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E-unitary covers and varieties of inverse semigroups

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1. Introduction and summary

E-unitary inverse semigroups have attracted considerable attention as a result of the remarkable work of McAlister [5], [6] concerning their structure and properties. He proved, inter alia, that every inverse semigroup S has an E-unitary cover, in the sense that there exists an E-unitary inverse semigroup P and an idempotent separating homomorphism of P onto S. Various properties and constructions of E-unitary covers were further established by McAlister and Reilly [7]. On the other hand, the lattice of varieties of inverse semigroups as algebras with a binary and a unary operation has been the focus of extensive investigations by several researchers; we mention only Kleĭman [3], [4].

The purpose of this note is to establish some surprising relationships between the two areas of research discussed above, viz., E-unitary covers and varieties of inverse semigroups. The main points of our consideration are: (i) which varieties admit E-unitary covers for their members, (ii) for a given variety of groups $\mathcal{U}$, which varieties of inverse semigroups $\mathcal{V}$ have E-unitary covers over $\mathcal{U}$, in the sense that every member S of $\mathcal{V}$ has an E-unitary cover P such that $P/\sigma\in\mathcal{U}$. The class $\mathcal{E}$ of all E-unitary inverse semigroups plays an important role in our investigation.

The content of the paper is briefly as follows. Some preliminary material is discussed in Section 2 in order to establish the notation and terminology. Several characterizations of varieties with E-unitary covers are established in Section 3. This is followed, in Section 4, by a description of subhomomorphisms in terms of homomorphisms of inverse semigroups, a result needed in the next section. The principal result of the paper, proved in Section 5 along with some consequences, provides several criteria for the existence of an E-unitary cover of an inverse semigroup S over

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a group variety $\mathscr{U}$. All varieties of inverse semigroups having E-unitary covers over a fixed group variety $\mathscr{U}$ are described in Section 6 in several ways. The relation v_3 defined on the lattice of varieties of inverse semigroups by: $\mathscr{U} v_3 \mathscr{V}$ if $\mathscr{U} \cap \mathscr{E} = \mathscr{V} \cap \mathscr{E}$ is discussed briefly in Section 7.

2. Preliminaries

We will follow the notation and terminology of Howie [2]. For background concerning inverse semigroups, we also refer the reader to this book.

Let S be an inverse semigroup. Then S is *E-unitary* if it satisfies the implication $xy=y \Rightarrow x^2=x$. The semilattice of idempotents of S will be denoted by E_S, the least group congruence by σ, the universal congruence by ω. The closure of a nonempty set A of S will be denoted by $A\omega$. An inverse semigroup P is an *E-unitary cover* of S if P is E-unitary and there is an idempotent separating homomorphism of P onto S; if $P/\sigma \cong G$ then P is an *E-unitary cover of S over G*.

Let ϱ be a congruence on S. The set

$$\ker \varrho = \{s \in S \mid s \varrho e \quad \text{for some} \quad e \in E_S\}$$

is the *kernel* of ϱ, $\operatorname{tr} \varrho = \varrho|_{E_S}$ is the *trace* of ϱ. The least congruence on S with the same trace as ϱ will be denoted by $\varrho_{\min}$. For a full discussion of these concepts, see Petrich [9]. The natural homomorphism $S \to S/\varrho$ will be denoted by $\varrho^\natural$. If $\varphi: S \to T$ is a homomorphism, we will denote by *ker* φ the kernel of the congruence on S induced by φ.

For any nonempty set X, we will denote the free inverse semigroup on X by I_X and the free group on X by G_X. The variety of all inverse semigroups will be denoted by $\mathscr{I}$, that of all groups by $\mathscr{G}$ and the lattice of all varieties of inverse semigroups by $\mathscr{L}(\mathscr{I})$. The variety generated by the semigroup S will be denoted by $\langle S \rangle$.

For a countably infinite set X and any $\mathscr{V} \in \mathscr{L}(\mathscr{I})$, let $\varrho(\mathscr{V})$ denote the fully invariant congruence on I_X corresponding to $\mathscr{V}$.

3. Varieties with E-unitary covers

The principal result here gives several characterizations of the varieties of inverse semigroups which have E-unitary covers. These characterizations involve free objects, E-unitary inverse semigroups and the kernel of the corresponding fully invariant congruence on the free object.

We start with a simple useful result.

Lemma 3.1. *Let ϱ be a congruence on an inverse semigroup S. Then S/ϱ is E-unitary if and only if* $\ker \varrho$ *is closed.*

Proof. Suppose that S/ϱ is E-unitary and let $a \in (\ker \varrho)\omega$. Then $ea \in \ker \varrho$ for some $e \in E_S$ and thus $ea\,\varrho\,(ea)^2$ which implies that $a\,\varrho\,a^2$ since S/ϱ is E-unitary. But then $a \in \ker \varrho$ and thus $\ker \varrho$ is closed.

Conversely, assume that $\ker \varrho$ is closed, and let $xy\,\varrho\,x$. Then $(x^{-1}x)y\,\varrho\,x^{-1}x$ so that $y \in (\ker \varrho)\omega = \ker \varrho$ and thus $y^2\,\varrho\,y$. Hence S/ϱ is E-unitary.

The following concept is basic for our considerations.

Definition 3.2. A variety $\mathscr{V}$ of inverse semigroups *has E-unitary covers* if, for every $S \in \mathscr{V}$, there is an E-unitary cover of S in $\mathscr{V}$.

We can now establish the first highlight of the paper.

