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On Jónsson modules over a commutative ring

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1. Introduction. Let R be a commutative ring with identity, let M be a unitary module over R, and let α be an infinite cardinal. Following the terminology of universal algebra [5], [3], we call M a *Jónsson α-module* over R if $|M|=\alpha$, while $|N|<\alpha$ for each proper submodule N of M. Our attention to this topic was attracted by a recent paper of SHELAH [13], who answered affirmatively the following old question of Kurosh: does there exist a Jónsson ω_1-group — that is, a group G of cardinality ω_1 such that each proper subgroup is countable? Like Shelah, we concentrate primarily on the cases where $\alpha\in\{\omega_0, \omega_1\}$ in this paper, because these are the cases of principal interest within our context.

If I is an ideal of R and if I, considered as an R-module, is a Jónsson α-module, then we refer to I as a *Jónsson α-ideal* of R. By passage to the idealization of R and an R-module M, the theory of Jónsson α-modules is equivalent to the corresponding theory for ideals, but we shall only occasionally make this transition to ideals via idealization.

Section 2 of the paper deals with Jónsson α-modules, Section 3 with Jónsson ω_0-modules, and Section 4 presents some pertinent examples. Corollary 3.2 shows that a finitely generated Jónsson α-module is simple, and hence the set of such modules over a given ring R is easily determined. Theorem 2.4 shows that if the cardinal α is countably inaccessible from below and if R belongs to the class $\mathscr{F}$ of rings over which each (**)-module is finitely generated (see Section 2 for terminology; in particular, $\mathscr{F}$ includes the class of Noetherian rings and the class of finite-dimensional chained rings), then each Jónsson α-module over R is finitely generated, hence simple; in particular, this result applies to Jónsson ω_1-modules over a ring in $\mathscr{F}$. Proposition 2.5 is in this context a useful result; it states that if M is a non-finitely generated

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Jónsson α-module over R, then $\operatorname{Ann}(M)$ is a prime ideal and $rM=M$ for each $r\in R-\operatorname{Ann}(M)$.

Assume that M is a non-finitely generated Jónsson ω_0-module over the ring R. Theorem 3.1 shows that there exists a maximal ideal Q of R such that $\operatorname{Ann}(x)$ is a Q-primary ideal of finite index for each nonzero element x of R; moreover, the powers of Q properly descend and $\bigcap_{i=1}^{\infty} Q^i$ is a prime ideal of R. It follows from Theorem 3.1 that in considering Jónsson ω_0-modules over R, there is no loss of generality in assuming that the module is faithful and R is a quasi-local integral domain. Proposition 3.2 shows that M can be expressed as the union of a strictly ascending sequence of cyclic submodules, and this leads both to a construction of classes of non-finitely generated Jónsson ω_0-modules by means of generators and relations (Theorem 3.5) and to a determination of the isomorphism class of non-finitely generated Jónsson ω_0-modules over a Prüfer domain J (Proposition 3.7 and the paragraph preceding that result).

The examples of Section 4 indicate certain restrictions on what can be said about the structure of a quasi-local domain D such that D admits a non-finitely generated Jónsson ω_0-module. Such a domain D need not be Noetherian, for example, and even for a Noetherian domain D, no restrictions can be placed on the (Krull) dimension of D.

All rings considered in this paper are assumed to be commutative and to contain an identity element; all modules considered are assumed to be unitary.

2. Jónsson modules. If R is a commutative ring with identity and M is a maximal ideal of R such that $|R/M|=\alpha$ is infinite, then R/M is a Jónsson α-module over R. One of our purposes in this section is to attempt to determine the class of rings S such that each Jónsson module over S arises essentially in this way — that is, as S/M for some maximal ideal M of S with infinite residue field.

The main results of this section are Corollary 2.3 and Theorem 2.4. In particular, Theorem 2.4 resolves the question of Jónsson modules over the rings normally encountered in commutative algebra. While the proof of Proposition 2.5 is not difficult, this result is an important tool in the development of Section 3 material.

According to the terminology of [2, Ex. 17, p. 245], the infinite cardinal α is said to be *regular* if $\alpha\neq\sum_{i\in I}\alpha_i$ for each nonempty family $\{\alpha_i\}_{i\in I}$ of cardinals with $|I|<\alpha$ and $\alpha_i<\alpha$ for each i. As noted by Simis [14], this condition is equivalent to the statement that there is no cofinal set of cardinality less than α in the set of ordinals preceding the first ordinal of cardinality α.

Proposition 2.1. *Assume that M is a Jónsson α-module over R, where α is a regular cardinal. If $\{M_i\}_{i\in I}$ is a nonempty family of proper submodules of M, where*

$|I|<\alpha$, then $M \neq \sum_{i \in I} M_i$. In particular, M is indecomposable and M has at most one maximal submodule.

Proof. Since $|M_i|<\alpha$ for each i and since α is regular, it follows that $|\sum_{i \in I} M_i|<$ $<\alpha$, and hence $\sum_{i \in I} M_i \neq M$. The statements in the second sentence of the proposition follow immediately from the first sentence.

