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Mal'cev conditions for regular and weakly regular subalgebras of the square

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1. Introduction. At the beginning of the seventies, Mal'cev conditions characterizing varieties of algebras with regular congruences were given by B. CSÁKÁNY [3], [4], G. GRÄTZER [7], and R. WILLE [16]. Recently, algebras with regular tolerances (=compatible symmetric and reflexive relations) were introduced and a Mal'cev condition for varieties of algebras with regular tolerances was derived by I. CHAJDA [2]. Since the concept of regularity can easily be extended for other sorts of compatible relations we have also varieties of algebras with regular compatible reflexive relations and varieties of algebras with regular quasiorders (=compatible transitive and reflexive relations). The aim of this paper is to show that all the above mentioned varieties form exactly two well-known classes of varieties. Moreover, Mal'cev conditions for these two classes of varieties simplify the Mal'cev characterizations presented in some former papers. In the second part of this paper, analogous results for weakly regular subalgebras of the square are derived.

2. Algebras with regular subalgebras of the square. Throughout this paper, the same symbol stands for an algebra and its base set. Let A be an algebra and let S be a subset of the square $A\times A$. We denote by

$R(S)$ the compatible reflexive relation on A generated by S;

$T(S)$ the tolerance on A generated by S;

$Q(S)$ the quasiorder on A generated by S; and

$[a]S$ the subset $\{x\in A;\ \langle a, x\rangle\in S\}$, where a is some element of A.

Notice that $[a]S$ is called a class of S. The rest of this section is formulated in terms of compatible reflexive relations only; for tolerances, quasiorders, and congruences the Definition and the Lemma below are modified in an evident way.

Definition. We say that an algebra A has regular compatible reflexive relations if any two compatible reflexive relations coincide whenever they have a class in com-

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mon. A variety V of algebras has regular compatible reflexive relations provided each algebra $A \in V$ has this property.

Lemma. *For any algebra A, the following conditions are equivalent:*

(a) *A has regular compatible reflexive relations;*

(b) *For every compatible reflexive relation Ψ on A, $\Psi = R(\{a\} \times [a]\Psi)$ holds for any element a of A;*

(c) *For every compatible reflexive relation Ψ on A and for each element a of A, $\Psi = R(\{a\} \times B)$ holds for some subset $B \subseteqq A$.*

Proof. (a)⇒(b). Apparently, for any compatible reflexive relation Ψ on A, $\{a\} \times [a]\Psi \subseteqq R(\{a\} \times [a]\Psi) \subseteqq \Psi$ hold and thus also $[a](\{a\} \times [a]\Psi) \subseteqq [a]R(\{a\} \times [a]\Psi) \subseteqq [a]\Psi$. However, $[a](\{a\} \times [a]\Psi) = [a]\Psi$, which implies $[a]R(\{a\} \times [a]\Psi) = [a]\Psi$. By applying the hypothesis, the equality $\Psi = R(\{a\} \times [a]\Psi)$ follows.

(b)⇒(a). If Ψ and Φ are two compatible reflexive relations on A with the same class $[a]\Psi = [a]\Phi$ then $\Psi = R(\{a\} \times [a]\Psi) = R(\{a\} \times [a]\Phi) = \Phi$.

(b)⇒(c) is trivial.

(c)⇒(b). It is enough to verify the inclusion $\Psi \subseteqq R(\{a\} \times [a]\Psi)$. By hypothesis, $R(\{a\} \times B) = \Psi$ holds for some B and so we have $\{a\} \times B \subseteqq R(\{a\} \times B) = \Psi$. This yields $B \subseteqq [a]\Psi$ and the conclusion $\Psi = R(\{a\} \times B) \subseteqq R(\{a\} \times [a]\Psi)$ follows.