Theorem 3.3. *The following conditions on a variety $\mathscr{V}$ of inverse semigroups are equivalent.*

(i) *$\mathscr{V}$ has E-unitary covers.*

(ii) *The free objects in $\mathscr{V}$ are E-unitary.*

(iii) *$\mathscr{V}$ is generated by its E-unitary members.*

(iv) $\ker \varrho(\mathscr{V})$ *is closed.*

Proof. (i) implies (ii). Let F be a $\mathscr{V}$-free inverse semigroup and S be an E-unitary cover for F in $\mathscr{V}$. There is an (idempotent separating) epimorphism $\varphi\colon S \to F$. Let $X \subseteq F$ be a set of $\mathscr{V}$-free generators of F, and let T be a cross section of the congruence on S induced by φ. Define a bijection $\psi\colon X \to T$ by $x\psi = t$ if $t \in T$ and $t\varphi = x$. Then ψ extends uniquely to a homomorphism ψ of F into S. For any $x \in X$, we have $x\psi\varphi = x$ so that $\psi\varphi$ is an endomorphism on F which restricts to the identity on X. Since X is a set of $\mathscr{V}$-free generators of F it follows that $\psi\varphi$ is the identity map on F. But then ψ is one-to-one and thus a monomorphism of F into S. Since S is E-unitary, so also is $F\psi$. Since ψ is a monomorphism, it follows that F is E-unitary.

(ii) implies (iii) trivially.

(iii) implies (i). Let $S \in \mathscr{V}$. By the general theory of varieties and the hypothesis, there exist E-unitary inverse semigroups T_α in $\mathscr{V}$, an inverse semigroup T which is a subdirect product of T_α's and an epimorphism $\varphi\colon T \to S$. Let ϱ be the congruence on T induced by φ. Letting $\varrho_{\min}$ be the least congruence on T with the same trace as ϱ, we obtain the following diagram of epimorphisms:

$$\begin{array}{ccc} T & \xrightarrow{\ \varphi\ } & S \\ {\scriptstyle \varrho_{\min}^{\natural}}\downarrow & & \uparrow{\scriptstyle \psi} \\ T/\varrho_{\min} & \xrightarrow{\ \tau\ } & T/\varrho \end{array}$$

where $\tau: t\varrho_{\min} \to t\varrho$ $(t \in T)$, and ψ is an isomorphism. Since ϱ and $\varrho_{\min}$ have the same trace, τ is one-to-one on idempotents, that is to say, it is idempotent separating. In view of ([10], Theorem 4.2), $a\, \varrho_{\min}\, b$ if and only if $ae=be$ and $e\, \varrho\, a^{-1}a\, \varrho\, b^{-1}b$ for some $e \in E_S$. Thus $\sigma \supseteq \varrho_{\min}$. This together with the fact that T is E-unitary implies

$$\ker \varrho_{\min} \subseteq \ker \sigma = E_T$$

and thus $\ker \varrho_{\min} = E_T = E_T\omega$. This implies by Lemma 3.1 that $T/\varrho_{\min}$ is E-unitary. Since $T/\varrho_{\min} \in \mathscr{V}$, we have proved that S has an E-unitary cover in $\mathscr{V}$.

The equivalence of items (ii) and (iv) follows by Lemma 3.1.

Remark. Part of Theorem 3.3 has been obtained independently by F. Pastijn [8].

4. Subhomomorphisms

The results proved in this section contain a description of subhomomorphisms in terms of homomorphisms and will be used in the construction of subdirect products which in turn will be needed in a construction of E-unitary covers.

We start with a concept which will prove quite useful.

Definition 4.1. Let S and T be inverse semigroups. Then a mapping $\varphi: S \to 2^T$ is a *subhomomorphism* of S into T if, for all $s, t \in S$,

(i) $s\varphi \neq \emptyset$;

(ii) $(s\varphi)(t\varphi) \subseteq (st)\varphi$;

(iii) $s^{-1}\varphi = (s\varphi)^{-1}$,

where, for any subset A of T, $A^{-1} = \{a^{-1} | a \in A\}$.

From (ii) and (iii) it follows that $S\varphi = \cup \{s\varphi : s \in S\}$ is an inverse subsemigroup of T and φ is said to be *surjective*, if $S\varphi = T$.

If T is a group, then the subhomomorphism φ above is *unitary* if for any $s \in S$, $1 \in s\varphi$ implies $s \in E_S$.

The following result will be needed.

Proposition 4.2. [7] *Let S and T be inverse semigroups and let φ be a (surjective) subhomomorphism of S into T. Then*

$$\Pi(S, T, \varphi) = \{(s, t) \in S \times T \,|\, t \in s\varphi\}$$

is an inverse semigroup (which is a subdirect product of S and T).

Conversely, suppose that V *is an inverse semigroup which is a subdirect product of* S *and* T *and let* ψ *be the induced monomorphism of* V *into* $S\times T$. *Then* φ *defined by*

$$s\varphi = \{t\in T\,|\,(s, t)\in V\psi\}$$

is a surjective subhomomorphism of S *into* T. *Furthermore,* $V\psi=\Pi(S, T, \varphi)$.

Theorem 4.3. *Let* R, S *and* T *be inverse semigroups. Let* $\alpha: R\to S$ *be an epimorphism and* $\beta: R\to T$ *a homomorphism. Then* $\varphi=\alpha^{-1}\beta$ *is a subhomomorphism of* S *into* T *and every such subhomomorphism is obtained in this way. If, in addition,* T *is a group, then* φ *is unitary if and only if* $\ker\beta\subseteq\ker\alpha$.

Proof. (i) It is clear that $s\varphi\neq\emptyset$ $(s\in S)$, since α is an epimorphism.

(ii) Let $x\in s\varphi$, $y\in t\varphi$. Then there exist $x', y'\in R$ with $x'\alpha=s$, $x'\beta=x$, $y'\alpha=t$, $y'\beta=y$. Hence $(x'y')\alpha=st$ while $(x'y')\beta=xy$ and $xy\in(st)\varphi$. Therefore $(s\varphi)(t\varphi)\subseteq(st)\varphi$.

