Acta Sci. Math. 46 (1983), 77—83

Disjoint sublattices of lattices

M. E. ADAMS and J. SICHLER

1. Introduction

M. Sekanina asked whether there exist lattices A and B such that A contains an arbitrarily large finite number of pairwise disjoint sublattices isomorphic to B but does not contain infinitely many pairwise disjoint sublattices isomorphic to B. Independently, I. KOREC [2] and V. KOUBEK [3] have shown that such lattices do indeed exist. In fact, Koubek has shown that both A and B may be chosen to be distributive.

The aim of the present paper is to strengthen Koubek's result by showing that the distributive lattices A and B may be chosen to be totally ordered sets. Actually more will be shown. The principal result will be the following:

Theorem. *There exist totally ordered sets A and B_α, for $\alpha<2^{2^{\aleph_0}}$, such that* (i) $|A|=2^{\aleph_0}$, (ii) $B_\alpha\cong B_\beta$ *if and only if* $\alpha=\beta$, *and* (iii) *if* $\alpha<2^{2^{\aleph_0}}$ *then, for* $n<\omega$, *A contains n disjoint copies of B_α, but it does not contain infinitely many such copies.*

That A is uncountable is no coincidence. A routine proof, using Hausdorff's classification of the countable order types, shows that if A is a countable totally ordered set that contains an arbitrarily large number of finite disjoint copies of a totally ordered set B then A contains infinitely many disjoint copies of B. (We shall not include the details.)

It is a pleasure to acknowledge the helpful suggestions made by the referee.

Received September 16, 1981, and in revised form February 15, 1982.

The support of the Research Foundation of the State University of New York and the National Research Council of Canada is gratefully acknowledged.

2. The construction

The construction of the totally ordered set A involves a new variation of a technique first introduced by B. DUSHNIK and E. W. MILLER [1].

Let λ denote the real line $[0, 1)$ and η its rational members. The Dedekind completion of a totally ordered set C will be given by C^+. Observe that for two totally ordered sets C and D any order preserving injection of C into D can be extended to an order preserving injection of C^+ into D^+. Since a monotone function on λ has at most countably many discontinuities, it is readily seen that there are $2^{\aleph_0}$ order preserving injections of λ into itself. With the exception of the identity function, let $G=\{g_\beta \mid 1\leqq\beta<2^{\aleph_0}\}$ be a list of all the order preserving injections of λ into itself.

We now define a distinguished countable subset of G. For $1\leqq i<\omega$ and $1\leqq k\leqq i!$ define

$$I_{ik} = [(k-1)/(i!),\ k/(i!));$$

that is, for each i, $\{I_{ik}: 1\leqq k\leqq i!\}$ is a system of pairwise disjoint intervals of length $1/(i!)$ covering λ. If $1\leqq j\leqq i+1$, define an order preserving injection $f_{ij}: \lambda\to\lambda$ by

$$f_{ij}(x) = x/(i+1)+((k-1)i+(j-1))/(i+1!)$$

for $x\in I_{ik}$ and $k=1, \ldots, i!$. Observe that $f_{ij}(I_{ik})=[(k-1)/(i!)+(j-1)/(i+1!), (k-1)/(i!)+j/(i+1!))=J_{ijk}\subseteq I_{ik}$ for every $j=1, \ldots, i+1$. The function f_{ij} is said to be of *type* i.