3. Varieties with regular subalgebras of the square. The main fact we will need about varieties with regular congruences is the following

Theorem 1 (B. Csákány [3]). *For any variety V, the following conditions are equivalent:*

(1) *V has regular congruences;*

(2) *There exist ternary polynomials $p_1, \ldots, p_n$ such that*

$$\big(z = p_i(x, y, z), \quad 1 \leqq i \leqq n\big) \Leftrightarrow x = y.$$

In [5] we announced

Theorem 2. *For any variety V, the following conditions are equivalent:*

(1) *V has regular and permutable congruences;*

(2) *V has regular tolerances;*

(3) *V has regular compatible reflexive relations;*

(4) *There exist ternary polynomials $p_1, \ldots, p_n$ and an $(n+3)$-ary polynomial r such that*

$$x = r(x, y, z, z, \ldots, z), \quad y = r\big(x, y, z, p_1(x, y, z), \ldots, p_n(x, y, z)\big),$$
$$z = p_i(x, x, z) \quad for \quad 1 \leqq i \leqq n.$$

Proof. (1)⇒(3) follows directly from the Theorem of H. Werner [11].

(3)$\Rightarrow$(4). Let $F_3(x, y, z)$ be the free algebra in V with free generators x, y, z. The compatible reflexive relation $R(x, y)$ on $F_3(x, y, z)$ is finitely generated, so by Lemma (c) from Section 2 there is a finite subset $\{p_i;\ 1 \leqq i \leqq n\} \subseteqq F_3(x, y, z)$ with the property $R(x, y) = R(\{z\} \times \{p_i;\ 1 \leqq i \leqq n\}) = R(\{\langle z, p_i \rangle;\ 1 \leqq i \leqq n\})$. Now, condition $\langle x, y \rangle \in R(\{\langle z, p_i \rangle;\ 1 \leqq i \leqq n\})$ yields

$$x = \varrho(z, \ldots, z) \quad \text{and} \quad y = \varrho(p_1, \ldots, p_n)$$

for some n-ary algebraic function ϱ over $F_3(x, y, z)$ and thus there are ternary polynomials $p_1, \ldots, p_n$ and an $(n+3)$-ary polynomial r such that

$$x = r(x, y, z, z, \ldots, z) \quad \text{and} \quad y = r\big(x, y, z, p_1(x, y, z), \ldots, p_n(x, y, z)\big).$$

Finally, the identities $z = p_i(x, x, z)$, $1 \leqq i \leqq n$, follow immediately from the above equality $R(x, y) = R(\{\langle z, p_i \rangle;\ 1 \leqq i \leqq n\})$.

(4)$\Rightarrow$(1). Regularity: Apparently, the ternary polynomials $p_1, \ldots, p_n$ satisfy condition (2) of Theorem 1, i.e., V has regular congruences.

Permutability: It is easily seen that $p(a, b, c) := r\big(c, a, b, p_1(b, a, b), \ldots, \ldots, p_n(b, a, b)\big)$ is the well-known Mal'cev polynomial and thus, by [10], the permutability of V follows.

(1)$\Rightarrow$(2) again by [15].

(2)$\Rightarrow$(1). Similarly as in the proof (3)$\Rightarrow$(4), the formula

$$T(x, y) = T(\{z\} \times \{p_i;\ 1 \leqq i \leqq n\}) \quad \text{for some} \quad \{p_i;\ 1 \leqq i \leqq n\} \subseteqq F_3(x, y, z)$$

implies the existence of ternary polynomials $p_1, \ldots, p_n$ and of a $(2n+3)$-ary polynomial t with

$$x = t\big(x, y, z, z, \ldots, z, p_1(x, y, z), \ldots, p_n(x, y, z)\big),$$

$$y = t\big(x, y, z, p_1(x, y, z), \ldots, p_n(x, y, z), z, \ldots, z\big),$$

and

$$z = p_i(x, x, z) \quad \text{for} \quad 1 \leqq i \leqq n.$$

Now, the regularity of V is trivial since any congruence is a tolerance; the permutability of V is entailed by the Mal'cev polynomial

$$p(a, b, c) := t\big(c, a, b, p_1(b, a, b), \ldots, p_n(b, a, b), p_1(c, b, b), \ldots, p_n(c, b, b)\big).$$

In this way, varieties with regular tolerances and also varieties with regular compatible reflexive relations are sufficiently described. For varieties with regular congruences and for varieties with regular quasiorders, the following theorem holds.