(iii) With x, x' as in (ii), $(x')^{-1}\alpha=s^{-1}$, $(x')^{-1}\beta=x^{-1}$. Hence $x^{-1}\in s^{-1}\varphi$, $(s\varphi)^{-1}\subseteq s^{-1}\varphi$ and conversely. Thus φ is a subhomomorphism.

Conversely, if φ is a subhomomorphism of S into T, let $R=\Pi(S, T, \varphi)$. Let $\alpha: (s, t)\to s$ and $\beta: (s, t)\to t$ be the projections of R onto S and onto T, respectively. Now, $(s, t)\in R$ if and only if $t\in s\varphi$ while $t\in s\alpha^{-1}\beta$ if and only if $(s, t)\in R$ which gives $\varphi=\alpha^{-1}\beta$.

Let T be a group, φ be unitary and $r\in\ker\beta$. Then $r\beta=1$ and $1\in(r\alpha)\varphi$. Since φ is unitary, $r\alpha\in E_S$, $r\in\ker\alpha$ and so $\ker\beta\subseteq\ker\alpha$. Conversely, if this inclusion holds and $1\in s\varphi$, then for some $r\in R$, $r\alpha=s$ and $r\beta=1$. Hence $r\in\ker\beta\subseteq\ker\alpha$ so that $s^2=s$ and φ is unitary.

The usefulness of Theorem 4.3 lies in the fact that by choosing R appropriately, for example to be a free inverse semigroup, it is possible to generate subhomomorphisms. This technique will be used in the next section.

In fact, in order to obtain all subhomomorphisms it suffices to let R range over all free inverse semigroups, as we now show.

Proposition 4.4. *Let* $\theta: S\to T$ *be a subhomomorphism of the inverse semigroup* S *into the inverse semigroup* T. *Then there exist a free inverse semigroup* F, *an epimorphism* $\alpha: F\to S$, *and a homomorphism* $\beta: F\to T$ *with* $\theta=\alpha^{-1}\beta$.

Proof. By Theorem 4.3, there exist an inverse semigroup R, an epimorphism $\gamma: R\to S$ and a homomorphism $\delta: R\to T$ with $\theta=\gamma^{-1}\delta$. Let I_R be the free inverse semigroup on the set R and let $\pi: I_R\to R$ be the homomorphism defined by the identity mapping on the set of generators R. Let $\alpha=\pi\gamma$, $\beta=\pi\delta$ and let $x\in S$.

If $y \in x\theta$, then $x=z\gamma$, $y=z\delta$, for some $z \in R$ and so, considering z as a generator of I_R, we have $x=z\pi\gamma=z\alpha$, $y=z\pi\delta=z\beta$ and so $y \in x\alpha^{-1}\beta$. Conversely, if $y \in x\alpha^{-1}\beta$, then $x=z\alpha=(z\pi)\gamma$, $y=z\beta=(z\pi)\delta$, for some $z \in I_R$, and so $y \in x\gamma^{-1}\delta=\theta$. Therefore $\theta=\alpha^{-1}\beta$.

5. *E*-unitary covers over a group variety

The question that we now wish to consider is the following: for a given inverse semigroup S, or variety of inverse semigroups $\mathcal{V}$, and a given group variety $\mathcal{U}$, when will S or every member of $\mathcal{V}$ possess an E-unitary cover over some member of $\mathcal{U}$?

For the purposes of the following discussion, we consider inverse semigroups and groups as algebras in the variety of unary semigroups, that is as algebras with a binary operation $((x, y)\to xy)$ and unary operation $(x\to x^{-1})$.

Notation 5.1. Let X be a countably infinite set. We denote the free unary semigroup on X by U_X.

Any law in a unary semigroup is of the form $u=v$, for some $u, v \in U_X$. A construction for U_X was recently given by CLIFFORD [1].

For each set X, there exist fully invariant congruences $\varkappa, \lambda$ on U_X such that I_X and G_X are isomorphic to $U_X/\varkappa$ and U_X/λ, respectively, since I_X and G_X are free objects in their respective varieties. We will identify I_X and G_X with $U_X/\varkappa$ and U_X/λ, respectively.

Notation 5.2. Let X be any countably infinite set. For any variety of inverse semigroups $\mathcal{V}$, let $K_{\mathcal{V}}=\ker \varrho(\mathcal{V})$ and for any variety of groups $\mathcal{U}$, let $N_{\mathcal{U}}$ denote the corresponding fully invariant subgroup of G_X.

Definition 5.3. Let $\mathcal{U}$ be a variety of groups, S an inverse semigroup and $\mathcal{V}$ a variety of inverse semigroups. We will say that S (respectively, $\mathcal{V}$) *has E-unitary covers over* $\mathcal{U}$ if (for every $S \in \mathcal{V}$) there is a group $G \in \mathcal{U}$ for which there is an E-unitary cover of S over G.

It follows that $\mathcal{V}$ has E-unitary covers if and only if it has E-unitary covers over $\mathcal{V} \cap \mathcal{G}$.

Recall that an inverse monoid S with a group of units G is called *factorizable* if for each $s \in S$, there exists $g \in G$ such that $s \leqq g$. We will need the following results.