By way of example, it follows that there are exactly two functions of type one: f_{11} is an order preserving bijection of $[0, 1)$ to $[0, 1/2)$ given by $f_{11}(x)=\frac{1}{2}x$; $f_{12}(x)=\frac{1}{2}x+\frac{1}{2}$ is an order preserving bijection of $[0, 1)$ to $[1/2, 1)$. There are three functions of type two: f_{21} is the order preserving bijection of $[0, 1)$ to $\left[0, \frac{1}{6}\right)\cup\left[\frac{1}{2}, \frac{2}{3}\right)$ defined by $f_{21}(x)=\frac{1}{3}x$, for $0\leqq x<\frac{1}{2}$, and $f_{21}(x)=\frac{1}{3}x+\frac{1}{3}$ for $\frac{1}{2}\leqq x<1$; f_{22} is the order preserving bijection from $[0, 1)$ to $\left[\frac{1}{6}, \frac{1}{3}\right)\cup\left[\frac{2}{3}, \frac{5}{6}\right)$ given by, for $0\leqq x<\frac{1}{2}$, $f_{22}(x)=\frac{1}{3}x+\frac{1}{6}$ and, for $\frac{1}{2}\leqq x<1$, $f_{22}(x)=\frac{1}{3}x+\frac{1}{2}$; finally, f_{23} is the order preserving bijection from $[0, 1)$ to $\left[\frac{1}{3}, \frac{1}{2}\right)\cup\left[\frac{5}{6}, 1\right)$ such that $f_{23}(x)=\frac{1}{3}x+\frac{1}{3}$, for $0\leqq x<\frac{1}{2}$, and $f_{23}(x)=\frac{1}{3}x+\frac{2}{3}$ for $\frac{1}{2}\leqq x<1$.

Let $F=\{f_{ij} \mid 1\leqq i<\omega$ and $1\leqq j\leqq i+1\}$; for $x\in\lambda$, denote $F(x)=\{f(x) \mid f\in F\}$;

and, for $X \subseteqq \lambda$, let $F(X)=\bigcup(F(x)\colon x\in X)$. Note that, for every $f\in F$, x is rational if and only if $f(x)$ is rational. Since F is countable, we may conclude the following:

Lemma 1. $|\{x\in\lambda | x\in F(x)\}|=\aleph_0$.

We shall also need the following lemma.

Lemma 2. *For* $X, Y\subseteqq\lambda$, *if* $|X|=2^{\aleph_0}$ *and* $|Y|<2^{\aleph_0}$ *then there exists* $x\in X$ *such that* $F(x)\cap Y=\emptyset$.

Proof. Suppose that for every $x\in X$ there exists an $f\in F$ with $f(x)\in Y$. For $y\in Y$, let $X_y=\{x\in X \mid y\in F(x)\}$. Thus, $X\subseteqq\bigcup(X_y\colon y\in Y)$. Since $|X|=2^{\aleph_0}$ and $|Y|<2^{\aleph_0}$, it follows that X_y is uncountable for some $y\in Y$. However, F is countable. Hence, there are two distinct elements x of X_y such that $f(x)=y$ for the same $f\in F$. Since each $f\in F$ is one-to-one, this is a contradiction. The proof is complete.

Some further notation is necessary. For $g\in G$, define $g_F=\{x\in\lambda \mid g(x)\notin F(x)\}$. Then set $G_F=\{g\in G \mid |g_F|<2^{\aleph_0}\}$. Clearly, $F\subseteqq G_F$ follows from $f_F=\emptyset$ for every $f\in F$; it is also easy to see that the inclusion is proper.

We are now ready to define the totally ordered sets A and B_α for $\alpha<2^{\aleph_0}$. As will transpire, the totally ordered set A will be a subset of λ that contains η; the definition will be given by transfinite induction. For $\beta<2^{\aleph_0}$, sets $A_\beta, C_\beta, D_\beta\subseteqq\lambda$ will be defined; subsequently, A will be the set $\lambda\setminus\bigcup(A_\beta\colon \beta<2^{\aleph_0})$ and, for $\alpha<2^{2^{\aleph_0}}$, $\bigcup(C_\beta\colon \beta<2^{\aleph_0})\subseteqq B_\alpha\subseteqq\bigcup(C_\beta\cup D_\beta\colon \beta<2^{\aleph_0})$. Intuitively, the mappings from F will be used to exhibit arbitrarily many finite disjoint copies of B_α in A and the construction will ensure that no $g\notin G_F$ can be used to provide an order preserving injection of B_α into A.