Proposition 2.2. *Assume that each proper ideal of the ring R has cardinality less than $|R|$. Then either R is finite or R is a field.*

Proof. We prove that if $|R|=\alpha$ is infinite, then R is a field. Proposition 2.1 shows that R has a unique maximal ideal P. Since $|P|<\alpha$, it follows that $|R/P|=\alpha$; let $\{r_\beta\}_{\beta \in B}$ be a complete set of representatives of the residue classes of P in R. If $x \in P$, then $\{r_\beta x\} \subseteqq P$, so there exist distinct $\beta, \gamma \in B$ so that $r_\beta x = r_\gamma x$. Since $r_\beta - r_\gamma$ is a unit of R, then $x=0$, so $P=(0)$ and R is a field, as asserted.

Corollary 2.3. *Let M be an infinite, finitely generated R-module and let $\alpha=|M|$. Then M is a Jónsson module if and only if M is cyclic and* Ann (M) *is a maximal ideal of R such that $|R/\mathrm{Ann}\,(M)|=\alpha$.*

Proof. It's clear that the stated conditions are sufficient for M to be a Jónsson module. Conversely, if M is a Jónsson module and $M=Rm_1+Rm_2+\ldots+Rm_n$, then Proposition 2.1 implies that $M=Rm_i$ for some i. Thus, M and $R/\mathrm{Ann}\,(M)$ are isomorphic modules over R and over $R/\mathrm{Ann}\,(M)$, so that $R/\mathrm{Ann}\,(M)$ is a field of cardinality α by Proposition 2.2.

Following the terminology of [1], we call a module M a (**)-module if M cannot be expressed as the union of a strictly ascending sequence $M_1<M_2<\ldots<M_n\ \ldots$ of submodules; we denote by $\mathscr{F}$ the class of rings R such that each (**)-module over R is finitely generated (clearly a finitely generated module is a (**)-module for any R). Theorems 4.2, 4.7, and 4.10 of [1] show that $\mathscr{F}$ contains the subclasses of Noetherian rings, finite-dimensional chained rings, and W^*-rings; Theorem 6.1 of [10] shows that $\mathscr{F}$ also contains each ring R such that (1) R has Noetherian spectrum, (2) the descending chain condition for prime ideals is satisfied in R, and either (3) each ideal of R is countably generated, or (4) each ideal of R contains a power of its radical.

If α is an infinite cardinal, we say that α *is countably inaccessible from below* if $\alpha \neq \sum_{i \in I} \alpha_i$ for each nonempty countable family $\{\alpha_i\}_{i \in I}$ of cardinals $\alpha_i<\alpha$. According to this terminology, ω_0 is countably accessible from below, while each infinite cardinal with an immediate predecessor (in particular, ω_1) is countably inaccessible from below. The next result deals both with the concept of countable inaccessibility from below and with the class $\mathscr{F}$.

Theorem 2.4. *Assume that R is in the class $\mathscr{F}$ and that the cardinal α is countably inaccessible from below. To within isomoprhism, the set of Jónsson α-modules is $\{R/M_i\}_{i\in I}$, where $\{M_i\}_{i\in I}$ is the set of maximal ideals of R whose associated residue class field has cardinality α.*

Proof. Clearly each R/M_i is a Jónsson α-module over R. Conversely, let L be a Jónsson α-module over R. If $\{L_j\}_{j=1}^{\infty}$ is an ascending sequence of proper submodules of L, then as in the proof of Proposition 2.1, it follows that $L\neq \sum_{j=1}^{\infty} L_j$. Thus, L is a (**)-module over R, and since $R\in\mathscr{F}$, then L is finitely generated. It then follows from Corollary 2.3 that as an R-module, $L\cong R/M_i$ for some $i\in I$.

In our further consideration of Jónsson α-modules, we shall begin in Section 3 to concentrate our attention on the cases where $\alpha=\omega_0$ or $\alpha=\omega_1$. Even for ω_1, Theorem 2.4 resolves the question of Jónsson modules over the rings normally encountered in commutative algebra. Because ω_0 is countably accessible from below, however, Theorem 2.4 does not apply to this case. We know, in fact, that a Jónsson ω_0-module over a principal ideal domain need not be finitely generated; the p-quasicyclic group $Z(p^\infty)$, considered as a Z-module, illustrates this statement. (It is well-known, in fact, that the p-quasicyclic groups are the only Jónsson ω_0-modules over Z [6, Ex. 4, p. 105].)

We conclude Section 2 with a proposition and a corollary that are valid for arbitrary cardinals α. In particular, Proposition 2.5 is used frequently in the rest of this paper.