Theorem 5.4. [7] *Let G be a group and let S be an inverse semigroup. Let F be a factorizable inverse monoid with group of units G which contains S as an* *inverse subsemigroup. Suppose that, for each* $g \in G$, *there exists* $s \in S$ *such that* $s \leqq g$. *Then*

$$\{(s, g) \in S \times G | s \leqq g\}$$

is an E-unitary cover of S *over* G. *Conversely, each E-unitary cover is isomorphic to a cover obtained in this way.*

Proposition 5.5. [7] *Let* S *be an inverse semigroup and let* G *be a group. Suppose that* φ *is a surjective unitary subhomomorphism of* S *into* G. *Then* $\Pi(S, G, \varphi)$ *is an E-unitary cover of* S *over* G. *Conversely, let* P *be an E-unitary cover of* S *over* G *with associated homomorphisms* $\alpha: P \to S$, $\beta: P \to G$ *and let* $\psi: P \to S \times G$ *be the induced monomorphism. Then* φ *defined by*

$$s\varphi = \{g \in G | (s, g) \in P\psi\}$$

is a surjective unitary subhomomorphism of S *into* G *and* $P \cong \Pi(S, G, \varphi)$.

We are now ready for one of the main results of the paper.

Theorem 5.6. *Let* S *be an inverse semigroup,* $\mathscr{U}$ *be a variety of groups and* X *be a countably infinite set. The following are equivalent.*

(i) S *has an E-unitary cover over* $\mathscr{U}$.
(ii) *If* $u^2 = u$ *is a law in* $\mathscr{U}$, *then it is also a law in* S.
(iii) *For all homomorphisms* $\alpha: I_X \to S$, $K_{\mathscr{U}} \subseteqq \ker \alpha$.

Proof. (i) implies (ii). Let $G \in \mathscr{U}$ and P be an E-unitary cover of S over G. By Theorem 5.4, P is isomorphic to an inverse subsemigroup of a factorizable inverse monoid F with group of units G. Let $u^2 = u$ be a law in $\mathscr{U}$, say $u = u(x_1, \dots, x_n)$. Let $s_1, \dots, s_n \in S$. Since F is factorizable, there exist $g_1, \dots, g_n \in G$, with $s_i \leqq g_i$ $(i = 1, \dots, n)$. Then

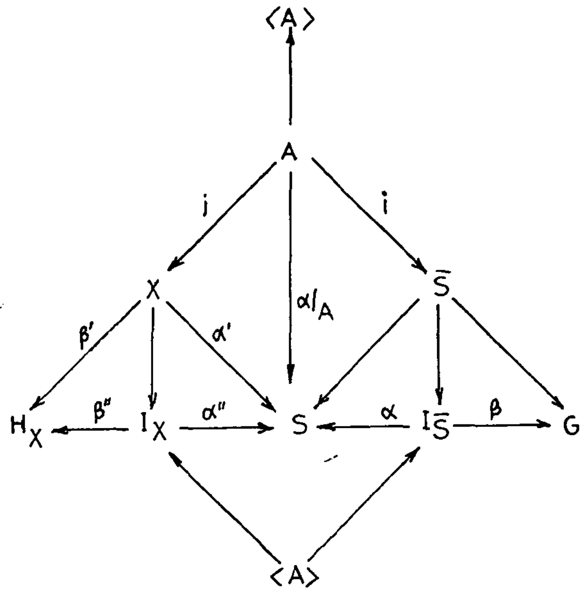
$$u(s_1, \dots, s_n) \leqq u(g_1, \dots, g_n)$$

where $u(g_1, \dots, g_n)$ is the identity of G, since $G \in \mathscr{U}$ and $u^2 = u$ is a law in $\mathscr{U}$. Hence $u(s_1, \dots, s_n)$ is an idempotent and $u^2 = u$ is a law in S.

(ii) implies (iii). Let $u \in U_X$ be such that $u\varkappa \in K_{\mathscr{U}}$. Then $u\lambda \in N_{\mathscr{U}}$ so that $u^2 = u$ is a law in $\mathscr{U}$ and so, by assumption, also in S. Hence, for any homomorphism $\beta: U_X \to S$, we have $u^2\beta = u\beta$. In particular, for any $\alpha: I_X \to S$, $u^2(\varkappa^\natural\alpha) = u(\varkappa^\natural\alpha)$ or $(u^2\varkappa, u\varkappa) \in \alpha \circ \alpha^{-1}$. Hence $u\varkappa \in \ker \alpha$.

(iii) implies (i). Let $\alpha: I_S \to S$ be the homomorphism defined on the generators of I_S by $s \to s$, let G be the free group in $\mathscr{U}$ on the set of generators S and let $\beta: I_S \to G$ be the natural homomorphism. By Theorem 4.3, $\theta = \alpha^{-1}\beta$ is a subhomomorphism of S into G. Since β is surjective so also is θ.

We next show that $\ker \beta \subseteqq \ker \alpha$. The following diagram illustrates the proof.



Since it will help to clarify the discussion, we will denote by $\bar{S}$ the underlying set of S.

Let $a \in \ker \beta$. Then there exists a finite subset $A = \{x_1, \ldots, x_n\}$ of $\bar{S}$ such that a is contained in the inverse subsemigroup $\langle A \rangle$ of $I_{\bar{S}}$ generated by A. Let us identify A with a subset of X and extend $\alpha|_A$ arbitrarily to a mapping $\alpha' : X \to S$. Let $\alpha'' : I_X \to S$ be the unique extension of α' to a homomorphism of I_X into S. Then $\alpha''|_{\langle A \rangle} = \alpha|_{\langle A \rangle}$.

Let H_X be the relatively free group in $\mathcal{U}$ on the set X and let $\beta' : X \to H_X$ embed X identically. Let $\beta'' : I_X \to H_X$ be the unique extension of β' to a homomorphism of I_X into H_X. Then $\ker \beta'' = K_{\mathcal{U}}$. Furthermore, since $\beta'|_A = \beta|_A$ we have $\beta''|_{\langle A \rangle} = \beta|_{\langle A \rangle}$. Since $a \in \ker \beta$, we have $a \in \ker \beta'' = K_{\mathcal{U}}$. Hence, by (iii), $a \in \ker \alpha''$ and so $a \in \ker \alpha$. Thus $\ker \beta \subseteq \ker \alpha$.