Let $A_0=\emptyset$, $A_0'=\eta$, $C_0=\eta$, $C_0'=\emptyset$, and $D_0=\emptyset$. By transfinite induction we will define, for $\beta<2^{\aleph_0}$, $A_\beta, A_\beta', C_\beta, C_\beta', D_\beta\subseteqq\lambda$ such that (i) $|A_\beta|, |A_\beta'|, |C_\beta|, |C_\beta'|, |D_\beta|<2^{\aleph_0}$, (ii) for $\gamma<\beta$, $A_\gamma\subseteqq A_\beta$, $A_\gamma'\subseteqq A_\beta'$, $C_\gamma\subseteqq C_\beta$, $C_\gamma'\subseteqq C_\beta'$, and $D_\gamma\subset D_\beta$, (iii) $A_\beta\cap A_\beta'=\emptyset$, $C_\beta\cap C_\beta'=\emptyset$, and $(C_\beta\cup C_\beta')\cap D_\beta=\emptyset$ and (iv) $F(C_\beta)\subseteqq A_\beta'$ and $F(D_\beta)\subseteqq A_\beta'$. (Note that these conditions are satisfied for $\beta=0$.) Suppose that, for $\gamma<\beta<2^{\aleph_0}$, A_γ, A_γ', C_γ, C_γ', D_γ are defined and satisfy (i), (ii), (iii), and (iv).

Since $g_\beta\colon\lambda\to\lambda$ is not the identity and is order preserving, there are $2^{\aleph_0}$ elements $x\in\lambda$ such that $x\neq g_\beta(x)$. Thus, because g_β is injective, the set of all elements $x\in\lambda$ such that $x\neq g_\beta(x)$, $x\notin\bigcup(C_\gamma'\colon \gamma<\beta)\cup\bigcup(D_\gamma\colon\gamma<\beta)$, and $g_\beta(x)\notin\bigcup(C_\gamma\colon\gamma<\beta)\cup \bigcup(D_\gamma\colon\gamma<\beta)$ has cardinality $2^{\aleph_0}$. By Lemma 2, choose such an $x\in\lambda$ for which $F(x)\cap\bigcup(A_\gamma\colon\gamma<\beta)=\emptyset$. Let $C_\beta'=\{g_\beta(x)\}\cup\bigcup(C_\gamma'\colon\gamma<\beta)$.

By Lemma 2, there exists $y\in\lambda\setminus(\{x\}\cup\bigcup(C_\lambda\colon\lambda<\beta)\cup C_\beta'\cup\bigcup(D_\gamma\colon\gamma<\beta)$ such that $F(y)\cap\bigcup(A_\gamma\colon\gamma<\beta)=\emptyset$. Choose such a $y\in\lambda$. Let $D_\beta=\{y\}\cup\bigcup(D_\gamma\colon\gamma<\beta)$.

There are now two cases to consider.

First, suppose $g_\beta \in G_F$. Let $A_\beta = \bigcup(A_\gamma: \gamma<\beta)$, $A'_\beta = F(x) \cup F(y) \cup \bigcup(A'_\gamma: \gamma<\beta)$, and $C_\beta = \{x\} \cup \bigcup(C_\gamma: \gamma<\beta)$. Clearly (i) and (ii) are satisfied. By the choice of $x \in \lambda$, $C_\beta \cap C'_\beta = \emptyset$ and, by the choice of $x, y \in \lambda$, $A_\beta \cap A'_\beta = \emptyset$ and $(C_\beta \cup C'_\beta) \cap D_\beta = \emptyset$; thus, (iii) holds. Obviously, by definition, (iv) is also valid.

Second, suppose $g_\beta \notin G_F$. Thus, $|(g_\beta)_F| = 2^{\aleph_0}$. Thus there are $2^{\aleph_0}$ elements $z \in (g_\beta)_F$ such that $z \notin C'_\beta \cup D_\beta$ and, since g_β is an injection, $g_\beta(z) \notin F(x) \cup F(y) \cup \bigcup(A'_\gamma: \gamma<\beta)$. By Lemma 2, we may choose the element z such that, in addition, $F(z) \cap \bigcup(A_\gamma: \gamma<\beta) = \emptyset$. Let $A_\beta = \{g_\beta(z)\} \cup \bigcup(A_\gamma: \gamma<\beta)$, $A'_\beta = F(x) \cup F(y) \cup F(z) \cup \bigcup(A'_\gamma: \gamma<\beta)$, and $C_\beta = \{x\} \cup \{z\} \cup \bigcup(C_\gamma: \gamma<\beta)$. Clearly, (i) and (ii) are valid. The choice of $z \in \lambda$ is such that $g_\beta(z) \notin F(z)$; thus, since $(F(x) \cup F(y) \cup F(z)) \cap \bigcup(A_\gamma: \gamma<\beta) = \emptyset$, it follows that $A_\beta \cap A'_\beta = \emptyset$. By choice, $C_\beta \cap C'_\beta = \emptyset$. As in the first case $C'_\beta \cap D_\beta = \emptyset$ and, by inspection, $C_\beta \cap D_\beta = \emptyset$; thus (iii) also holds. Once more it is clear that (iv) is valid.

