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Tolerance trivial algebras and varieties

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Tolerances on algebras and varieties were studied in many papers, see e.g. [1], [4], [5], [6] and numerous references there. An importance and suitability of tolerances in algebraic constructions mainly for congruence investigations was shown in [2], [3] and [10]. In particular, the paper [10] uses the concept of tolerance trivial algebra for characterizations of order polynomial completeness of ordered algebras. The aim of this paper is to give necessary and sufficient conditions under which (principal) tolerances and (principal) congruences on a given algebra coincide.

0. Preliminaries. Let $\mathfrak{A}=(A, F)$ be an algebra. A binary relation R on A, i.e., $R\subseteq A\times A$, has the *substitution property* (briefly SP) on $\mathfrak{A}$ if for each n-ary, $f\in F$ we have $\langle f(a_1, ..., a_n), f(b_1, ..., b_n)\rangle\in R$ whenever $\langle a_i, b_i\rangle\in R$ for $i=1, ..., n$. in other words, it is a subalgebra of the direct product $\mathfrak{A}\times\mathfrak{A}$.

A *tolerance* on an algebra $\mathfrak{A}=(A, F)$ is a reflexive and symmetric binary relation on A having SP (on $\mathfrak{A}$). Denote by $\mathrm{LT}(\mathfrak{A})$ the set of all tolerances on $\mathfrak{A}$. Clearly, $LT(\mathfrak{A})$ is a complete lattice with respect to set inclusion [4]. Denote by $\vee$ the join in $LT(\mathfrak{A})$. The meet evidently coincides with set intersection. Let $a, b\in A$. By $T(a, b)$ is denoted the least tolerance of $LT(\mathfrak{A})$ containing the pair $\langle a, b\rangle$. It is called a *principal tolerance (generated by $\langle a, b\rangle$).* The principal congruence generated by $\langle a, b\rangle$ will be denoted by $\Theta(a, b)$.

The following lemma is clear (see e.g. [4]):

Lemma 1. *For every algebra $\mathfrak{A}$ and each $T\in LT(\mathfrak{A})$,*

$$T=\vee\{T(a, b);\ \langle a, b\rangle\in T\}.$$

The next lemma is proved in [1] (see also [2]):

Lemma 2. *Let $\mathfrak{A}=(A, F)$ be an algebra and $a_i, b_i\in A$ for $i=1, ..., n$. Then*

$$\langle x, y\rangle\in\vee\{T(a_i, b_i);\ i=1, ..., n\}$$

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if and only if there exists a 2n-ary algebraic function φ *such that*

$$x = \varphi(a_1, \ldots, a_n, b_1, \ldots, b_n), \quad y = \varphi(b_1, \ldots, b_n, a_1, \ldots, a_n).$$

As usual, Con $(\mathfrak{A})$ denotes the congruence lattice of $\mathfrak{A}$. Although every congruence is a tolerance, in general, Con $(\mathfrak{A})$ is not a sublattice of $LT(\mathfrak{A})$ (see Section 3 below).

1. Tolerance trivial varieties. *Definition.* An algebra $\mathfrak{A}$ is *(principal) tolerance trivial* if every (principal) tolerance on $\mathfrak{A}$ is a congruence. A variety $\mathcal{V}$ of algebras is *(principal) tolerance trivial* if each $\mathfrak{A}\in\mathcal{V}$ has this property.

H. WERNER [11] proved that for each algebra $\mathfrak{A}$ in a variety $\mathcal{V}$ every reflexive binary relation having SP on $\mathfrak{A}$ is a congruence on $\mathfrak{A}$ if and only if the variety $\mathcal{V}$ is congruence permutable. Hence congruence permutable varieties are tolerance trivial. The following theorem shows that also the converse statement is valid:

Theorem 1. *For a variety* $\mathcal{V}$ *of algebras, the following conditions are equivalent:*

(1) $\mathcal{V}$ *is tolerance trivial;*

(2) $\mathcal{V}$ *is congruence permutable.*

Proof. Taking into account Werner's theorem [11] mentioned above, it remains to prove only (1)$\Rightarrow$(2). Let $\mathcal{V}$ be a variety of algebras and $\mathfrak{A}=F_3(x, y, z)$ the $\mathcal{V}$-free algebra with the set of free generators $\{x, y, z\}$. Since $\langle x, y\rangle\in T(x, y)$, $\langle y, z\rangle\in T(y, z)$ and, by (1), all tolerances are transitive, we obtain $\langle x, z\rangle\in T(x, y)\vee \vee T(y, z)$. By Lemma 2, there exists a 4-ary algebraic function φ over $\mathcal{V}$ such that $x=\varphi(x, y, y, z)$, $z=\varphi(y, x, z, y)$. Since $\mathfrak{A}=F_3(x, y, z)$, there exists a 7-ary polynomial p over $\mathcal{V}$ such that