Proposition 2.5. *Let M be a Jónsson α-module over the ring R.*

(1) *If $r\in R$, then either $rM=M$ or $rM=(0)$.*

(2) Ann (M) *is a prime ideal of R.*

Proof. To prove (1), assume that $rM\neq M$ and let $N=\{m\in M|rm=0\}$. We show that $N=M$. We write rM as $\{rm_i\}_{i\in I}$, where $|I|<\alpha$. If $m\in M$, then $rm=rm_i$ for some i so that $m\in m_i+N$. It follows that $M=\bigcup_{i\in I}(m_i+N)$, and hence $|M|\leqq |I|\cdot|N|$. By hypothesis on M and I, we conclude that $|N|=\alpha$ so that $N=M$ as we wished to prove. It follows from (1) that if $x, y\in R-\mathrm{Ann}\,(M)$, then $M=xM=yM$, and hence $M=xyM$. Thus $xy\notin \mathrm{Ann}\,(M)$, and Ann (M) is prime in R, as asserted.

Corollary 2.6. *Assume that I is a Jónsson α-ideal of the ring R. If $I^2\neq(0)$, then I is a field, and hence I is a direct summand of R.*

Proof. Take $r, s\in I$ such that $rs\neq 0$. Then $rI=I=sI$ by Proposition 2.5, and since $r, s\in I$, then $I=(r)=(s)$. By Corollary 2.3, it follows that I is a simple R-module, so $(rs)=I=I^2$. We conclude that as an ideal of R, I is principal and is generated by an idempotent. Hence I is a direct summand of R and the structure of I as an R-module is the same as its structure as a ring. Consequently, I is a field, as asserted.

It's clear that the converse of Corollary 2.6 is also valid. Namely, if K is an infinite field of cardinality α and if S is a nonzero ring, then K is a Jónsson α-ideal of the ring $S\oplus K$ and $K^2\neq(0)$.

3. Jónsson ω_0-modules. We restrict our consideration in this section to the case where $\alpha=\omega_0$, the first infinite cardinal, and in view of Corollary 2.3, we consider only Jónsson ω_0-modules that are not finitely generated. Such a module M has a particularly simple description: M is not finitely generated, is countably infinite, and each proper submodule of M is finite[3].

Assume that M is a non-finitely generated Jónsson ω_0-module over the ring R. What restrictions are imposed on the structure of R and M? Theorem 3.1 and Proposition 3.2 provide some answers to this question. In particular, these two results allow us to restrict to the case where the module M is faithful and the ring R is a quasilocal integral domain. In the case of a Prüfer domain R, we determine the isomorphism class of non-finitely generated Jónsson ω_0-modules over R.

If N is an R-module, we say that N is a *torsion module* if $\operatorname{Ann}(n)\neq(0)$ for each $n\in N$. On the other hand, the module N is *torsion-free* if $\operatorname{Ann}(n)=(0)$ for each nonzero element $n\in N$. The statement of Theorem 3.1 uses this terminology.

Theorem 3.1. *Let M be a Jónsson ω_0-module over the ring R, where M is not finitely generated. Then M is a torsion R-module, and there exists a maximal ideal Q of R such that the following conditions are satisfied:* (1) $\operatorname{Ann}(x)$ *is a Q-primary ideal of finite index for each* $x\in M-\{0\}$, (2) R/Q *is finite,* (3) *the powers of Q properly descend,* (4) $\bigcap_{i=1}^{\infty} Q^i$ *is a prime ideal, and* (5) *if* $H_i=\{x\in M\mid Q^ix=(0)\}$, *then* $\{H_i\}_{i=1}^{\infty}$ *is a strictly ascending sequence of submodules of M such that* $M=\bigcup_{i=1}^{\infty} H_i$.

Proof. As the first step in the proof, we show that $PM=M$ for each maximal ideal P of R. Thus, if $PM\neq M$, then Proposition 2.5 shows that $PM=(0)$, and hence M is a Jónsson ω_0-module over the field R/P. Since M is indecomposable, M is a one-dimensional vector space over R/P. This implies, however, that M is a cyclic R-module, contradicting the fact that M is not finitely generated. Therefore $PM=M$ for each maximal ideal P of R.

For P_α maximal in R, let M_α be the set of elements x of M such that $P_\alpha\subseteq\sqrt{\operatorname{Ann}(x)}$. Then M_α is a submodule of M since the inclusion $\operatorname{Ann}(x-y)\supseteq$

[3]) We remark that "countably infinite" is redundant in this definition — if M is not finitely generated and each proper submodule of M is finite, then M is countably infinite.