Hence by Theorem 4.3, θ is a unitary subhomomorphism and by Proposition 5.5, there exists an E-unitary cover of S over G.

Theorem 5.6 has an obvious analogue for any variety of inverse semigroups $\mathcal{V}$, obtained by letting S range over $\mathcal{V}$.

Corollary 5.7. *Let $\mathcal{V}$ be a variety of inverse semigroups and $\mathcal{U}$ be a variety of groups. The following are equivalent.*

(i) *$\mathcal{V}$ has E-unitary covers over $\mathcal{U}$.*

(ii) *If* $u^2=u$ *is a law in* $\mathcal{U}$, *then it is also a law in* $\mathcal{V}$.

(iii) $K_{\mathcal{U}}\subseteq K_{\mathcal{V}}$.

Corollary 5.8. *Let* S *be an inverse semigroup and* $\mathcal{U}$ *be a group variety. If* S *has an E-unitary cover over* $\mathcal{U}$, *then* $\langle S\rangle$ *has E-unitary covers over* $\mathcal{U}$.

Proof. Let $u^2=u$ be a law in $\mathcal{U}$. By Theorem 5.6 (ii), $u^2=u$ is also a law in S. But then $u^2=u$ is also a law in $\langle S\rangle$, and the desired conclusion follows from Corollary 5.7.

As an application of the above theory, we now produce a variety of inverse semigroups which has E-unitary covers over almost all varieties of groups, but which does not itself have E-unitary covers.

Proposition 5.9 . *Let* B_2 *denote the 5-element Brandt semigroup with 3 idempotents. Then* $\langle B_2\rangle$ *has E-unitary covers over any nontrivial group variety.*

Proof. Let I_1 denote the free inverse semigroup on one generator. It follows from [9] that, for each integer $n>1$, there is a congruence ϱ_n on I_1 such that $P_n=I_1/\varrho_n$ is an ideal extension of the cyclic group Z_n of order n by B_2 which is E-unitary. Furthermore, the projection of P_n onto B_2 is idempotent separating, since the ideal is a group. Hence each P_n is an E-unitary cover for B_2. Now $\mathcal{G}\cap\langle P_n\rangle$ is simply the variety $\mathcal{A}_n$ of abelian groups of exponent n. Thus B_2 and so, by Corollary 5.8, $\langle B_2\rangle$ has E-unitary covers over each variety $\mathcal{A}_n$ $(n>1)$, of abelian groups of exponent n, and so over every nontrivial variety of groups.

We shall now see how the equivalence of (iv) and (i) in Theorem 3.3 can be used to establish that varieties have E-unitary covers.

In $\mathcal{L}(\mathcal{I})$, the various varieties generated by groups, semilattices and Brandt semigroups constitute an ideal isomorphic to the product of $\mathcal{L}(\mathcal{G})$ and a three element chain. (See KLEĬMAN [3].) Following [9], we will call any semigroup in any of these varieties a *strict inverse semigroup*. Each variety of strict inverse semigroups which is not a variety of groups and semilattices of groups is generated by a single Brandt semigroup. Moreover, if $\mathcal{V}=\langle B\rangle$ where $B=\mathcal{M}^0(I,G,I;\Delta)$, then $\mathcal{V}=\langle G\rangle\vee\langle B_2\rangle$ where $\langle G\rangle$ is now a variety of groups. Similarly, any variety $\mathcal{V}$ of semilattices of groups which is not a variety of groups is of the form $\mathcal{U}\vee\mathcal{S}$, where $\mathcal{U}$ is a variety of groups and $\mathcal{S}$ is the variety of semilattices. For more details on this subject, see KLEĬMAN [3]:

Proposition 5.10. *If* $\mathcal{V}$ *is a variety of strict inverse semigroups containing nontrivial groups, then* $\mathcal{V}$ *has E-unitary covers.*

Proof. First let $\mathcal{V}=\mathcal{U}\vee\langle B_2\rangle$, where $\mathcal{U}$ is a nontrivial variety of groups and let $S\in\mathcal{V}$. By the general theory of varieties, there exist T, A, B where $A\in\mathcal{U}$,

$B\in\langle B_2\rangle$ and $T\subseteq A\times B$ is a subdirect product of A and B together with an epimorphism φ of T onto S. Since $\mathcal{U}$ is nontrivial, by Proposition 5.9 there exists an E-unitary cover P, say, of B over $\mathcal{U}$. Then $P\in\mathcal{U}\vee\langle B_2\rangle=\mathcal{V}$ by ([7], Corollary 1.8). Let $\alpha: P\to B$ be an idempotent separating epimorphism and let $T'=\{(a,p)|(a,p\alpha)\in T\}\subseteq A\times P$. Since A is a group and P is E-unitary, $A\times P$ is E-unitary. Hence T' is also E-unitary. Moreover, $T'\in\mathcal{V}$ and $(a,p)\to(a,p\alpha)\varphi$ is an epimorphism of T' onto S. By Theorem 3.3 (iv), $\mathcal{V}$ has E-unitary covers (over $\mathcal{U}$). A similar argument will show that any variety of semilattices of groups has E-unitary covers and clearly varieties of groups do also.

Remark 5.11. The arguments of Proposition 5.10 would also apply to any variety of the form $\mathcal{U}\vee\langle B_2^1\rangle$, where $\mathcal{U}$ is a non-trivial variety of groups.

6. The Malcev product

For any group variety $\mathcal{U}$ we will now characterize the class of all inverse semigroups $\mathcal{V}$ which have E-unitary covers over $\mathcal{U}$. It will turn out that the variety generated by the Malcev product $\mathcal{S}\circ\mathcal{U}$, where $\mathcal{S}$ denotes the variety of semilattices, is the greatest variety of inverse semigroups having E-unitary covers over $\mathcal{U}$. The variety generated by $\mathcal{S}\circ\mathcal{U}$ will be characterized in several ways.

