\alpha,\beta)}(x)=(-1)^nP_n^{(\beta,\alpha)}(-x). \tag{4.7}$$

(Here $x_j=\cos\vartheta_j$ is the nearest root to x $(j=j(n))$; for the symbol "$\sim$", which may depend on α,β,ε and η, see [3], 1.1; the sources of (4.3)—(4.6) can be found in [4]; ($\varepsilon>0$ and $\eta>0$ are arbitrary fixed values).)

If $|k-j|>1$, we can write

$$l_k(x)+l_{k+1}(x)=P_n(x)\{[P_n'(x_k)(x-x_k)]^{-1}+[P_n'(x_{k+1})(x-x_{k+1})]^{-1}\}. \tag{4.8}$$

It is easy to see that

$$\{\dots\}=\frac{P_n'(x_k)+P_n'(x_{k+1})}{P_n'(x_k)P_n'(x_{k+1})(x-x_{k+1})}+\frac{x_k-x_{k+1}}{P_n'(x_k)(x-x_k)(x-x_{k+1})}=I_1+I_2.$$

If $k\geqq M_0$, $\vartheta_k\leqq\pi-\varepsilon$, then by (4.5) and $K(\vartheta)=2\left(\sin\frac{\vartheta}{2}\right)^{\alpha+3/2}\left(\cos\frac{\vartheta}{2}\right)^{\beta+3/2}$ we obtain after a simple calculation, that

$$|P_n'(x_k)+P_n'(x_{k+1})|=\sqrt{\frac{n}{\pi}}\left|\frac{[1+O(k^{-1})]K(\vartheta_{k+1})-[1+O(k^{-1})]K(\vartheta_k)}{K(\vartheta_k)K(\vartheta_{k+1})}\right|\sim\frac{\sqrt{n}}{kK(\vartheta_k)},$$

if k is big enough. I.e., if $k\geqq M_1$ and $\vartheta_k\leqq\pi-\varepsilon$ then $I_1\sim[kP_n'(x_k)(x-x_k)]^{-1}$. On the other hand, by (4.3), $|(x_k-x_{k+1})/(x-x_{k+1})|\sim k[(k+j)(|k-j|+1)]^{-1}$, so $I_2\sim k\{[(k+j)(|k-j|+1)]|P_n'(x_k)|\,|x-x_k|\}^{-1}$. Now, using (4.8), we obtain (4.2) if $|k-j|>1$, $k\geqq M_1$, $\vartheta_k\leqq\pi-\varepsilon$. (We used that $|P_n'(x_k)(x-x_k)|\sim|P_n'(x_{k+1})(x-x_{k+1})|$.) The statement is obvious if $|k-j|\leqq1$.

By (4.2) we get as in [4]: If $\psi=\min(2;1.5-\alpha)$, then

$$\left|\sum_{k=1}^{n-1}(f-f_k)(l_k+l_{k+1})\right|=O(1)\sum_{i=1}^{n}\omega\left(f,\frac{\sin\vartheta}{n}\,i+\frac{i^2}{n^2}\right)\frac{1}{i^\psi} \tag{4.9}$$

uniformly in $x\in[-1+\eta,,1]$

(see, e.g., [4], 4.10, where $\sum|f-f_k|\,|l_kk^{-1}|$ (which, by (4.2), is analogous to $\sum|f-f_k|\,|l_k+l_{k+1}|$) is estimated).

4.2. By (3.3) the second sum of (4.1) can be written as follows ($\delta_k=\delta(f, x_k-x_{k+1})$, $l_k=f_k\equiv 0$ if $k\neq 1, 2, \ldots, n$):

$$\left|\sum_{k=1}^{n-1}(f_k-f_{k+1})l_{k+1}\right| \leqq \left|\sum_{k=1}^{n-1}(f_k-f_{k-1}-\delta_k)|l_{k+1}|\right|+\sum_{k=1}^{n-1}\delta_k|l_{k+1}| \leqq$$

$$\leqq \left|\sum_{k=1}^{n-1}(f_k-f_{k+1})|l_{k+1}|\right|+2\sum_{k=1}^{n-1}\delta_k|l_{k+1}| \leqq \left|\sum_{k=j-m+1}^{j+m-1}(f_k-f_{k+1})|l_{k+1}|\right|+$$