Theorem 3. *For any variety V, the following conditions are equivalent:*

(1) *V has regular congruences;*

(2) *V has regular quasiorders;*

(3) *There exist ternary polynomials* $p_1, \ldots, p_n$ *and* $(n+3)$*-ary polynomials* $r_1, \ldots, r_k$ *such that*

$$x = r_1(x, y, z, z, \ldots, z),$$

$$r_j(x, y, z, p_1(x, y, z), \ldots, p_n(x, y, z)) = r_{j+1}(x, y, z, z, \ldots, z) \quad \textit{for} \quad 1 \leqq j < k,$$

$$y = r_k(x, y, z, p_1(x, y, z), \ldots, p_n(x, y, z)),$$

$$z = p_i(x, x, z) \quad \textit{for} \quad 1 \leqq i \leqq n.$$

Proof. (1)⇒(2). By Theorem 3 of J. HAGEMANN [9; p. 11], varieties with regular congruences are n-permutable for some $n>1$. Then, using Corollary 4 of J. HAGEMANN [9; p. 7], quasiorders coincide with congruences.

(2)⇒(3). The identities (3) are derived from the formula

$$Q(x, y) = Q(\{z\}\times\{p_i;\ 1 \leqq i \leqq n\}) \quad \text{for some} \quad \{p_i;\ 1 \leqq i \leqq n\} \subseteq F_3(x, y, z)$$

in a similar way as above.

(3)⇒(1). Evidently, the polynomials $p_1, \ldots, p_n$ satisfy condition (2) of Theorem 1, i.e., V has regular congruences.

Remarks. (i) As it was already noted in [13], [14], congruence regularity and congruence permutability are independent conditions.

(ii) The Mal'cev condition from Theorem 2 simplifies the identities given in [1] and [2].

(iii) Part (3) of Theorem 3 is a slightly improved version of [3; p. 188].

4. Varieties with weakly regular subalgebras of the square. Let V be a variety having distinguished nullary operations $c_1, \ldots, c_m$. We say that V has weakly regular congruences with respect to $c_1, \ldots, c_m$ if $[c_i]\Theta=[c_i]\Psi$, $1\leqq i\leqq m$, imply $\Theta=\Psi$ for any two congruences Θ and Ψ on $A\in V$. Analogously we introduce the concept of varieties with weakly regular tolerances, with weakly regular compatible reflexive relations, etc. This Section contains the variations on theorems of Section 3; the proofs are very similar to those of Section 3, so they can be omitted. For brevity we denote the sequences $c_i, \ldots, c_i$ (n times) and $q_{i1}(x, y), \ldots, q_{in}(x, y)$ by $\vec{c}_i$ and $\vec{q}_{ij}(x, y)$ respectively.

Weakly regular varieties were first investigated by K. FICHTNER; the following theorem is a paraphrase of his result [6; Theorem 1 (I), (IV)].

Theorem 4. *For any variety V with nullary operations $c_1, \ldots, c_m$, the following conditions are equivalent:*

(1) *V has weakly regular congruences with respect to $c_1, \ldots, c_m$;*

(2) *There exist an integer $n\geqq 1$ and binary polynomials q_{ij}, $1\leqq i\leqq m$, $1\leqq j\leqq n$, such that*

$$\left(\vec{c}_i = \vec{q}_{ij}(x, y),\ 1 \leqq i \leqq m\right) \Leftrightarrow x = y.$$

Example. The variety of implicative semilattices (see, e.g., [11], [12] for this concept) has weakly regular congruences with respect to the nullary operation 1: For $n=2$, $q_{11}(x, y)=x*y$, $q_{12}(x, y)=y*x$, we have $(1=x*y=y*x)\Leftrightarrow x=y$.