$\supseteq \operatorname{Ann}(x) \cap \operatorname{Ann}(y)$ implies that $\sqrt{\operatorname{Ann}(x-y)} \supseteq \sqrt{\operatorname{Ann}(x)} \cap \sqrt{\operatorname{Ann}(y)}$. We show that M is the direct sum of the family $\{M_\alpha\}$, taken over all maximal ideals P_α of R. If $x \in M - \{0\}$, then $Rx \subset M$ implies that Rx is finite, so $R/\operatorname{Ann}(x)$ is a finite ring. Therefore, $\operatorname{Ann}(x)$ is uniquely expressible as a finite intersection $\bigcap_{i=1}^{n} C_{\alpha_i}$ of primary ideals with distinct (maximal) radicals $P_{\alpha_i} = \sqrt{C_{\alpha_i}}$. If $B_j = \bigcap_{i \neq j} C_{\alpha_i}$ for $1 \leq j \leq n$, then no maximal ideal of R contains each B_j, so $R = B_1 + \ldots + B_n$, and $1 = b_1 + b_2 + \ldots + b_n$ with $b_i \in B_i$ for each i. Then $x = \sum_{j=1}^{n} b_j x$, where $C_{\alpha_i} b_j x = (0)$ for each j, and hence $b_j x \in M_j$. This proves that $M = \sum_\alpha M_\alpha$. The sum is direct, for if $m \in M_\alpha \cap (M_{\alpha_1} + \ldots + M_{\alpha_k})$, with $\alpha \neq \alpha_j$ for each j, then $\operatorname{Ann}(m) \supseteq P_\alpha + (P_{\alpha_1} \cap \ldots \cap P_{\alpha_k}) = R$, so $m = 0$. Because M is indecomposable, we conclude that $M = M_\alpha$ for some α. Let $Q = P_\alpha$; by definition of M_α, $\operatorname{Ann}(x)$ is a Q-primary ideal of finite index for each $x \in M - \{0\}$; in particular, Q has finite index in R. Let H_i be defined as in the statement of Theorem 3.1. Clearly each H_i is a submodule of M, and $H_i \subseteq H_{i+1}$ for each i. Moreover, for $x \in M$, $\operatorname{Ann}(x)$ contains a power of Q since $R/\operatorname{Ann}(x)$ is finite, so that $x \in H_i$ for some i; that is $H = \bigcup_{i=1}^{\infty} H_i$. Observe that H_i is a proper submodule of M for each i since $M = Q^i M \neq (0)$. Finally, we note that the assumption $H_i = H_{i+1}$ leads to the contradiction that $M = H_i$; it suffices to show that $H_i = H_{i+1}$ implies that $H_{i+1} = H_{i+2}$. Thus, if $x \in H_{i+2}$, then $Qx \subseteq H_{i+1} = H_i$, so $Q^i Q x = (0)$ and $x \in H_{i+1}$, as was to be proved. The fact that $H_i < H_{i+1}$ for each i shows that $Q^i > Q^{i+1}$ for each i; in particular, $Q^i \neq (0)$ for each i so that $\operatorname{Ann}(x) \neq (0)$ for each $x \in M - \{0\}$, and M is a torsion module. The equality $M = \bigcup_{i=1}^{\infty} H_i$ implies that $\bigcap_{i=1}^{\infty} Q^i = \operatorname{Ann}(M)$, and Proposition 2.5 shows that $\operatorname{Ann}(M)$ is prime in R. This completes the proof of Theorem 3.1.

If M is a non-finitely generated Jónsson ω_0-module over R, then replacing R by $R/\operatorname{Ann}(M)$, there is no loss of generality in assuming that M is faithful, and Proposition 2.5 shows that $R/\operatorname{Ann}(M)$ is an integral domain. Under these assumptions on R and M, let Q be as in the statement of Theorem 3.1 It is then possible to consider M as a module over the quasi-local domain R_Q. To wit, for $m \in M$ and $r/s \in R_Q$, we define the product $(r/s) \cdot m$ to be rm_1, where $sm_1 = m$. The product is well-defined, for Proposition 2.5 and Theorem 3.1 show that left multiplication by s induces an R-automorphism of M. It is somewhat lengthy, but routine, to verify that M is an R_Q-module under this definition, and we omit the details. We note that $Rm = R_Q m$ for each $m \in M$; for a proof, we need only show that $R_Q m \subseteq Rm$ — that is, we need to show that if $s \in R - Q$ and if $sm_1 = m$, then $m_1 \in Rm$. This statement follows since Rm is finite and since left multiplication by s induces an injection of Rm into Rm

so that $Rm=sRm$. We conclude that the structure of M as an R_Q-module is essentially the same as the structure of M as an R-module [7, Ex. 2, p. 8]. In particular, M is a Jónsson ω_0-module over R_Q. Thus, in considering non-finitely generated Jónsson ω_0-modules M over a ring R, we are led to consider the case where R is a quasi-local domain and M is faitful. The next result is stated for this hypothesis, and is somewhat analogous to Theorem 3.1.

Proposition 3.2. *Assume that M is a non-finitely generated faithful Jónsson ω_0-module over the quasi-local domain (D, P). For $x\in P-\{0\}$, denote by $M(x)$ the submodule of M consisting of elements annihilated by x. Then $M(x)$ is finite and non-zero, $M(x)<M(x^2)<M(x^3)<\dots$, and $M=\bigcup_{i=1}^{\infty} M(x^i)$. Moreover, if $m_1\in M(x)-\{0\}$ and if elements $m_2, m_3, \dots\in M$ are chosen successively so that $m_i=xm_{i+1}$ for each i, then $Dm_1<Dm_2<\dots$ and $M=\bigcup_{i=1}^{\infty} Dm_i$.*

Proof. Since M is faithful, then $M(x)\neq M$, and hence $M(x)$ is finite. Pick $m\in M-\{0\}$. Since $x\in P=\sqrt{\mathrm{Ann}\,(m)}$, there exists a positive integer k so that $x^k m=0$ while $x^{k-1}m\neq 0$. Thus $x^{k-1}m$ is a nonzero element of $M(x)$. For a given i, we assume that $s\in M(x^{i+1})-M(x^i)$. Then $s\in xM$ implies $s=xt$ for some $t\in M$. Thus $x^{i+2}t=x^{i+1}s=0$, but $x^{i+1}t=x^i s\neq 0$ so that $t\in M(x^{i+2})-M(x^{i+1})$. Since $\bigcup_{i=1}^{\infty} M(x^i)$ is an infinite submodule of M, we conclude that $M=\bigcup_{i=1}^{\infty} M(x^i)$.