$$+\left|\sum_{k=1}^{j-m-1}\left[\sum_{i=1}^{k}(f_i-f_{i+1})\right](|l_{k+1}|-|l_{k+2}|)\right|+\left||l_{j-m+1}|\sum_{i=1}^{j-m}(f_i-f_{i+1})\right|+$$

$$+\left|\sum_{k=j+m}^{n-2}\left[\sum_{i=j+m}^{k}(f_i-f_{i+1})\right](|l_{k+1}|-|l_{k+2}|)\right|+\left||l_n|\sum_{i=j+m}^{n-1}(f_i-f_{i+1})\right|+$$

$$+2\sum_{k=1}^{n-1}\delta_k|l_{k+1}| = K_1+K_2+K_3+K_4+K_5+K_6,$$

where $1\leqq m=m(n)\leqq n$ will be determined later.

4.3. To estimate K_1 we need

Lemma 4.2. *Let* $-1<\alpha, \beta$ *and* $\eta>0$ *be fixed. Then*

$$\sum_{k=j-m}^{j+m}|l_{kn}^{(\alpha,\beta)}(x)| = O(1)[\ln 2m+m^{\alpha+1/2}] \tag{4.10}$$

uniformly in $x\in[-1+\eta, 1]$ *and* m, $1\leqq m\leqq n$.

If $n=O(m)$, (4.10) is well known ([3], 14.4). So let $m=o(n)$.

a) If $j\leqq m$, we obtain

$$\sum_{k=j-m}^{j+m} \leqq \sum_{k=1}^{j/2}+\sum_{k=j/2}^{2j}+\sum_{k=2j}^{2m} = J_1+J_2+J_3.$$

(Here and later $\sum_{k=a}^{b}$ stands for $\sum_{k=[a]}^{[b]}$.)

By (4.3)—(4.6), $l_k(x)=O(1)k^{\alpha+3/2}[j^{\alpha+1/2}(k+j)(|k-j|+1)]^{-1}$, which implies $J_1=O(1)$, $J_2=O(1)\ln 2j=O(1)\ln 2m$ and $J_3=O(1)(m/j)^{\alpha+1/2}=O(1)(1+m^{\alpha+1/2})$.

b) If $m\leqq j<2m$, we have

$$\sum_{k=j-m}^{j+m} \leqq J_1+J_2+\sum_{k=2j}^{3m},$$

which can be estimated as above.

c) Finally, if $j\geqq 2m$, we have

$$\sum_{k=j-m}^{j+m}|l_k(x)| = O(1)\sum_{k=j-m}^{j+m}(|k-j|+1)^{-1} = O(\ln 2m).$$

4.4. So by Lemma 4.2 we have

$$K_1 = O\left(\omega\left(f, \frac{1}{n}\right)\right)(\ln 2m + m^{\alpha+1/2}). \tag{4.11}$$

4.5. To estimate K_2 and K_4 we prove

Lemma 4.3. *Let* $-1<\alpha, \beta$ *and* $\eta>0$ *be fixed. Then*

$$\sum_{k=1}^{n} |l_{kn}^{(\alpha,\beta)}(x)+l_{k+1,n}^{(\alpha,\beta)}(x)| = O(1)\left(1+\frac{r(\alpha, n)}{j^{\alpha+1/2}}\right), \tag{4.12}$$

$$\sum_{k=j-m}^{j+m} |l_{kn}^{(\alpha,\beta)}(x)+l_{k+1,n}^{(\alpha,\beta)}(x)| = O(1)\left(1+\frac{r(\alpha, m)}{j^{\alpha+1/2}}\right), \tag{4.13}$$

$$\sum_{k=1}^{j-m} |l_{kn}^{(\alpha,\beta)}(x)+l_{k+1,n}^{(\alpha,\beta)}(x)| = O(1)\frac{\ln 2m}{m}, \tag{4.14}$$

$$\sum_{k=j+m}^{n} |l_{kn}^{(\alpha,\beta)}(x)+l_{k+1,n}^{(\alpha,\beta)}(x)| = O(1)\left(\frac{\ln 2m}{m}+\frac{r(\alpha, n)}{j^{\alpha+1/2}}\right) \tag{4.15}$$

uniformly in $x\in[-1+\eta, 1]$ *and* m, $1\leqq m\leqq n$. *Here*

$$r(u, v) = \begin{cases} v^{u-1/2} & \text{if } u \neq 1/2, \\ \ln v & \text{if } u = 1/2. \end{cases}$$