Notation 6.1. We will denote by $\mathcal{S}$ the variety of all semilattices. For any variety of groups $\mathcal{U}$,

$$\mathcal{S}\circ\mathcal{U} = \{P\in\mathcal{I}\,|\,P \text{ is } E\text{-unitary and } P/\sigma\in\mathcal{U}\}$$

is the *Malcev product* of $\mathcal{S}$ and $\mathcal{U}$. For any family of laws $u_\alpha=v_\alpha$, $\alpha\in A$, we write $\langle u_\alpha=v_\alpha|\alpha\in A\rangle$ for the variety of inverse semigroups determined by these laws.

Another highlight of the paper can now be established.

Theorem 6.2. *The following statements are valid for any group variety $\mathcal{U}$.*

(i) $\langle\mathcal{S}\circ\mathcal{U}\rangle=\langle u^2=u|u^2$ *is a law in* $\mathcal{U}\rangle$.

(ii) $\langle\mathcal{S}\circ\mathcal{U}\rangle=\{S\in\mathcal{I}|S$ *has an E-unitary cover over* $\mathcal{U}\}$.

(iii) $\langle\mathcal{S}\circ\mathcal{U}\rangle$ *is the largest variety of inverse semigroups with E-unitary covers over* $\mathcal{U}$.

(iv) $\mathcal{U}$ *is the smallest variety of groups over which* $\langle\mathcal{S}\circ\mathcal{U}\rangle$ *has E-unitary covers.*

Proof. (i) Let $\mathcal{V}=\langle\mathcal{S}\circ\mathcal{U}\rangle$ and $\mathcal{W}=\langle u^2=u|u^2=u$ is a law in $\mathcal{U}\rangle$. First let $S\in\mathcal{S}\circ\mathcal{U}$ and let $u^2=u$ be a law in $\mathcal{U}$. By the definition of $\mathcal{S}\circ\mathcal{U}$, we have

$S/\sigma \in \mathcal{U}$ and thus $u^2=u$ is a law in S/σ. Hence, for any substitution $\bar{u}$ of u in S, it follows that $\bar{u}^2\sigma\bar{u}$, whence $\bar{u}\in\ker\sigma=E_S$. Thus $u^2=u$ is a law in S. Consequently, $S\in\mathcal{W}$ and thus $\mathcal{S}\circ\mathcal{U}\subseteqq\mathcal{W}$. But then also $\mathcal{V}=\langle\mathcal{S}\circ\mathcal{U}\rangle\subseteqq\mathcal{W}$.

Conversely, let $S\in\mathcal{W}$. Then by Theorem 5.6, S has an E-unitary cover P over G for some $G\in\mathcal{U}$. It follows that $P\in\mathcal{S}\circ\mathcal{U}$ and hence $S\in\langle\mathcal{S}\circ\mathcal{U}\rangle=\mathcal{V}$. Therefore $\mathcal{W}\subseteqq\mathcal{V}$ and equality prevails.

(ii) This is a direct consequence of part (i) and Theorem 5.6.

(iii) This is an obvious consequence of part (ii).

(iv) Let $\mathcal{V}$ be a variety of groups over which $\langle\mathcal{S}\circ\mathcal{U}\rangle$ has E-unitary covers, and let $G\in\mathcal{U}$. Then $G\in\langle\mathcal{S}\circ\mathcal{U}\rangle$ and hence has an E-unitary cover P over $\mathcal{V}$. Now, P being an E-unitary cover of a group must itself be a group. Since G is a homomorphic image of P, we obtain that $G\in\mathcal{V}$. Consequently $\mathcal{U}\subseteqq\mathcal{V}$, as required.

An interesting property of the varieties $\mathcal{V}$ between $\mathcal{U}$ and $\mathcal{S}\circ\mathcal{U}$ is provided by the next result.

Proposition 6.3. *For any variety of groups $\mathcal{U}$ and any variety $\mathcal{V}$ of inverse semigroups, the following holds:*

$$\ker\varrho(\mathcal{U})=\ker\varrho(\mathcal{V})\Leftrightarrow\mathcal{U}\subseteqq\mathcal{V}\subseteqq\langle\mathcal{S}\circ\mathcal{U}\rangle.$$

Proof. First assume that $\ker\varrho(\mathcal{U})=\ker\varrho(\mathcal{V})$. This means that $w^2=w$ is a law in $\mathcal{U}$ if and only if $w^2=w$ is a law in $\mathcal{V}$. It follows from Theorem 6.2 (i) that $\mathcal{V}\subseteqq\langle\mathcal{S}\circ\mathcal{U}\rangle$. Since $\mathcal{U}$ is a group variety, $\operatorname{tr}\varrho(\mathcal{U})=\omega$ and thus $\operatorname{tr}\varrho(\mathcal{U})\supseteqq\operatorname{tr}\varrho(\mathcal{V})$. This together with the hypothesis that $\ker\varrho(\mathcal{U})=\ker\varrho(\mathcal{V})$ implies that $\varrho(\mathcal{U})\supseteqq\varrho(\mathcal{V})$ and thus $\mathcal{U}\subseteqq\mathcal{V}$.

Conversely, assume that $\mathcal{U}\subseteqq\mathcal{V}\subseteqq\langle\mathcal{S}\circ\mathcal{U}\rangle$. The first inclusion implies $\varrho(\mathcal{U})\supseteqq\varrho(\mathcal{V})$ and thus $\ker\varrho(\mathcal{U})\supseteqq\ker\varrho(\mathcal{V})$. The second inclusion implies $\ker\varrho(\mathcal{U})\subseteqq\ker\varrho(\mathcal{V})$ by Theorem 6.2 (i), as above. Therefore $\ker\varrho(\mathcal{U})=\ker\varrho(\mathcal{V})$.