$$\varphi(x_1, x_2, x_3, x_4) = p(x_1, x_2, x_3, x_4, x, y, z),$$

i.e. $x=p(x, y, y, z, x, y, z)$, $z=p(y, x, z, y, x, y, z)$. Evidently, $t(x, y, z)= =p(x, y, z, y, x, y, z)$ is the Mal'cev polynomial, i.e., $t(x, x, z)=t(z, x, x)=z$, whence $\mathcal{V}$ is congruence permutable.

2. Principal tolerance trivial algebras and varieties. Clearly, every tolerance trivial algebra is also principal tolerance trivial (but not vice versa). However, a characterization of principal tolerance trivial varieties and algebras is more complicated than that of tolerance trivial varieties.

Proposition 1. *If an algebra* $\mathfrak{A}=(A, F)$ *is principal tolerance trivial, then for each* $x, y\in A$ *there exist binary algebraic functions* ψ_1, ψ_2 *such that*

(1) $T(x, y)\supseteq T(\psi_1(x, y), \psi_2(x, y))$;

(2) *if* $\langle x, y\rangle\in\Theta(a, b)$, *then* $\psi_1(x, y)=\psi_1(\psi_1(a, b), \psi_2(a, b))$ *and* $\psi_2(x, y)=\psi_1(\psi_2(a, b), \psi_1(a, b))$.

Proof. If $\langle x, y\rangle\in\Theta(a, b)=T(a, b)$, by Lemma 2 there exists a binary algebraic function φ over $\mathfrak{A}$ such that $x=\varphi(a, b)$, $y=\varphi(b, a)$. Put $\psi_1(x_1, x_2)=\varphi(x_1, x_2)$, $\psi_2(x_1, x_2)=\varphi(x_2, x_1)$. Hence $x=\psi_1(a, b)$, $y=\psi_2(a, b)$ and

$$\psi_1(x, y) = \varphi(x, y) = \psi_1(\psi_1(a, b), \psi_2(a, b)),$$
$$\psi_2(x, y) = \varphi(y, x) = \psi_1(\psi_2(a, b), \quad \psi_1(a, b)),$$

proving (2). Moreover,

$$\langle\psi_1(x, y),\ \psi_2(x, y)\rangle = \langle\varphi(x, y),\ \varphi(y, x)\rangle\in T(x, y),$$

whence (1) is evident.

Now, we give a sufficient condition for principal tolerance triviality in a form closely connected with that of Proposition 1.

Proposition 2. *Let $\mathfrak{A}=(A, F)$ be an algebra such that there exist binary algebraic functions ψ_1, ψ_2 over $\mathfrak{A}$ with*

(1) $T(x, y)=T(\psi_1(x, y), \psi_2(x, y))$;

(2) *if* $\langle x, y\rangle\in\Theta(a, b)$, *then there exists a binary algebraic function φ over $\mathfrak{A}$ such that* $\psi_1(x, y)=\varphi(\psi_1(a, b), \psi_2(a, b))$ *and* $\psi_2(x, y)=\varphi(\psi_2(a, b), \psi_1(a, b))$.

Then $\mathfrak{A}$ is principal tolerance trivial.

Proof. Clearly $T(a, b)\subseteqq\Theta(a, b)$ for each $a, b\in A$. Prove the reverse inclusion. Let $\langle x, y\rangle\in\Theta(a, b)$. By (2) and (1), we obtain

$$\langle\psi_1(x, y), \psi_2(x, y)\rangle = \langle\varphi(\psi_1(a, b), \psi_2(a, b)), \varphi(\psi_2(a, b), \psi_1(a, b))\rangle\in$$
$$\in T(\psi_1(a, b), \psi_2(a, b)) = T(a, b),$$

hence, by (1), $T(x, y)=T(\psi_1(x, y), \psi_2(x, y))\subseteqq T(a, b)$, which implies $\langle x, y\rangle\in$ $\in T(a, b)$.