If $m_1, m_2, \dots$ are as described in the hypothesis of Proposition 3.2, then the proof above shows that $m_{i+1}\in M(x^{i+1})-M(x^i)$ for each i so that $Dm_i<Dm_{i+1}$ and $M=\bigcup_{i=1}^{\infty} Dm_i$, as asserted.

The next result is a partial converse of Proposition 3.2. The proof of this result is routine and will be omitted.

Proposition 3.3. *Let M be an R-module that can be expressed as the union of an infinite strictly ascending sequence $\{M_i\}_{i=1}^{\infty}$ of finite submodules. The following conditions are equivalent.*

(1) *M is a Jónsson ω_0-module.*
(2) *Each proper submodule of M is contained in some M_i.*
(3) *If $x_i\in M-M_i$ for each i, then $\{x_i\}_{i=1}^{\infty}$ generates M.*

If the notation and hypothesis are as in the statement of Proposition 3.2, if F is a free D-module on the countably infinite set $\{y_i\}_{i=1}^{\infty}$, and if φ is the natural surjection of F onto M induced by the mapping $y_i\to m_i$, then, of course, $M\cong F/\ker\varphi$, where $\ker\varphi$ contains the submodule generated by the set $\{y_i-xy_{i+1}\}_{i=1}^{\infty}$. This observation provided the original motivation for Theorem 3.5. The next result provides some motivation for the hypothesis in the statement of Theorem 3.5.

Proposition 3.4. *Assume that P is a maximal ideal of the ring R such that the powers of P properly descend and such that $P=P^2+tR$ for some $t\in R$. Then $P^i:(t)=$ $=P^{i-1}$ for each i.*

Proof. Since P/P^2 is a one-dimensional vector space over R/P, there are no ideals of R strictly between P and P^2. It is known that this implies that $P=P^n+tR$ and that $\{P^j\}_{j=1}^n$ is the set of ideals between P and P^n for each n [7, (38,2)]. If $i>1$ then the inclusion $P^{i-1}\subseteq P^i:(t)$ is clear. Moreover, $t\notin P^i$ implies that $P^i:(t)\subseteq P$. Now $p^{i-2}\nsubseteq P^i:(t)$, for otherwise, $P^{i-1}=P^{i-2}[P^2+(t)]\subseteq P^i$, contrary to the hypothesis that the powers of P properly descend. We conclude that $P^i:(t)=P^{i-1}$, as asserted.

Theorem 3.5. *Assume that $P=A_1, A_2, A_3, \dots$ is a sequence of ideals of R and $\{t_i\}_{i=2}^{\infty}$ is a sequence of elements of R such that the following conditions are satisfied: (1) P is a maximal ideal of R and R/P is finite, (2) the powers of P properly descend, and (3) for each $i>1$, $P=A_1+(t_i)$, $A_i\supseteq P^i$, and $A_i:(t_i)\subseteq P^{i-1}$. Then there exists a non-finitely generated Jónsson ω_0-module M over R such that* Ann (x) *is P-primary for each $x\in M-\{0\}$.*

Proof. Let F be a free R-module on the set $\{x_i\}_{i=1}^{\infty}$, let A be the submodule of F generated by $\{A_ix_i\}_{i=1}^{\infty}\cup\{x_i-t_{i+1}x_{i+1}\}_{i=1}^{\infty}$, and let $M=F/A$; we prove that M has the required properties. Let $y_i=x_i+A$ for each i. It is clear that $\{y_i\}_{i=1}^{\infty}$ generates M and that $\langle y_i\rangle\subseteq\langle y_{i+1}\rangle$ for each i. We prove that the inclusion $\langle y_i\rangle\subseteq\langle y_{i-1}\rangle$ is proper by establishing the following property of the submodule A: if $a\in A-\{0\}$ and if $a=\sum_{j=1}^{k} r_jx_j$, where $r_k\neq 0$, then $r_k\in P$. For some n, we can write $a=a_1x_1+$ $+\dots+a_nx_n+h_2(x_1-t_2)+\dots+h_n(x_{n-1}-t_nx_n)$, where $a_i\in A_i$ and $h_j\in R$. If $k=n$, then $r_k=a_n-h_nt_n\in P$. Otherwise, we obtain a sequence of equations

$$a_n-h_nt_n=0$$

$$h_n+a_{n-1}-h_{n-1}t_{n-1}=0$$

$$\vdots$$

$$h_{k+2}+a_{k+1}-h_{k+1}t_{k+1}=0.$$

The first equation implies that $h_n\in A_n:(t_n)\subseteq P^{n-1}$, and hence, from the second equation, $h_{n-1}t_{n-1}=h_n+a_{n-1}\in A_{n-1}$ so that $h_{n-1}\in A_{n-1}:(t_{n-1})\subseteq P^{n-2}$. Inductively, we obtain $h_{k+1}\in P^k$. If $k>1$, it follows that $r_k=h_{k+1}+a_k-h_kt_k\in P$, and if $k=1$, then $r_k=h_2+a_1$ is also in P. This establishes the assertion concerning A, and hence

$\langle y_i \rangle \neq \langle y_{i+1} \rangle$ for each i. Thus, no finite subset of $\{y_i\}_{i=1}^{\infty}$ generates M, and this implies that M is not finitely generated.