As indicated earlier, we set $A = \lambda \setminus \bigcup(A_\beta: \beta<2^{\aleph_0})$, $A' = \bigcup(A'_\beta: \beta<2^{\aleph_0})$, $C = \bigcup(C_\beta: \beta<2^{\aleph_0})$, $D = \bigcup(D_\beta: \beta<2^{\aleph_0})$, and $B = C \cup D$. It follows, by (iii), that $A' \subseteqq A$. However, by (iv), $F(B) \subseteqq A' \subseteqq A$. Thus $f \restriction B$ is an order preserving injection from B into A for each $f \in F$. By (ii), $|D| = 2^{\aleph_0}$. Let $(S_\alpha: \alpha<2^{2^{\aleph_0}})$ be an indexing of the power set of D, let $B_\alpha = C \cup S_\alpha$ for $\alpha<2^{2^{\aleph_0}}$. Since $B_\alpha \subseteqq B$, the mapping $f \restriction B_\alpha$ is an order preserving injection of B_α into A for $\alpha<2^{2^{\aleph_0}}$ and $f \in F$.

3. Proof of the theorem

We first show that, for distinct $\alpha, \beta<2^{2^{\aleph_0}}$, $B_\alpha \ncong B_\beta$. If $\alpha \neq \beta$, then $S_\alpha \neq S_\beta$. Suppose, with no loss of generality, that there exists $s \in S_\alpha \setminus S_\beta$. If $S_\alpha \cong S_\beta$ then there is an order preserving injection $g: B_\alpha \to B_\beta$. In which case, g extends to an order preserving injection $g^+: B^+_\alpha \to B^+_\beta$. Since $\eta \subseteqq B_\alpha, B_\beta \subseteqq \lambda$, it follows that $g^+: \lambda \to \lambda$. By (iii), $C \cap D = \emptyset$; thus, $s \notin B_\beta$. Consequently, g^+ is not the identity function and, hence, $g^+ \in G$. Whence, for some $\gamma<2^{\aleph_0}$, $g^+ = g_\gamma$. However, for g_γ, there is $x \in \lambda$ for which $x \in C$ and $g_\gamma(x) \in C'$. By (iii), $C \cap C' = \emptyset$ and $D \cap C' = \emptyset$. Since $C \subseteqq B_\alpha, B_\beta \subseteqq C \cup D$, we conclude that $x \in B_\alpha$ and $g_\gamma(x) \notin B_\beta$. However, g_γ is an extension of $g: B_\alpha \to B_\beta$; that is to say, $g_\gamma(x) = g(x) \in B_\beta$. By contradiction, we conclude that there is no order preserving injection $g: B_\alpha \to B_\beta$. We have shown the following:

Lemma 3. *For* $\alpha, \beta<2^{2^{\aleph_0}}$, $B_\alpha \cong B_\beta$ *if and only if* $\alpha = \beta$.

For the interested reader, we remark that, in the construction, a more judicious choice of subsets of D yields the following stronger result: for distinct $\alpha, \beta<2^{2^{\aleph_0}}$, B_α is not a sublattice of B_β and B_β is not a sublattice of B_α.