Corollary 1. *The variety of all distributive lattices is principal tolerance trivial but not tolerance trivial.*

Proof. By Theorem 1 (or by [6]), $\mathscr{V}$ is not tolerance trivial. We prove by Proposition 2 that $\mathscr{V}$ is principal tolerance trivial. Put $\psi_1(x, y)=x\wedge y$, $\psi_2(x, y)=x\vee y$. Let $T\in\mathrm{LT}(\mathfrak{A})$ for $\mathfrak{A}\in\mathscr{V}$. If $\langle x, y\rangle\in T$, then also $\langle x, x\vee y\rangle = \langle x\vee x, x\vee y\rangle\in T$ and, analogously, $\langle y, x\vee y\rangle\in T$. Hence

$$\langle x\wedge y, x\vee y\rangle = \langle x\wedge y, (x\vee y)\wedge(x\vee y)\rangle\in T.$$

Conversely, let $\langle x\wedge y, x\vee y\rangle\in T$. Then $\langle x, x\vee y\rangle = \langle x\vee(x\wedge y), x\vee(x\vee y)\rangle\in T$ and, similarly, $\langle y, x\vee y\rangle\in T$, i.e., $\langle x\vee y, y\rangle\in T$. Hence

$$\langle x, y\rangle = \langle x\wedge(x\vee y), (x\vee y)\wedge y\rangle\in T.$$

Accordingly, $T(x, y)=T(x\wedge y, x\vee y)$ is proved, i.e., ψ_1, ψ_2 satisfy (1) of Proposition 2.

By [8], $\langle x, y\rangle\in\Theta(a, b)$ if and only if

$$x\wedge y = [(a\wedge b)\vee(x\wedge y)]\wedge(x\vee y) \quad \text{and} \quad x\vee y = [(a\vee b)\vee(x\wedge y)]\wedge(x\vee y).$$

Putting $\varphi(x_1, x_2)=[x_1\vee(x\wedge y)]\wedge(x\vee y)$ we have immediately (2) of Proposition 2 which finishes the proof.

Now, we give a characterization of principal tolerance trivial algebras based on a description of $\Theta(a, b)$ by Mal'cev's lemma (see [9]) and that of $T(a, b)$ by Lemma 2:

Theorem 2. *For an algebra $\mathfrak{A}=(A, F)$, the following conditions are equivalent:*

(1) *$\mathfrak{A}$ is principal tolerance trivial;*

(2) *for each $a, b\in A$ and for all unary algebraic functions $\tau_1, \dots, \tau_n$ over $\mathfrak{A}$, if*

$$\{\tau_i(a), \tau_i(b)\}\cap\{\tau_{i+1}(a), \tau_{i+1}(b)\} \neq \emptyset \quad \text{for} \quad i = 1, \dots, n-1,$$

then there exists a binary algebraic function φ over $\mathfrak{A}$ such that $\tau_1(a)=\varphi(a, b)$, $\tau_n(b)=\varphi(b, a)$;

(3) *For each $a, b\in A$ and for all binary algebraic functions φ_1, φ_2 over $\mathfrak{A}$, if $\varphi_1(b, a)=\varphi_2(a, b)$, then there exists a binary algebraic function ψ over $\mathfrak{A}$ such that $\psi(a, b)=\varphi_1(a, b)$, $\psi(b, a)=\varphi_2(b, a)$.*

Proof. (2)$\Rightarrow$(1). Let $a, b\in A$, $\langle x, y\rangle\in\Theta(a, b)$. By Mal'cev's lemma (see [9] or [7]), there exist elements $e_0, \dots, e_n\in A$ and unary algebraic functions (so called translations) $\tau_1, \dots, \tau_n$ over $\mathfrak{A}$ such that $\{\tau_i(a), \tau_i(b)\}=\{e_{i-1}, e_i\}$ for $i=1, \dots, n$, and either $\{\tau_1(a), \tau_n(b)\}=\{x, y\}$ or $\{\tau_1(b), \tau_n(a)\}=\{x, y\}$. By (2), there exists a binary algebraic function φ over $\mathfrak{A}$ such that $x=\varphi(a, b)$, $y=\varphi(b, a)$, whence $\langle x, y\rangle\in T(a, b)$. The reverse inclusion in evident.

(3)$\Rightarrow$(1). Let $\langle x, y\rangle\in T(a, b)$, $\langle y, z\rangle\in T(a, b)$. By Lemma 2, there exist binary algebraic functions φ_1, φ_2 over $\mathfrak{A}$ such that $\langle x, y\rangle=\langle\varphi_1(a, b), \varphi_1(b, a)\rangle$, $\langle y, z\rangle=\langle\varphi_2(a, b), \varphi_2(b, a)\rangle$. By (3), $\langle x, z\rangle=\langle\psi(a, b), \psi(b, a)\rangle$, whence $\langle x, z\rangle\in T(a, b)$, proving the transitivity of $T(a, b)$, i.e., $T(a, b)=\Theta(a, b)$.