We show next that each $\langle y_i \rangle$ is finite. Since $P \subseteq \operatorname{Ann}(y_1)$ and R/P is finite, the submodule $\langle y_1 \rangle$ is finite. Assume that $\langle y_i \rangle$ is finite. To prove that $\langle y_{i+1} \rangle$ is finite, it suffices to prove that $\langle y_{i+1} \rangle / \langle y_i \rangle$ is finite. The annihilator of $\langle y_{i+1} \rangle / \langle y_i \rangle$ contains A_{i+1} and the element t_{i+1}, hence the ideal $A_{i+1} + (t_{i+1}) = P$. Therefore $\langle y_{i+1} \rangle / \langle y_i \rangle$ is finite, and $\langle y_{i+1} \rangle$ is finite.

To complete the proof, we show that $y \notin \langle y_i \rangle$ implies that $y_i \in \langle y \rangle$. Choose k so that $y \in \langle y_{k+1} \rangle$, $y \notin \langle y_k \rangle$; thus $k \geqq i$. Then $y = r y_{k+1}$, and since $P y_{k+1} \subseteq \langle y_k \rangle$, it follows that $r \notin P$. Hence $R = A_{k+1} + rR$ and we write $1 = q + rs$ for some $q \in A_{k+1}$ and $s \in R$. Then $y_{k+1} = q y_{k+1} + rs y_{k+1} = sy$ and $y_i \in \langle y_{k+1} \rangle \subseteq \langle y \rangle$. This is sufficient to show that each proper submodule of M is finite, for if L is a submodule of M that is contained in no $\langle y_i \rangle$, then L contains $\{y_j\}_1^{\infty}$, and hence $L = M$. It is clear from the construction that $\operatorname{Ann}(x)$ is P-primary for each $x \in M - \{0\}$.

Assume that (R, P) is a quasi-local domain such that $P = tR$ is principal and R/P is finite. Then the hypothesis of Theorem 3.5 is satisfied for $A_i = P^i$ and $t_i = t$ for each i. In this case, the module M constructed in the proof of Theorem 3.5 is isomorphic to $R[1/t]/R$, and in the case where this module is faithful (that is, where $\bigcap_{i=1}^{\infty} P^i = (0)$), then R is a rank-one discrete valuation ring and $R[1/t]$ is the quotient field of R. The next result determines equivalent conditions in order that the D-module K/D, where D is an integral domain and K is the quotient field of D, should be a Jónsson ω_0-module. The statement of Theorem 3.6 uses the following terminology from [12]. The ring R is said to have the *finite norm property* (FNP) if R/A is finite for each nonzero ideal A of R (such a ring is said to be *residually finite* in [4]).

Theorem 3.6. *Let D be an integral domain with quotient field $K \neq D$. Let D^* be the integral closure of D. Then K/D is a Jónsson ω_0-module over D if and only if the following conditions are satisfied.*

(1) *D has the finite norm property,*

(2) *D^* is a rank-one discrete valutation ring, and*

(3) *D^* is a finite D-module.*

Proof. Assume that K/D is a Jónsson ω_0-module. If d is a nonzero nonunit of D, then Dd^{-1}/D is a proper submodule of K/D, and hence is finite. Since Dd^{-1}/D and D/dD are isomorphic D-modules, it follows that dD has finite norm, and D has the finite norm property. Let $J \neq K$ be an overring of D. Since J/D is finite, J is integral over D; hence $J \subseteq D^*$ and K is the only proper overring of D^*. Therefore D^* is a rank-one valuation ring finitely generated over D, a ring with (FNP), and hence D^* is rank-one discrete with (FNP).