For $\alpha<2^{2^{\aleph_0}}$, we have already observed that, for $1\leqq i<\omega$ and $1\leqq j\leqq i+1$, $f_{ij}\restriction B_\alpha$ is an order preserving injection from B_α into A. We now show that, for $n<\omega$, A contains n disjoint copies of B_α. As stated previously, for $1\leqq i<\omega$ and $1\leqq j\leqq i+1$, $f_{ij}\colon I_{ik}\to J_{ijk}$ is an order preserving bijection for every $1\leqq k\leqq i!$. Since, for distinct $1\leqq j, l\leqq i+1, J_{ijk}\cap J_{ilk}=\emptyset$, it follows that $f_{ij}(\lambda)\cap f_{il}(\lambda)=\emptyset$. Consequently, the restrictions of the functions of type i to B_α yield $i+1$ order preserving injections of B_α into A such that, for distinct $1\leqq j, l\leqq i+1, f_{ij}\restriction(B_\alpha)\cap \cap f_{il}\restriction(B_\alpha)=\emptyset$. Thus, we have shown:

Lemma 4. *Let* $\alpha<2^{2^{\aleph_0}}$. *For* $n<\omega$, *the totally ordered set* A *contains* n *disjoint copies of* B_α.

It remains to show that, for $\alpha<2^{2^{\aleph_0}}$, A does not contain infinitely many disjoint copies of B_α. Since, for every $\alpha<2^{2^{\aleph_0}}, C\subseteqq B_\alpha$, it is sufficient to show that A does not contain infinitely many disjoint copies of C.

Suppose that $g\colon C\to A$ is an order preserving injection. Then g extends to an order preserving injection $g^+\colon C^+\to A^+$. Again, since $\eta\subseteqq A, C\subseteqq\lambda$, it follows that $g^+\colon\lambda\to\lambda$; that is to say, if g^+ is not the identity function then $g^+\in G$.

Lemma 5. *Let* $g\colon C\to A$ *be an order preserving injection. If* g *is not the identity function, then* $g^+\in G_F$.

Proof. Suppose $g^+\notin G_F$. By the above comments, there exists $1\leqq\beta<2^{\aleph_0}$ such that $g^+=g_\beta$; thus, $g_\beta\notin G_F$. Hence, by the definition of A_β and C_β, there is $z\in(g_\beta)_F$ such that $z\in C_\beta$ and $g_\beta(z)\in A_\beta$. Consequently, $z\in C$ and $g_\beta(z)\notin A$. However, g_β is an extension of g; whence, $g_\beta(z)\in A$. By contradiction, we conclude $g^+\in G_F$.

Before considering infinitely many order preserving injections from C into A we must derive Lemma 8.

Let $g\in G_F$ and I be a nonempty open interval of λ. Since $g\in G_F$, $|\{x\in I\,|\,g(x)\notin \notin F(x)\}|\leqq|g_F|<2^{\aleph_0}$. Hence, $|\{x\in I\,|\,g(x)\in F(x)\}|=2^{\aleph_0}$ and, by Lemma 1, $|\{x\in I\,|\,x\neq g(x)\}|=2^{\aleph_0}$. Consequently, there exists $x\in I$ such that $x\neq g(x)$ and $x\neq(k-1)/(i!)$ for any $1\leqq i<\omega$ and $1\leqq k\leqq i!$. Select such an x. Since I is open there exists $d>0$ such that $(x-d, x+d)\subseteqq I$. For $d'=|g(x)-x|$, choose $1\leqq p<\omega$ such that $1/(p!)<\min\{d, d'\}$. Hence, there exists $1\leqq r\leqq p!$ such that $x\in I_{pr}\subseteqq I$ but $g(x)\notin I_{pr}$.

Lemma 6. *There is a nonempty open interval* $I'\subseteqq I_{pr}$ *such that, for* $y\in I'$, *either* $y\in g_F$ *or* $g(y)=f_{ij}(y)$ *for some* $1\leqq i<p$ *and* $1\leqq j\leqq i+1$.