A. Indeed, to prove (4.12) we write the sum as follows:

$$\sum_{k=1}^{n} \dots = \sum_{k=1}^{M_1-1} \dots + \sum_{k=M_1}^{cn} \dots + \sum_{k=cn+1}^{n} \dots = I_1+I_2+I_3 \quad (0<c<1).$$

By (4.2)—(4.6), if $1\leqq k\leqq n-1$, then

$$|l_k(x)+l_{k+1}(x)| = O(1)\frac{k^{\alpha+1/2}}{j^{\alpha+1/2}(k+j)(|k-j|+1)}+\frac{k^{\alpha+5/2}}{j^{\alpha+1/2}(k+j)^2(|k-j|+1)^2}, \tag{4.16}$$

i.e., $I_1=O(j^{-\alpha-5/2})$. For I_2 we can write (if, say, $2j<cn$),

$$I_2 = \sum_{k=M_1}^{j/2} + \sum_{k=j/2+1}^{2j} + \sum_{k=2j+1}^{cn} = O(1)\left[\frac{1}{j}+1+\frac{r(\alpha, n)}{j^{\alpha+1/2}}\right].$$

If, say, $\eta=2\varepsilon$, then by (4.7),

$$I_3 = O(1)\frac{n^\alpha}{j^{\alpha+1/2}}\sum_{k=1}^{n}\frac{k^{\beta+1/2}}{n^{\beta+2}} = O(1)\frac{n^{\alpha-1/2}}{j^{\alpha+1/2}}.$$

By these estimations we obtain (4.12). Similar arguments apply for the other cases including the estimation of the n-th term. Now we sketch the remaining three formulae.

B. To prove (4.13), we write

$$\sum_{k=j-m}^{j+m} = \sum_{k=j-m}^{j} + \sum_{k=j+1}^{j+m} = J_1+J_2.$$

Here $J_1<\sum_{k=1}^{2j}$, which can be estimated by $O(1)$ (see I_2, above). Now, by (4.12), we can suppose that $m=o(n)$. To estimate J_2, we proceed as follows.

a) First let $m\leqq 2j$. If $n=O(j)$, then $\sum_{k=j}^{j+m}\leqq\sum_{k=j}^{(1+\varrho)j}$ for arbitrary $\varrho>0$ if n is big enough. But this sum can be estimated by $O(1)$, if ϱ is small enough. On the other hand, if $j=o(n)$, then $\sum_{k=j}^{j+m}\leqq\sum_{k=j}^{3j}$ which again can be estimated by $O(1)$.

b) If $m\geqq 2j$, then we have $\sum_{k=j}^{j+m}\leqq\sum_{k=j}^{2j}+\sum_{k=2j}^{m}$, which can be estimated by $O(1)(1+r(\alpha, m)j^{-\alpha-1/2})$ (see the previous estimations for I_2 and I_3).

C. To obtain (4.14), we argue as follows. If $m\leqq j/2$, then $\sum_{k=1}^{j-m}\leqq\sum_{k=1}^{j/2}+\sum_{k=j/2}^{j-m}$, which can be estimated by $O(j^{-1}\ln 2j)=O(m^{-1}\ln 2m)$ (see (4.16) and the above considerations). On the other hand, if $j>m>j/2$, we can estimate as follows:

$$\sum_{k=1}^{j-m} < \sum_{k=1}^{j/2} = O(j^{-1}) = O(m^{-1}) \quad \text{(see (4.16))}.$$

D. Now we estimate $\sum_{k=j+m}^{n}$.

a) First let $m\leqq 2j$. Then, if, say, $3j<cn$ $(0<c<1)$, we can write

$$\sum_{k=j+m}^{n} = \sum_{k=j+m}^{3j} + \sum_{k=3j+1}^{n} = J_3+J_4.$$

Here $J_3=O(j^{-1}(\ln 2j-\ln 2m))=O(m^{-1}\ln 2m)$ (see (4.16)), moreover $J_4=$ $=O(r(\alpha, n)j^{-\alpha-1/2})$ (see the corresponding parts of I_2 and I_3).

b) If $m>2j$, then

$$\sum_{k=j+m}^{n} < \sum_{k=m}^{2m} + \sum_{k=2m}^{n} = J_5+J_6.$$

By (4.16), $J_5=O(m^{\alpha-1/2}j^{-\alpha-1/2})$. Further, if, say, $3m<cn$ $(0<c<1)$, we can write $J_6=\sum_{k=2m}^{cn}+\sum_{k=cn+1}^{n}=J_7+I_3$. Here by (4.16) $J_7=O(r(\alpha, m)j^{-\alpha-1/2})$, moreover using the estimation for I_3, finally we get (4.15).