7. An equivalence relation on $\mathcal{L}(\mathcal{I})$

We introduce a relation on $\mathcal{L}(\mathcal{I})$ which relates any two varieties if they have the same E-unitary members and consider some associated properties.

In order to put the relation we are introducing into the proper perspective, we include two known relations ν_1 and ν_2 in our scheme. For any $\mathcal{U},\mathcal{V}\in\mathcal{L}(\mathcal{I})$, let

$$\mathcal{U}\nu_1\mathcal{V}\Leftrightarrow\mathcal{U}\cap\mathcal{A}=\mathcal{V}\cap\mathcal{A},\quad\mathcal{U}\nu_2\mathcal{V}\Leftrightarrow\mathcal{U}\cap\mathcal{G}=\mathcal{V}\cap\mathcal{G},\quad\mathcal{U}\nu_3\mathcal{V}\Leftrightarrow\mathcal{U}\cap\mathcal{E}=\mathcal{V}\cap\mathcal{E}.$$

Here $\mathscr{A}$, $\mathscr{G}$, and $\mathscr{E}$ denote the classes of all antigroups (fundamental inverse semigroups), groups and E-unitary inverse semigroups. The relations ν_1 and ν_2 were introduced by KLEĬMAN [3], who showed that they are congruences. He defined ν_1 as follows: $\mathscr{U}\nu_1\mathscr{V}\Leftrightarrow\mathscr{U}\vee\mathscr{G}=\mathscr{V}\vee\mathscr{G}$, and then proved the above equivalence. The relation ν_3 is new and the subject of our study in this section.

We can say that $\mathscr{U}\nu_3\mathscr{V}$ precisely when $\mathscr{U}$ and $\mathscr{V}$ have the same E-unitary members.

Proposition 7.1. $\nu_1\cap\nu_2\subseteq\nu_3\subseteq\nu_2$.

Proof. Let $\mathscr{U}(\nu_1\cap\nu_2)\mathscr{V}$ and $S\in\mathscr{U}\cap\mathscr{E}$. Since S is E-unitary, $\mathscr{H}\cap\sigma=\varepsilon$, the equality relation. Hence $\mu\cap\sigma=\varepsilon$ and thus S is a subdirect product of S/μ and S/σ. Here $S/\mu\in\mathscr{U}\cap\mathscr{A}$ and $S/\sigma\in\mathscr{U}\cap\mathscr{G}$. Since $\mathscr{U}\nu_1\mathscr{V}$, we have $S/\mu\in\mathscr{V}\cap\mathscr{A}$, and since $\mathscr{U}\nu_2\mathscr{V}$, we get $S/\sigma\in\mathscr{V}\cap\mathscr{G}$. But then $S\in(\mathscr{V}\cap\mathscr{A})\vee(\mathscr{V}\cap\mathscr{G})\subseteq\mathscr{V}$, which proves that $\mathscr{U}\cap\mathscr{E}\subseteq\mathscr{V}\cap\mathscr{E}$. By symmetry, we conclude that $\mathscr{U}\nu_3\mathscr{V}$. This proves that $\nu_1\cap\nu_2\subseteq\nu_3$. If $\mathscr{U}\cap\mathscr{E}=\mathscr{V}\cap\mathscr{E}$, then intersecting by $\mathscr{G}$, we get $\mathscr{U}\cap\mathscr{G}=\mathscr{V}\cap\mathscr{G}$. Hence $\nu_3\subseteq\nu_2$.

Remark 7.2. It should be noted that ν_3 is not a congruence on $\mathscr{L}(\mathscr{I})$. If $\mathscr{W}=\langle B_2\rangle$, $\mathscr{W}'=\langle B_2^1\rangle$, then $\mathscr{W}\nu_3\mathscr{W}'$. However, $(\mathscr{W}\vee\mathscr{G})\cap\mathscr{E}\subset(\mathscr{W}'\vee\mathscr{G})\cap\mathscr{E}$.

Proposition 5.9 shows that, in general, for a given variety of inverse semigroups $\mathscr{V}$, there is no minimum variety $\mathscr{U}$ of groups such that $\mathscr{V}$ has E-unitary covers over $\mathscr{U}$. This may be contrasted with the next result.

Proposition 7.3. *The following statements are true for any variety of inverse semigroups* $\mathscr{V}$.

(i) $\langle\mathscr{V}\cap\mathscr{E}\rangle$ *is the smallest member of the* ν_3*-class containing* $\mathscr{V}$.

(ii) $\langle\mathscr{V}\cap\mathscr{E}\rangle$ *is the largest variety contained in* $\mathscr{V}$ *having E-unitary covers.*

(iii) $\langle\mathscr{V}\cap\mathscr{E}\rangle=\{S\in\mathscr{I}\mid S$ *has an E-unitary cover in* $\mathscr{V}\}$.

Proof. (i) First note that

$$\langle\mathscr{V}\cap\mathscr{E}\rangle\cap\mathscr{E}\subseteq\mathscr{V}\cap\mathscr{E}\subseteq\langle\mathscr{V}\cap\mathscr{E}\rangle\cap\mathscr{E}$$

which shows that $\langle\mathscr{V}\cap\mathscr{E}\rangle\nu_3\mathscr{V}$. Now let $\mathscr{W}\nu_3\mathscr{V}$. Then $\mathscr{W}\cap\mathscr{E}=\mathscr{V}\cap\mathscr{E}$ which implies that $\langle\mathscr{V}\cap\mathscr{E}\rangle=\langle\mathscr{W}\cap\mathscr{E}\rangle\subseteq\mathscr{W}$, as required.