(1)$\Rightarrow$(3). If $\{\varphi_i(a, b), \varphi_i(b, a)\}=\{c_i, c_{i+1}\}$ for $i=1, 2$, then, by Lemma 2, $\langle c_1, c_2\rangle\in T(a, b)$, $\langle c_2, c_3\rangle\in T(a, b)$. Since $T(a, b)=\Theta(a, b)$, also $\langle c_1, c_3\rangle\in T(a, b)$ and (3) is an easy consequence of Lemma 2.

(1)$\Rightarrow$(2) is analogous to that of (1)$\Rightarrow$(3), only the Mal'cev's lemma is used instead of Lemma 2.

The situation can be essentially simplified for a variety having a *uniform restricted congruence scheme* (for the definition, see [7]) and such principal tolerance trivial varieties can be characterized by a Mal'cev condition:

Theorem 3. *Let $\mathscr{V}$ be a variety of algebras having a uniform restricted congruence scheme* $\{p_0, \ldots, p_n; f\}$. *The following conditions are equivalent:*

(1) *$\mathscr{V}$ is principal tolerance trivial;*

(2) *There exists a* 6-ary *polynomial q over $\mathscr{V}$ such that*

$$q(x_0, x_1, x_0, x_1, y_0, y_1) = p_0(x_{f(0)}, x_0, x_1, y_0, y_1),$$
$$q(x_1, x_0, x_0, x_1, y_0\, y_1) = p_n(x_{1-f(n)}, x_0, x_1, y_0, y_1).$$

Proof. (1)$\Rightarrow$(2). Let $\{p_0, \ldots, p_n; f\}$ be a restricted congruence scheme satisfied by $\mathscr{V}$. Let $\mathfrak{A}\in\mathscr{V}$ be a $\mathscr{V}$-free algebra generated by the four-element set of free generators $\{x_0, x_1, y_0, y_1\}$. Then $\langle y_0, y_1\rangle\in\Theta(x_0, x_1)$ if and only if (see [7])

$$y_0 = p_0(x_{f(0)}, x_0, x_1, y_0, y_1),$$
$$p_i(x_{1-f(i)}, x_0, x_1, y_0, y_1) = p_{i+1}(x_{f(i+1)}, x_0, x_1, y_0, y_1) \quad \text{for} \quad i = 0, \ldots, n-1,$$
$$y_1 = p_n(x_{1-f(n)}, x_0, x_1, y_0, y_1).$$

According to (1), $\langle y_0, y_1\rangle\in T(x_0, x_1)$, i.e., Lemma 2 yields the existence of a binary algebraic function φ over $\mathscr{V}$ such that $y_0=\varphi(x_0, x_1)$, $y_1=\varphi(x_1, x_0)$. Since $\mathfrak{A}$ is a $\mathscr{V}$-free algebra with generators x_0, x_1, y_0, y_1, there exists a 6-ary polynomial q with

$$\varphi(x, y) = g(x, y, x_0, x_1, y_0, y_1)$$

whence (2) is evident.

The converse implication (2)$\Rightarrow$(1) is a direct consequence of Lemma 2.

3. Tolerance lattices of principal tolerance trivial algebras. It is easy to characterize whether the congruence lattice is a sublattice of the tolerance lattice for a principal tolerance trivial algebra:

Theorem 4. *Let $\mathfrak{A}$ be a principal tolerance trivial algebra. The following conditions are equivalent:*

(1) Con$(\mathfrak{A})$ *is a sublattice of* $LT(\mathfrak{A})$;

(2) $\mathfrak{A}$ *is tolerance trivial, i.e.,* Con$(\mathfrak{A})=LT(\mathfrak{A})$.

Proof. (2)$\Rightarrow$(1) is trivial. To prove (1)$\Rightarrow$(2), let $T\in LT(\mathfrak{A})$. By Lemma 1, T is the join of the tolerances $\{T(a, b);\ \langle a, b\rangle\in T\}$. Since $\mathfrak{A}$ is principal tolerance trivial, T is the join of the congruences $\{\Theta(a, b);\ \langle a, b\rangle\in T\}$ in $LT(\mathfrak{A})$ and, by (1), also in Con$(\mathfrak{A})$; thus $T\in$Con$(\mathfrak{A})$, proving Con$(\mathfrak{A})=LT(\mathfrak{A})$.

Corollary 2. *Let $\mathscr{V}$ be a principal tolerance trivial variety. The following conditions are equivalent:*

(1) *For each* $\mathfrak{A}\in\mathscr{V}$, Con$(\mathfrak{A})$ *is a sublattice of* $LT(\mathfrak{A})$;

(2) *$\mathscr{V}$ is congruence permutable.*

This is a direct consequence of Theorems 1 and 4.

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