Proof. For $1\leqq q\leqq p+1$, $f_{pq}(I_{pr})=J_{pqr}\subseteqq I_{pr}$. Furthermore, by definition, for $p\leqq i<\omega$ and $1\leqq j\leqq i+1$, $f_{ij}(I_{pr})\subseteqq I_{pr}$. Since, by hypothesis, $x\neq(r-1)/(p!)$

and $x \neq g(x)$, there is a nonempty open interval $I' \subseteq I_{pr}$ such that $g(I') \cap I_{pr} = \emptyset$. Thus, for $y \in I'$, either $g(y) \notin F(y)$ (in which case, $y \in g_F$), or there exists $1 \leqq i < \omega$ and $1 \leqq j \leqq i+1$ such that $f_{ij}(y) = g(y) \notin I_{pr}$. Since $y \in I_{pr}$, it follows that $i < p$. The proof is complete.

Since $g \in G_F$ is assumed, it follows that the set of all $y \in I'$ with $g(y) = f_{ij}(y)$ for some $1 \leqq i < p$ and $i \leqq j \leqq i+1$ has cardinality $2^{\aleph_0}$. Furthermore, any nonempty open interval contained in I' has the same property.

Lemma 7. *There is a nonempty open interval $I'' \subseteq I'$, $1 \leqq i < p$, and $1 \leqq j \leqq i+1$ such that, for $y \in I''$, $g(y) = f_{ij}(y)$.*

Proof. Since I' is nonempty and open, $I' = (u_0, v_0)$ for some distinct $u_0, v_0 \in \lambda$. Let $I_0 = I'$. For $n < \omega$, we inductively define a nonempty open interval $I_n = (u_n, v_n)$ such that, for $n \leqq m < \omega$, $I_n \supseteqq I_m$. Assume that I_n has been defined and choose, if possible, distinct $u_{n+1}, v_{n+1} \in I_n$ such that, for some $y \in I_n$, either there exist $1 \leqq i < p$ and $1 \leqq j \leqq i+1$ such that $g(y) = f_{ij}(y)$ but, for all $z \in (u_{n+1}, v_{n+1})$, $g(z) \neq f_{ij}(z)$, or $y \in g_F$ but, for all $z \in (u_{n+1}, v_{n+1})$, $z \notin g_F$. If u_{n+1} and v_{n+1} exist then set $I_{n+1} = (u_{n+1}, v_{n+1})$; otherwise, let $I_{n+1} = I_n$. Since there are only finitely many possibilities for i and j, there exists some $n < \omega$ such that $I_n = I_m$ for all $n \leqq m < \omega$. Let $I'' = I_n$. We must show that I'' satisfies the requirements of the lemma. By the remark preceding Lemma 7, there exists $y \in I''$ such that, for some $1 \leqq i < p$ and $1 \leqq j \leqq i+1$, $\langle y, g(y) \rangle \in f_{ij}$. Hence, by construction, for any distinct $u, v \in I''$, there exists $u < z < v$ such that $\langle z, g(z) \rangle \in f_{ij}$ for the same i and j; that is to say, the set of all elements $z \in I''$ such that $g(z) = f_{ij}(z)$ is dense in I''. Since g is order preserving and f_{ij} is continuous on I'' (recall that $I'' \subseteq I' \subseteq I_{pr}$ and f_{ij} is continuous on I_{pr}), it follows that $g(z) = f_{ij}(z)$ for all $z \in I''$. The lemma is verified.

The statement of the next lemma is immediate from the discussion following Lemma 5 together with Lemma 6 and Lemma 7.

Lemma 8. *Let $g \in G_F$ and let I be a nonempty open interval of λ. Then there exist a nonempty open interval $J \subseteq I$ and $f \in F$ such that $g(x) = f(x)$ for all $x \in J$.*

Suppose that, for $n < \omega$, $h_n \colon C \to A$ is an order preserving injection.

Lemma 9. *There exists a nonempty open interval $I \subseteq \lambda$ such that if $y \in I$ is rational then $y = h_0(x)$ for some rational x.*

Proof. If h_0 is the identity function then, since $\eta \subseteq C$, any open interval $I \subseteq \lambda$ will satisfy the lemma. If h_0 is not the identity then, by Lemma 5, $h_0^+ \in G_F$. Thus, by Lemma 8, there is a nonempty open interval $J \subseteq \lambda$, $1 \leqq i < \omega$, and $1 \leqq j \leqq i+1$ such that, for $x \in J$, $h_0^+(x) = f_{ij}(x)$. Since $\lambda = \bigcup (I_{ik} \colon 1 \leqq k \leqq i!)$, there is some $1 \leqq k \leqq i!$ such that $I_{ik} \cap J \neq \emptyset$. Choose a nonempty interval $I' \subseteq I_{ik} \cap J$.

