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Asymptotically commuting finite rank unitary operators without commuting approximants

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Dedicated to Professor Béla Szőkefalvi-Nagy on the occasion of his 70th birthday

The following is an old unsolved problem: Given selfadjoint operators $A_n, B_n \in \mathscr{L}(\mathscr{H}_n)$, $\dim \mathscr{H}_n < \infty$ $(n=1,2,\ldots)$, such that $\sup_n (\|A_n\|+\|B_n\|)<\infty$ and $\lim_{n\to\infty} \|[A_n, B_n]\|=0$, do there exist selfadjoint operators $A'_n, B'_n \in \mathscr{L}(\mathscr{H}_n)$ so that $[A'_n, B'_n]=0$ and $\lim_{n\to\infty} (\|A_n-A'_n\|+\|B_n-B'_n\|)=0$? We present in this note an example showing that the answer to the corresponding question for unitaries instead of selfadjoints is negative.

We shall take $\mathscr{H}_n=\ell^2(\mathbf{Z}/n\mathbf{Z})$ consisting of functions $\xi\colon \mathbf{Z}/n\mathbf{Z}\to\mathbf{C}$ and consider the unitary operators

$$\left.\begin{aligned}(U_n\xi)(k+n\mathbf{Z}) &= \xi(k-1+n\mathbf{Z}),\\ (V_n\xi)(k+n\mathbf{Z}) &= \exp(2k\pi i/n)\,\xi(k+n\mathbf{Z})\end{aligned}\right. \quad (k=0,1,\ldots,n-1).$$

Proposition. *Let U_n, V_n be the unitary operators defined above. Then we have $\lim_{n\to\infty}\|[U_n, V_n]\|=0$, but there do not exist unitary operators $U'_n, V'_n \in \ell^2(\mathbf{Z}/n\mathbf{Z})$ such that $[U'_n, V'_n]=0$ and $\lim_{n\to\infty}(\|U_n-U'_n\|+\|V_n-V'_n\|)=0$.*

Proof. We have $U_nV_n=\exp(-2\pi i/n)V_nU_n$, which implies $\|[U_n, V_n]\|\to 0$ as $n\to\infty$. Assuming the existence of the commuting approximants U'_n, V'_n we will reach a contradiction.

Consider on the unit circle $\mathbf{T}=\{z\in\mathbf{C}\,\big|\,|z|=1\}$ the arcs $\Gamma, \Gamma', \Gamma'', \Phi^{(1)}, \Phi^{(2)}$ given respectively by

$$\Gamma\colon \frac{\pi}{5}\leqq \arg z<\frac{4\pi}{5},\quad \Gamma'\colon \frac{2\pi}{5}\leqq \arg z<\frac{3\pi}{5},\quad \Gamma''\colon 0\leqq \arg z<\pi,$$

$$\Phi^{(1)}\colon 0\leqq \arg z<\frac{2\pi}{5},\quad \Phi^{(2)}\colon \frac{3\pi}{5}\leqq \arg z<\pi.$$

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Let E_n be the spectral projection of V_n' corresponding to Γ and let $E_n', E_n'', F_n^{(1)}, F_n^{(2)}$ be the spectral projections of V_n corresponding to $\Gamma', \Gamma'', \Phi^{(1)}, \Phi^{(2)}$, respectively. Note that $E_n''=E_n'+F_n^{(1)}+F_n^{(2)}$. Also, since $[V_n', U_n']=0$, we have $[U_n', E_n]=0$ and hence

$$\|[U_n, E_n]\| \to 0 \quad \text{as} \quad n \to \infty. \tag{1}$$

We shall use the following folklore-type fact. If N_n, N_n' are normal operators, $\|N_n-N_n'\|\to 0$, $\|N_n\|<C$ and P_n, P_n' are spectral projections of N_n, respectively N_n', corresponding to Borel sets Ω, Ω' such that $\bar{\Omega}\cap\bar{\Omega}'=\emptyset$, then we have $\|P_nP_n'\|\to 0$. This gives, in particular,

$$\lim_{n\to\infty} \|(I-E_n'')E_n\| = \lim_{n\to\infty} \|(I-E_n)E_n'\| = 0.$$

It is also easily seen that $\lim_{n\to\infty} \|F_n^{(1)}E_nF_n^{(2)}\|=0$. So we find selfadjoint projections $\tilde{E}_n$ such that $E_n'\leqq\tilde{E}_n\leqq E_n''$ and $\lim_{n\to\infty}\|\tilde{E}_n-E_n\|=0$. One may define $\tilde{E}_n$ for instance as follows. Let $X_n=E_n'+F_n^{(1)}E_nF_n^{(1)}+F_n^{(2)}E_nF_n^{(2)}$ so that $\|X_n-E_n\|\to 0$ and hence $\|X_n^2-X_n\|\to 0$. Define $\tilde{E}_n$ (for n big enough) as the spectral projection of X_n for the interval [1/2, 2]. Remark also that $\tilde{E}_n=\tilde{F}_n^{(1)}+E_n'+\tilde{F}_n^{(2)}$ where $\tilde{F}_n^{(1)}\leqq$ $\leqq F_n^{(1)}$, $\tilde{F}_n^{(2)}\leqq F_n^{(2)}$ are selfadjoint projections.