The remaining cases can be treated analogously. Thus we have proved Lemma 4.2.

4.6. Let us estimate K_2. By 4.2 and (4.14), using the fact that $\big||\alpha|-|\beta|\big|\leqq|\alpha+\beta|$, we get

$$(4.17)\quad K_2=\left|\sum_{k=1}^{j-m-1}(f_1-f_{k+1})(|l_{k+1}|-|l_{k+2}|)\right|\leqq\sum_{k=1}^{j-m-1}|f_{k+1}-f_1|\,|l_{k+1}+l_{k+2}|\leqq$$

$$\leqq 2\|f\|\sum_{k=1}^{j-m-1}|l_{k+1}+l_{k+2}|=\|f\|\,O\left(\frac{\ln 2m}{m}\right).$$

Similarly,

$$(4.18)\qquad K_4\leqq 2\|f\|\sum_{k=j+m}^{n-2}|l_{k+1}+l_{k+2}|=\|f\|\,O(1)\left(\frac{\ln 2m}{m}+\frac{r(\alpha,n)}{j^{\alpha+1/2}}\right).$$

4.7. To estimate K_3 and K_5 we remark that for any $\alpha>-1$, $|l_k(x)|=$ $=O(1)k^{\alpha+3/2}j^{-\alpha-1/2}(k+j)^{-1}(|k-j|+1)^{-1}$ if $x\in[-1+\eta,1]$ and $\vartheta_k\leqq\pi-\varepsilon$, which can be obtained using the above arguments. I.e., if $j-m\geqq2$, we have

$$(4.19)\qquad K_3=O(1)\frac{(j-m)^{\alpha+3/2}}{j^{\alpha+1/2}(2j-m)m}|f_1-f_{j-m+1}|=\frac{O(1)\|f\|}{m}.$$

If $0\leqq j-m<2$, then $K_3=O(1)j^{-\alpha-3/2}m^{-1}\omega(f,n^{-2})=O(1)\|f\|m^{-1}$. For K_5 we have by (4.7)

$$(4.20)\qquad K_5=O(1)\frac{n^{\alpha-\beta-2}}{j^{\alpha+1/2}}|f_n-f_{j+m}|=\frac{O(1)\|f\|}{n^{1/2}}.$$

4.8. Now we estimate $|[f(x)-f(x_1)]l_1(x)|=K_7$ (see (4.1)). By $l_k(x)=$ $=O(1)k^{\alpha+3/2}[j^{\alpha+1/2}(k+j)(|k-j|+1)]^{-1}$,

$$(4.21)\quad K_7=O(1)\omega\left(f,\frac{j^2}{n^2}\right)\frac{1}{j^{\alpha+5/2}}=O(1)\omega\left(f,\frac{j^{3/2}}{n^{3/2}}\right)\frac{1}{j^{3/2}}=O(1)\omega\left(f,\frac{1}{n^{3/2}}\right).$$

Using similar estimations we get that

$$(4.22)\qquad K_8\overset{\text{def}}{=}|f(x)-f(x_n)|\,|l_n(x)|=O(n^{-3/2}).$$

4.9. Summarizing the estimations (4.1), (4.9), (4.11) and (4.17)—(4.22), we have that for $x\in[-1+\eta,1]$ and $-1<\alpha<1/2$,

(4.23)

$$|L_n^{(\alpha,\beta)}(f,x)-f(x)|=O(1)\left[\sum_{i=1}^{n}\omega\left(f,\frac{i}{n}\right)\frac{1}{i^{\psi}}+\omega\left(f,\frac{1}{n}\right)(\ln 2m+m^{\alpha+1/2}+n^{\alpha-1/2})+\right.$$

$$\left.+\|f\|\left(\frac{\ln 2m}{m}+\frac{n^{\alpha-1/2}}{j^{\alpha+1/2}}\right)+\delta\left(f,\frac{1}{n}\right)\sum_{k=1}^{n}|l_k(x)|+n^{-1/2}\right].$$