By definition, f_{ij} is continuous on I_{ik} and, hence, it is a continuous order preserving injection on I'. Thus, $f_{ij}(I')$ is a nonempty open interval of λ. Let $I=f_{ij}(I')$. If $y\in I$ then $y=f_{ij}(x)=h_0^+(x)$ for some $x\in I'$. By the definition of f_{ij}, if y is rational it follows that x is rational. Again, since $\eta\subseteqq C$, $h_0^+(x)=h_0(x)$ and the proof is complete.

Lemma 10. *There exist* $x, y\in\eta$ *and distinct* $n, m<\omega$ *such that* $h_n(x)=h_m(y)$.

Proof. Let I be given as in Lemma 9. Suppose that, for some $1\leqq n<\omega$, h_n is the identity function. In particular, for $y\in I$, $y=h_n^+(y)$. If y is rational $h_n^+(y)=h_n(y)$ and, by Lemma 9, the proof is complete. Thus, we assume that, for $1\leqq n<\omega$, h_n is not the identity function.

Choose $1\leqq p<\omega$ such that for some $1\leqq r\leqq p!$, $I_{pr}\subseteqq I$. Recall that, for all $f_{ij}=f\in F$ of type $i\geqq p$, $f_{ij}(I_{pr})\subseteqq I_{pr}\subseteqq I$.

By Lemma 5, all h_n^+ belong to G_F. Lemma 8 yields the existence of an open nonempty interval $I_1\subseteqq I_{pr}$ such that h_1^+ agrees with some $f_{(1)}\in F$ on I_1. Define inductively $I_{n+1}\subseteqq I_n$ as a nonempty open subinterval on which h_{n+1}^+ agrees with some $f_{(n+1)}\in F$. If some $f_{(n)}$ is of type $i\geqq p$, choose a rational $x\in I_n$. Then $h_n(x)=h_n^+(x)=f_{(n)}(x)\in I$ is rational, and, by Lemma 9, $h_n(x)=h_0(x')$ for some rational x'. Therefore, each $f_{(n)}$ for $1\leqq n<\omega$ is of type $i_n<p$. Since there are only finitely many of these functions, there exist $1\leqq m<n<\omega$ with $h_n^+\restriction I_n=f_{(n)}\restriction I_n=f_{(m)}\restriction I_n=h_m^+\restriction I_n$. For any rational $x\in I_n$ it follows that $h_n(x)=h_n^+(x)=h_m^+(x)=h_m(x)$. The proof is complete.

Since $\eta\subseteqq C$, Lemma 10 implies that there are distinct $n, m<\omega$ such that $h_n(C)\cap h_m(C)\neq\emptyset$.

Lemma 11. *If, for* $n<\omega$, $h_n\colon C\to A$ *is an order preserving injection then there exist distinct* $n, m<\omega$ *such that* $h_n(C)\cap h_m(C)\neq\emptyset$; *that is to say,* A *does not contain infinitely many disjoint copies of* C.

Lemmas 3, 4, and 11 yield the Theorem.

References

[1] B. Dushnik and E. W. Miller, Concerning similarity transformations of linearly ordered sets, *Bull. Amer. Math. Soc.*, **46** (1940), 322—326.

[2] I. Korec, On systems of isomorphic copies of an algebra in another algebra, *Acta Fac. Rerum Natur. Univ. Comenian. Math.*, **34** (1979), 213—222.

[3] V. Koubek, Sublattices of a distributive lattice, *Acta Sci. Math.*, **41** (1979), 137—150.

(M. E. A.)
DEPARTMENT OF MATHEMATICS
STATE UNIVERSITY OF NEW YORK
NEW PALTZ, NEW YORK 12561, U.S.A.

(J. S.)
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MANITOBA
WINNIPEG, MANITOBA R3T 2N2, CANADA