(ii) Since $\langle\mathscr{V}\cap\mathscr{E}\rangle$ is generated by E-unitary inverse semigroups, it has E-unitary covers by Theorem 3.3. Let $\mathscr{W}$ be a variety of inverse semigroups contained in $\mathscr{V}$ and having E-unitary covers. Again by Theorem 3.3, we get $\mathscr{W}=\langle\mathscr{W}\cap\mathscr{E}\rangle$. Since also $\langle\mathscr{W}\cap\mathscr{E}\rangle\subseteq\langle\mathscr{V}\cap\mathscr{E}\rangle$, we conclude that $\mathscr{W}\subseteq\langle\mathscr{V}\cap\mathscr{E}\rangle$, as required.

(iii) We have already observed that every S in $\langle\mathscr{V}\cap\mathscr{E}\rangle$ has an E-unitary cover in $\langle\mathscr{V}\cap\mathscr{E}\rangle$ and thus in $\mathscr{V}$. Conversely, let S have an E-unitary cover

P in $\mathscr{V}$. Hence $P\in\mathscr{V}\cap\mathscr{E}\subseteq\langle\mathscr{V}\cap\mathscr{E}\rangle$ and S is a homomorphic image of P so that $S\in\langle\mathscr{V}\cap\mathscr{E}\rangle$.

It can be verified easily that any group variety $\mathscr{U}$ alone constitutes a ν_3-class. If $\mathscr{V}$ is a variety of inverse semigroups contained in $\langle x^n=x^{n+1}\rangle$, then no S in $\mathscr{V}$ which is not a semilattice is E-unitary since $a^n=a^na$ and $a^2\neq a$ for any non-idempotent element a in S. In view of this and the results of KLEĬMAN [3], we conclude that the join of all varieties ν_3-equivalent to $\mathscr{S}$ is equal to $\mathscr{I}$.

Some additional information about $\langle\mathscr{S}\circ\mathscr{U}\rangle$ is provided by the following statement.

Proposition 7.4. *For any group variety* $\mathscr{U}$, *we have*

$$\langle\mathscr{S}\circ\mathscr{U}\rangle\cap\mathscr{G}=\mathscr{U},\quad \langle\mathscr{S}\circ\mathscr{U}\rangle\cap\mathscr{E}=\mathscr{S}\circ\mathscr{U}.$$

Proof. Let $G\in\langle\mathscr{S}\circ\mathscr{U}\rangle\cap\mathscr{G}$ and let $u^2=u$ be a law in $\mathscr{U}$. By Theorem 5.6, $u^2=u$ is also a law in G, and thus $G\in\mathscr{U}$ since every law in $\mathscr{U}$, except $xx^{-1}=yy^{-1}$, can be written in the form $u^2=u$. Consequently, $\langle\mathscr{S}\circ\mathscr{U}\rangle\cap\mathscr{G}\subseteq\mathscr{U}$; the opposite inclusion is obvious.

Let $S\in\langle\mathscr{S}\circ\mathscr{U}\rangle\cap\mathscr{E}$. Then $S/\sigma\in\langle\mathscr{S}\circ\mathscr{U}\rangle\cap\mathscr{G}=\mathscr{U}$ by the first formula. Since S is E-unitary, we obtain that $S\in\mathscr{S}\circ\mathscr{U}$. Therefore $\langle\mathscr{S}\circ\mathscr{U}\rangle\cap\mathscr{E}\subseteq\mathscr{S}\circ\mathscr{U}$; the opposite inclusion is trivial.

In connection with the congruences ν_1 and ν_2, and Theorem 3.3, the next proposition seems to be of some interest. For it, we need a known result.

Lemma 7.5. [3] *For any variety of inverse semigroups* $\mathscr{V}$, *the minimum element of* $\mathscr{V}(\nu_1\cap\nu_2)$ *is* $(\mathscr{V}\cap\mathscr{G})\vee\langle\mathscr{V}\cap\mathscr{A}\rangle$.

Proposition 7.6. *Let* $\mathscr{V}$ *be a variety of inverse semigroups. Consider the following conditions on* $\mathscr{V}$.

(i)—(iv) *The conditions of Theorem* 3.3.

(v) *For every* $S\in\mathscr{V}$, *there exists* $G\in\mathscr{V}\cap\mathscr{G}$, *an inverse semigroup* T *which is a subdirect product of* S/μ *and* G, *and an idempotent separating epimorphism* $\varphi: T\to S$.

(vi) $\mathscr{V}$ *is the minimum element of its* $\nu_1\cap\nu_2$*-class.*

Then (i) *implies* (v) *and* (v) *implies* (vi).

Proof. (i) implies (v). Let $S, T\in\mathscr{V}$ where T is an E-unitary cover of S. Then T is a subdirect product of T/μ and T/σ since $\mu\cap\sigma=\varepsilon$. Since T is an E-unitary cover of S it follows that $T/\mu\cong S/\mu$, so that T is a subdirect product of S/μ and S/σ, where the latter is in $\mathscr{V}\cap\mathscr{G}$.

(v) implies (vi). Let the notation be as in part (v). Then $S\in\langle S/\mu\times G\rangle\subseteqq\subseteqq\langle\mathcal{V}\cap\mathcal{A}\rangle\vee(\mathcal{V}\cap\mathcal{G})$ which proves that $\mathcal{V}\subseteqq\langle\mathcal{V}\cap\mathcal{A}\rangle\vee(\mathcal{V}\cap\mathcal{G})$; the opposite inclusion is trivial. By Lemma 7.5, we have that $\mathcal{V}$ is the minimum element of its $v_1\cap v_2$-class.

The first implication in the above proposition cannot be reversed. For example, the variety $\mathcal{V}=\langle x^3=x^2\rangle$ of inverse semigroups satisfies part (v) but not part (i). We have no counterexample for the converse of the second implication.

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