Conversely, assume that conditions (1)—(3) are satisfied, and write V instead of D^*. Assume that π is a generator of the maximal ideal of V. Since V is a finitely generated D-module, the conductor C of D in V is nonzero; say $C=\pi^k V$. We know that $K=\bigcup_{i=1}^{\infty} \pi^{-1}V$, where $\pi^{-1}V<\pi^{-2}V<\dots$. To prove that K/D is a Jónsson ω_0-module, it suffices to show that $\pi^{-i}V/D$ is finite for each i and that each proper submodule of K/D is a submodule of $\pi^{-i}V/D$ for some i. $\pi^{-i}V/D$ is a finitely generated D-module and π^{i+k} belongs to the annihilator of this module. Since the ring $D/\pi^{i+k}D$ is finite, it follows that $\pi^{-i}V/D$ is finite. To prove that each proper submodule of K/D is contained in some $\pi^{-i}V/D$, it suffices to show that if N is a D-submodule of K such that $N\subsetneqq\pi^{-i}V$ for each i, then $N=K$. Since $K=\bigcup_{i=1}^{\infty} \pi^{-i}D$, it is enough to show that $\pi^{-i}\in N$ for each positive integer i. Choose $n\in N-\pi^{-(i+k)}V$. We write n as $\pi^{-s}u$, where u is a unit of V and $s>i+k$. Then $\pi^{s-i}\in C$ and $\pi^{s-i}u^{-1}n=\pi^{-i}\in Dn\subseteq \subseteq N$. This established Theorem 3.6.

Considerations similar to those in the proof of Theorem 3.6 and in the paragraph preceding that result enable us to determine to within isomorphism the class $\mathscr{C}(J)$ of all non-finitely generated Jónsson ω_0-modules over a Prüfer domain J. In order for $\mathscr{C}(J)$ to be nomepty, we know from Theorem 3.1 that it is necessary that there should exist a maximal ideal M of J such that J/M is finite and the powers of M properly descend. Assume that J has such a maximal ideal and let $\{M_i\}_{i\in I}$ be the family of all such maximal ideals of J. Since J is a Prüfer domain, $P_i=\bigcap_{k=1}^{\infty} M_i^k$ is prime in J and there is no prime of J properly between P_i and M_i [7, Chap. 23]. Moreover, $V_i=(J/P_i)_{(M_i/P_i)}\cong J_{M_i}/P_iJ_{M_i}$ is a rank-one valuation ring with residue field J/M_i, and to within isomorphism. $\mathscr{C}(J)=\bigcup_{i\in I}\mathscr{C}(V_i)$. According to the next result, Proposition 3.7, the unique faithful, non-finitely generated Jónsson ω_0-module over V_i is K_i/V_i, where K_i is the quotient field of V_i, and this in turn yields a determination of $\mathscr{C}(J)$.

Proposition 3.7. *Let V be a rank-one discrete valuation ring with quotient field K and with finite residue field V/P. To within isomorphism, K/V is the unique faithful, non-finitely generated Jónsson ω_0-module over V.*

Proof. Let M be a non-finitely generated faithful Jónsson ω_0-module over V and assume that p generates P. According to Proposition 3.2, M can be expressed as $\bigcup_{i=1}^{\infty} Vx_i$, where $x_i\neq 0$, $px_1=0$, and $px_{i+1}=x_i$ for each i. Noting that the set $\{p^{-i}+V\}_{i+1}^{\infty}$ generates K/V, it is then routine to verify that the mapping $p^{-i}+V\to x_i$ can be extended to a V-module isomorphism of K/V onto M.

Assume that (D, P) is a quasi-local domain that admits a non-finitely generated faithful Jónsson ω_0-module. From Theorem 3.1 and Proposition 3.2, it follows that D/P is finite, that $\bigcap_{i=1}^{\infty} P^i=(0)$, and that (0) can be expressed as the intersection of a strictly decreasing sequence $\{Q_i\}_{i=1}^{\infty}$ of P-primary ideals such that each D/Q_i is finite. Based on considerations up to this point, it seems reasonable to ask if D must be one-dimensional, or Noetherian, or if the residue class rings D/P^i are finite. We present in Section 4 examples that show that each of these questions has a negative answer; moreover, if D is one-dimensional, then D need not be Noetherian, and conversely.

4. Examples. The examples in this section indicate certain limitations on what can be said about the structure of a quasi-local domain (D, P) such that D admits a non-finitely generated faithful Jónsson ω_0-module. In particular, the class of examples included in Example 4.1 is large enough to show that D need not be Noetherian, and that no restriction on the dimension of D is possible.

Example 4.1. *Assume that $(V, M(V))$ and $(W, M(W))$ are independent valuation rings on a field K, that V is rank-one discrete, and that there exists a finite field so that $V=k+M(V)$ and $W=k+M(W)$. Set $D=k+P$, where $P=M(V)\cap M(W)$. Then (D, P) is quasi-local,* $\dim D=\dim W$, *and W/D is a non-finitely generated faithful Jónsson ω_0-module over D.*

Proof. Corollary 5.6 of [8] shows that (D, P) is quasi-local and $\dim D=\dim W$. Let v be a valuation associated with V and choose, by the approximation theorem for independent valuations [7, (22.9)], an element $x\in W-V$ so that $v(x)=-1$. If $d\in D-\{0\}$ and if $v(d)=r\geqq 0$, then $dx^{r+1}\notin D$, so W/D is a faithful D-module. To prove that W/D is a non-finitely generated Jónsson ω_0-module, we show that the sequence $\{(D+Dx^i)/D\}_{i=1}^{\infty}$ of submodules of W/D satisfies the hypothesis and condition (2) of Proposition 3.3. To do so, we prove first the following assertion.