Consider now the projection $E_n^+=F_n^{(1)}+E_n'+\tilde{F}_n^{(2)}$ and assume from now on $n\geqq 10$. We have

$$E_n^+ \leqq E_n'' \tag{2}$$

and

$$(I-E_n^+)U_nF_n^{(1)} = (I-E_n^+)U_n\tilde{F}_n^{(1)} = 0,$$

so that

$$(I-E_n^+)U_nE_n^+ = (I-E_n^+)U_n\tilde{E}_n = (I-E_n^+)(I-\tilde{E}_n)U_n\tilde{E}_n.$$

Since, by (1), $\lim_{n\to\infty}\|(I-\tilde{E}_n)U_n\tilde{E}_n\|=0$, we infer that

$$\lim_{n\to\infty} \|(I-E_n^+)U_nE_n^+\| = 0. \tag{3}$$

Define the isometric operator $W_n: \ell^2(\mathbf{Z}/n\mathbf{Z})\to\ell^2(\mathbf{Z}_{\geqq 0})$, by

$$(W_n\xi)(k) = \begin{cases} 0 & \text{if} \quad k \geqq n, \\ \xi(k+n\mathbf{Z}) & \text{if} \quad 0 \leqq k < n. \end{cases}$$

Then for $P_n^+=W_nE_n^+W_n^*$ and the unilateral shift S on $\ell^2(\mathbf{Z}_{\geqq 0})$, we have

$$W_n(I-E_n^+)U_nE_n^+W_n^* = W_n(I-E_n^+)W_n^*W_nU_nW_n^*W_nE_n^+W_n^* =$$
$$= (W_nW_n^*-P_n^+)SP_n^+ = (I-P_n^+)SP_n^+$$

since, by (2), $(W_nU_nW_n^*-S)P_n^+=0$ and $(I-W_nW_n^*)SP_n^+=0$. Thus we have rank $P_n^+<\infty$, $\text{s-}\lim_{n\to\infty} P_n^+=I$ and, using (3), $\lim_{n\to\infty}\|(I-P_n^+)SP_n^+\|=0$. This contradicts the non-quasitriangularity of the unilateral shift [1] and hence concludes the proof.

Remark. The approximation problems for selfadjoint and unitary operators can be interpreted in terms of singular extensions (see [2], [3]). Consider the C^*-algebra

$$\mathscr{A} = \{(T_n)_1^\infty \mid T_n \in \mathscr{L}(\mathscr{H}_n), \sup_n \|T_n\| < \infty\}$$

and $\mathscr{I} \subset \mathscr{A}$, the ideal of sequences $(T_n)_1^\infty$ such that $\lim_{n\to\infty} \|T_n\| = 0$. Then the approximation problem for selfadjoint operators amounts to the question whether every $*$-homomorphism $C(X) \to \mathscr{A}/\mathscr{I}$ can be lifted to a $*$-homomorphisms $C(X) \to \mathscr{A}$, where $X = [0, 1] \times [0, 1]$ and the problem for unitary operators to the same question for $X = \mathbf{T}^2$, the 2-torus. In connection with this we should mention that from our strong non-splitting result in [4] for the singular extension in the C^*-algebra of the Heisenberg group one can construct a $*$-homomorphism $C_0(\mathbf{R}^2) \to \mathscr{A}/\mathscr{I}$ which does not lift (here $C_0(\mathbf{R}^2)$ denotes the continuous functions on $\mathbf{R}^2$ vanishing at infinity). Adjoining a unit to $C_0(\mathbf{R}^2)$ one gets a C^*-algebra isomorphic to $C(S^2)$, where S^2 is the two-sphere, and hence the answer to the lifting problem is negative also for $X = S^2$. Like $[0, 1] \times [0, 1]$, the spaces $\mathbf{T}^2$ and S^2 are two-dimensional, but it seems that the counterexamples for $\mathbf{T}^2$ and S^2 are not due only to the dimension of these spaces but rather to their non-zero two-dimensional cohomology and hence it seems improbable that these examples will have a direct bearing on the problem for selfadjoint operators.

References

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