Theorem 5. *For any variety V with nullary operations $c_1, \ldots, c_m$, the following conditions are equivalent:*

(1) *V has permutable and weakly regular congruences with respect to $c_1, \ldots, c_m$;*

(2) *V has weakly regular tolerances with respect to $c_1, \ldots, c_m$;*

(3) *V has weakly regular compatible reflexive relations with respect to $c_1, \ldots, c_m$;*

(4) *There exist an integer $n\geqq 1$, binary polynomials q_{ij}, $1\leqq i\leqq m$, $1\leqq j\leqq n$, and an $(mn+2)$-ary polynomial w such that*

$$x = w(x, y, \vec{c}_1, \ldots, \vec{c}_m), \quad y = w\big(x, y, \vec{q}_{1j}(x, y), \ldots, \vec{q}_{mj}(x, y)\big),$$

$$\vec{c}_i = \vec{q}_{ij}(x, x) \quad \textit{for} \quad 1 \leqq i \leqq m.$$

Theorem 6. *For any variety V with nullary operations $c_1, \ldots, c_m$, the following conditions are equivalent:*

(1) *V has weakly regular congruences with respect to $c_1, \ldots, c_m$;*

(2) *V has weakly regular quasiorders with respect to $c_1, \ldots, c_m$;*

(3) *There exist integers $n, k\geqq 1$, binary polynomials q_{ij}, $1\leqq i\leqq m$, $1\leqq j\leqq n$, and $(mn+2)$-ary polynomials $w_1, \ldots, w_k$ such that*

$$x = w_1(x, y, \vec{c}_1, \ldots, \vec{c}_m),$$

$$w_h\big(x, y, \vec{q}_{1j}(x, y), \ldots, \vec{q}_{mj}(x, y)\big) = w_{h+1}(x, y, \vec{c}_1, \ldots, \vec{c}_m) \quad \textit{for} \quad 1 \leqq h < k,$$

$$y = w_k\big(x, y, \vec{q}_{1j}(x, y), \ldots, \vec{q}_{mj}(x, y)\big),$$

$$\vec{c}_i = \vec{q}_{ij}(x, x) \quad \textit{for} \quad 1 \leqq i \leqq m.$$

Remarks. (i) The implication (1)$\Rightarrow$(2) is again a direct consequence of Theorem 6 and Corollary 4 from [9].

(ii) Part (3) of our Theorem 6 improves the identities exhibited in [6; Theorem 2].

References

[1] I. Chajda, Regularity and permutability of congruences, *Algebra Universalis*, **11** (1980), 159—162.

[2] I. Chajda, A Mal'cev characterization of tolerance regularity, *Acta Sci. Math.*, **42** (1980), 229—232.

[3] B. Csákány, Characterization of regular varieties, *Acta Sci. Math.*, **31** (1970), 187—189.

[4] B. Csákány, Congruences and subalgebras, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.*, **18** (1975), 37—44.

[5] J. Duda, Mal'cev conditions for congruence-regular and congruence-permutable varieties (Abstract), *Comm. Math. Univ. Carolinae*, **20**/4 (1979), 795.

[6] K. FICHTNER, Varieties of universal algebras with ideals, *Mat. Sbornik*, **75** (117) (1968), 445—453. (Russian)

[7] G. GRÄTZER, Two Mal'cev-type theorems in universal algebra, *J. Combin. Theory*, **8** (1970), 334—342.

[8] G. GRÄTZER, *Universal Algebra*, Second Expanded Edition, Springer-Verlag (Berlin—Heidelberg—New York, 1979).

[9] J. HAGEMANN, *On regular and weakly regular congruences*, Preprint Nr. 75, Technische Hochschule Darmstadt (1973).

[10] A. I. Mal'cev, On the general theory of algebraic systems, *Mat. Sbornik*, **35** (77) (1954), 3—20. (Russian)

[11] W. C. NEMITZ, Implicative semi-lattices, *Trans. Amer. Math. Soc.*, **31** (1965), 128—142.

[12] W. C. NEMITZ and T. WHALEY, Varieties of implicative semilattices, *Pacific J. Math.*, **37** (1971), 759—769.

[13] E. T. SCHMIDT, Über reguläre Mannigfaltigkeiten, *Acta Sci. Math.*, **31** (1970), 197—201.

[14] H. A. THURSTON, Derived operations and congruences, *Proc. London Math. Soc.*, (3), **8** (1958), 127—134.

[15] H. WERNER, A Mal'cev condition for admissible relations, *Algebra Universalis*, **3** (1973), 263.

[16] R. WILLE, *Kongruenzklassengeometrien*, Lecture Notes in Mathematics, No. 113, Springer-Verlag (Berlin, 1970).

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