(*) *If $r\in W$, if $s\in W-V$, and if $v(s)<v(r)$, then $r\in D+Ds$.*

To prove (*), consider first the case where s is a unit and r is a nonunit of W. Then $r/s\in M(W)$, and since $v(r/s)>0$, then $r/s\in M(V)$ as well. Hence $r\in Ds$ in this case. On the other hand, if s is a nonunit of W, then we can replace s by the unit $s_1=s+1$ without affecting the hypothesis or the conclusion since $s_1\in W-V$, $v(s)=v(s_1)$ and $D+Ds=D+Ds_1$. Similarly, if r is a unit of W, then $r_1=r-u\in M(W)$ for some nonzero element r of k, and replacing r by r_1 yields the desired conclusion. This establishes (*).

It follows from (*) that $W=\bigcup_{i=1}^{\infty}(D+Dx^i)$ and that $D+Dx^i\subseteqq D+Dx^{i+1}$. The minimum of the v-values of elements of $D+Dx^i$ is $-i$, so $x^{i+1}\notin D+Dx^i$ and the inclusion $D+Dx^i\subseteqq D+Dx^{i+1}$ is proper. Statement (*) also implies that if N is a proper D-submodule of W containing D, then the set of v-values of elements of N is bounded below, and hence $N \subseteq D+Dx^i$ for some i. Thus, to complete the proof of Example 4.1, we need only show that $(D+Dx^i)/D$ is finite for each i. It is clear, however, that $M(W) \cap (M(V))^i$ is contained in the annihilator of $(D+Dx^i)/D$. As $|V/(M(V))^i|=|k|^i$ is finite, the subring $D/[M(W) \cap (M(V))^i]$ is also finite. Since $(D+Dx^i)/D$ is a finitely generated D-module, we conclude that $(D+Dx^i)/D$ is finite.

If k is a finite field and $\{X_i\}_{i=1}^{\infty}$ is a set of indeterminates over k, then the field $K=k(\{X_i\}_1^{\infty})$ admits independent valuations v, w such that v is rank-one discrete, the valuation ring V of v is of the form $k+M(V)$, and the valuation ring W of w is of the form $k+M(W)$. Example 4.1 shows that W/D, where $D=k+(M(V) \cap M(W))$, is a Jónsson ω_0-module, and $\dim D = \dim W$ can be any positive integer or it can be infinite. Moreover, if W is chosen so that $M(W)$ is unbranched [7, p. 189], then no principal ideal of D is primary for $M(V) \cap M(W)$. Thus the assumption that a quasi-local domain admits a faithful non-finitely generated Jónsson ω_0-module does not imply that the domain is Noetherian, and it imposes no restriction on its dimension. We remark that the approximation theorem for independent valuations can be avoided in the proof of Example 4.1 and that the conclusion concerning W/D remains valid for any quasi-local domain $W=k+M(W)$ with quotient field K such that $W \subsetneqq V$. Using this fact, we see that if W is rank-one nondiscrete, if $B \neq M(W)$ is any $M(W)$-primary ideal and if $J=k+(M(V) \cap B)$, then J admits the non-finitely generated faithful Jónsson ω_0-module $(k+B)/J$, and yet $J/(M(V) \cap B)^n$ is infinite for each $n>1$.

There is an analogue, for generating sets, of the concept of a Jónsson α-module. Namely, we say that a unitary module M over a commutative ring R with identity is a *Jónsson α-generated module* if M has a generating set of cardinality α, no generating set of smaller cardinality, and each proper submodule of M has a generating set of cardinality less than α. We have developed a theory of Jónsson α-generated modules in [11]. This theory contains many similarities, but also some differences, with the theory of Jónsson α-modules. The differences stem frequently from the fact that, by definition, a Jónsson α-generated module is not finitely generated, whereas a Jónsson α-module may be cyclic. In particular, a modification of the proof of [11, Example 3.3] establishes the following result.

Example 4.2. *Assume that D is an integral domain with quotient field K, that (W, M) is a rank-one discrete valuation ring on K containing D, and that $W/M=D/P$ is a finite field, where P is the center of W on D. Then K/W is a Jónsson ω_0-module over D.*

Example 4.2 can be used to show that even in the case of a Noetherian domain D, existence of a non-finitely generated faithful Jónsson ω_0-module over D imposes no restriction on the dimension of D. For example, let k be a finite field, let n be a positive integer, and choose $x_1, x_2, \ldots, x_n \in Yk[[Y]]$ such that $\{x_i\}_{i=1}^n$ is algebraically independent over k. Then $D = k[x_1, \ldots, x_n]_{(x_1, \ldots, x_n)}$ is an n-dimensional regular local ring and $W = k[[Y]] \cap k(x_1, \ldots, x_n)$ is a rank-one discrete valuation overring of D such that D and W have residue field k. By Example 4.2, $k(x_1, \ldots, x_n)/W$ is a faithful Jónsson ω_0-module over D.

We remark that, in general, a Noetherian ring R admits a non-finitely generated Jónsson ω_0-module if and only if R contains a maximal ideal M of positive height such that the residue field R/M is finite. This result follows from Theorem 2.7 of [11].

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