Here $O(1)$ does not depend on f. Now let $m(n)$ tend to infinity (with n) so that, say, $\lim_{n\to\infty}m\omega\left(\frac{1}{n}\right)=0$. Using that for $\alpha>-\frac{1}{2}$, $\sum_{k=1}^{n}|l_k(x)|=O(n^{\alpha+1/2})$ and that

$\delta\left(f, \frac{1}{n}\right)=o(1/n^{\alpha+1/2})$, we obtain the statement for $x\in[-1+\eta, 1]$ in virtue of $\sum_{i=1}^{n}\omega\left(f, \frac{i}{n}\right)i^{-\psi}=o(1)$ (see [4], 3.2). If $-1<\alpha\leqq-1/2$, we can use the relations $\sum_{k=1}^{n}|l_k(x)|=O(\ln n)$ and $\delta(f, 1/n)=o(1/\ln n)$. Now by (4.7) we obtain the theorem for the whole interval $[-1, 1]$.

4.10. Proof of Theorem 3.2. First let $\alpha=1/2+2\varrho$, $\varrho>0$ and $\beta>-1$, say. Furthermore, let $\omega(t)$ be a modulus of continuity with $\lim_{t\to+0}\omega(t)t^{-1}=\infty$, $\omega_2(t):=t\,\omega(t)$, and $C(\omega_2)=\{f; f\in C \text{ and } \omega(f', t)\leqq a(f)\omega(t)\}$. We quote P. VÉRTESI [5], Theorem 8.1: There exists a function $h\in C(\omega_2)$ for which

$$\overline{\lim_{n\to\infty}}\frac{|L_n^{(\alpha,\beta)}(h, 1)-h(1)|}{n^{\alpha+1/2}\omega_2\left(\frac{1}{n}\right)}\geqq 1 \quad \text{for any} \quad \alpha, \beta>-1. \tag{4.24}$$

If $\omega_2(t)=t^{1+\varrho}$, then by (4.24), $|L_n(h, 1)-h(1)|\geqq n^{1+2\varrho}n^{-1-\varrho}=n^{\varrho}$ $(n=n_1, n_2, \ldots)$ i.e., $\overline{\lim_{n\to\infty}}|L_n(h, 1)|=\infty$. On the other hand, $h\in\text{Lip}\,1$, from where $h\in BC$, i.e., $h=h_1-h_2$, where $h_1, h_2\in C$ and are monotone decreasing. But then, say, $\overline{\lim_{n\to\infty}}|L_n(h_1, 1)|=\infty$.

4.11. Now let $\alpha=\beta=1/2$. To obtain Theorem 3.2, we use the next statement (see H. HAHN [1]): If for the arbitrary fixed interpolatory matrix $\{x_{kn}\}$ $(k=1,2,\ldots,n;\ n=1,2,\ldots)$ in $[-1, 1]$, the interpolatory polynomials $L_n(f, x)$ converge for every function f of bounded variation at any point where f is continuous, then if $t\in[-1, 1]$ and differs from the nodes x_{kn}, we have

$$\lim_{n=\infty}\sum_{x_{kn}<t} l_{kn}(x)=0 \quad \text{if} \quad t<x \tag{4.25}$$

$$\lim_{n=\infty}\sum_{x_{kn}>t} l_{kn}(x)=0 \quad \text{if} \quad t>x. \tag{4.26}$$

We shall see that, e.g., (4.25) does not hold if $\alpha=\beta=1/2$, $x=1$ and $t=0$. Indeed, if $n=4s$, then by [3], (4.1.7) and $x=\cos\vartheta$,

$$\sum_{x_k<0} l_k^{(1/2,1/2)}(1)=\sum_{x_k<0}\left[(-1)^{k-1}\frac{\sin(n+1)\vartheta}{(n+1)\sin\vartheta}\frac{\sin^2\vartheta_k}{(\cos-\cos\vartheta_k)}\right]_{\vartheta=0}=$$

$$=\sum_{x_k<0}(-1)^{k-1}(1+x_k)=(x_{2s+1}-x_{2s+2})+(x_{2s+3}-x_{2s+4})+\ldots+(x_{4s-1}-x_{4s})>1-x_s=$$

$$=1-\cos\frac{s\pi}{4s+1}>\frac{2-\sqrt{2}}{2}.$$

I.e., there exists an $f\in BC$ for which (3.5) does not hold. As in 4.10, we get the proper monotone decreasing function.

4.12. Finally, if $\alpha=1/2$ and $\beta>1/2$, then by (4.7) and the argument of 4.10 we obtain the statement.

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