Acta Sci. Math. 46 (1983), 73—75

Separation of the radical in ring varieties

M. V. VOLKOV

Varieties of associative rings in which the Jacobson radical of every member is 1) nil, 2) nilpotent, or 3) a direct summand were studied in [1]. Varieties satisfying 1) or 2) were described there; the same varieties were characterized independently by the author in [2]. As to condition 3), Theorem 19 from [1] states that varieties in which the Jacobson radical of every finitely generated ring is a direct summand may be given by a finite set of two-variable identities. However these identities cannot be found by the method from [1], and the problem of exact description of varieties satisfying condition 3) remained open. This note is devoted to solve that problem.

Theorem. *The following conditions on an associative ring variety $\mathfrak{X}$ are equivalent:*

(a) *the Jacobson radical of every member is a direct summand;*

(b) *the Jacobson radical of every finitely generated member is a direct summand;*

(c) $\mathfrak{X}$ *is generated by a finite (possibly empty) set of finite fields and by a nil-ring of restricted index;*

(d) *the identities*

$$X^k Y = YX^k = X^k Y^n \tag{*}$$

hold in $\mathfrak{X}$ for some natural numbers $k \geqq 1$ and $n \neq 1$.

Proof. (a)→(b) obviously.

(b)→(c). We consider for every prime number p the variety $\mathfrak{A}_p$ given by the identities $XY-YX=pX=0$. There are finitely generated rings in $\mathfrak{A}_p$ in which the radical is not a direct summand, for example, the ring S_p of all 2×2 matrices of the form $\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}$ is such where α and β run through the p-element field. Hence

Received April 13, 1982.

$\mathfrak{X}$ does not contain $\mathfrak{A}_p$ for any p, and some identity $X^m=X^n$, $m<n$, holds in $\mathfrak{X}$ by the main theorem from [2]. $\mathfrak{X}$ is generated by its finite rings ([2], Corollary 1) and therefore by its finite subdirectly irreducible rings. If R is such a ring then we get by (b) that either $R=J(R)$ (and R is nilpotent) or $J(R)=0$ (and R is simple). A finite simple ring is either a finite field or the ring of all $r\times r$ matrices over a finite field $(r>1)$. However, rings of the second type cannot be contained in $\mathfrak{X}$ since every such ring contains the ring S_p for some p as subring. We see that the variety is generated by its finite nilpotent rings and finite fields. It remains to note that only a finite number of finite fields may be contained in $\mathfrak{X}$ and all finite nilpotent rings from $\mathfrak{X}$ satisfy the identity $X^m=0$ (in view of the identity $X^m=X^n$ holding in $\mathfrak{X}$). Thus, the direct sum of all finite nilpotent rings from $\mathfrak{X}$ is the required nilring of restricted index.

(c)$\rightarrow$(d). Let $N, F_1, \ldots, F_s$ be rings generating $\mathfrak{X}$ where the identity $X^k=0$ holds in N, and $F_1, \ldots, F_s$ are finite fields. If F_i consists of m_i elements and $n=(m_1-1)\ldots(m_s-1)+1$, then the identity $X^n=X$ holds in every field F_i. We see that the identities (*) hold in all rings generating $\mathfrak{X}$, hence they hold in all rings from $\mathfrak{X}$.

(d)$\rightarrow$(a). Let R be a ring satisfying (*). It is easy to see that $J(R)$ is nil and the idempotents of R lie in its center. Further, since an arbitrary ring of $r\times r$ matrices over a field $(r>1)$ does not satisfy (*) a standard application of Kaplansky's theorem about primitive PI-rings shows that $R/J(R)$ is a subdirect sum of finite fields and satisfies therefore the identity $X^n=X$. Denote by E the ideal of R generated by all idempotents of R. Let $y=\sum_{i=1}^{m} r_ie_i\in J(R)\cap E$, where $r_i\in R$, e_i are idempotents. Let us consider the element

$$e=\sum_{i=1}^{m}e_i-\sum_{1\leq i<j\leq m}e_ie_j+\sum_{1\leq i<j<s\leq m}e_ie_je_s-\ldots+(-1)^{m+1}e_1\ldots e_m.$$

It can be immediately verified that $e^2=e$ and $e_ie=e_i$ for any i. Thus, $y=ye=$ $=ye^k=y^ne^k=y^{2n-1}e^k=\ldots=0$. On the other hand, the image of the element x^{n-1} in the ring $R/J(R)$ is an idempotent for every $x\in R$. We lift it to an idempotent e_x of the ring R; then $x-xe_x\in J(R)$ and $x=xe_x+(x-xe_x)\in E+J(R)$. We see that R is a direct sum of the ideals $J(R)$ and E.

The theorem is proved.

Let us recall that a ring R is called a semidirect sum of an ideal J and a subring S if $S+J=R$, $S\cap J=0$. In connection with our theorem we pose a natural

Question. What are the ring varieties in which the Jacobson radical of 1) every, 2) every finitely generated member is a semidirect summand?

Note that these classes of varieties are sufficiently large. Thus, all locally finite varieties of prime characteristic belong to the second of them by Wedderburn's classical theorem about separation of the radical.

The author expresses sincere gratitude to Professor L. N. Ševrin for helpful discussions.

References

[1] A. A. ISKANDER, The radical in ring varieties, *Acta Sci. Math.*, **39** (1977), 291—301.
[2] M. V. VOLKOV, Periodic varieties of associative rings, *Izv. Vysš. Učebn. Zaved. Matematika*, **1979**, no. 8, 3—13. (Russian)

URAL UNIVERSITY
620083 SVERDLOVSK, USSR