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On lightly compact spaces

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1. Introduction. A topological space X is called *lightly compact* if every locally finite family of open sets of X is finite. Several characterizations of light compactness are given in [1] and [2]. Two well-known characterizations of these spaces are: a space X is lightly compact iff every countable open cover of X contains a finite subfamily whose union is dense in X; and, every countable open filter base has an adherent point. The aim of this note is an investigation of lightly compact spaces. We give some characterizations of light compactness in term of regular-open, regular-closed sets. We also prove some structural properties of such spaces.

Recall that a set U is regular-open if $U=\overset{\circ}{\overline{U}}$ and a set F is regular-closed if $F=\overline{\overset{\circ}{F}}$ where $^{-}$ denotes the closure of a set and $^{\circ}$ denotes the interior of a set.

2. Results. We first prove a lemma.

Lemma 1. *The family of closures of members of a locally finite, infinite family is not finite.*

Proof. Let $\Psi=\{W_\alpha \mid \alpha\in\Delta\}$ be a locally finite, infinite family of subsets of a topological space X. Suppose $\overline{\Psi}=\{\overline{W}_\alpha \mid \alpha\in\Delta\}$ is finite, say only the sets $\overline{W}_{\alpha_1}, \overline{W}_{\alpha_2}, \ldots, \overline{W}_{\alpha_n}$ are distinct. Since $\Psi=\{W_\alpha \mid \alpha\in\Delta\}$ is an infinite family, then at least one of the sets $\overline{W}_{\alpha_1}, \overline{W}_{\alpha_2}, \ldots, \overline{W}_{\alpha_n}$ is the closure of infinitely many W_α. Suppose $\overline{W}_{\alpha_1}$ is the closure of infinitely many W_α. Take any $x\in\overline{W}_{\alpha_1}$. Then this implies that every neighbourhood of x meets infinitely many W_α. This is a contradiction with Ψ being a locally finite family.

The following theorem shows that the open sets in the definition of lightly compactness may be replaced with regular-closed sets.

Theorem 1. *A space X is lightly compact iff every locally finite family of regular-closed sets is finite.*

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Proof. Let X be a lightly compact space and $\mathcal{E}=\{F_\alpha \mid \alpha\in\Delta\}$ be a locally finite family of regular-closed sets. Since $F_\alpha=\overline{\mathring{F}_\alpha}$ for each $\alpha\in\Delta$, $\{\mathring{F}_\alpha \mid \alpha\in\Delta\}$ is a locally finite family of open sets of the lightly compact space X. Hence $\{\mathring{F}_\alpha \mid \alpha\in\Delta\}$ is finite. Thus the family $\mathcal{E}$ is finite.

Conversely, suppose $\{G_\alpha \mid \alpha\in\Delta\}$ is a locally finite family of open sets. Then $\{\overline{G}_\alpha \mid \alpha\in\Delta\}$ is a locally finite family of regular-closed sets. By hypothesis, $\{\overline{G}_\alpha \mid \alpha\in\Delta\}$ is finite. By Lemma 1, the family $\{G_\alpha \mid \alpha\in\Delta\}$ is finite. Hence X is lightly compact.

We next give another characterization theorem for light compactness.

Theorem 2. *In a topological space X the following are equivalent:*

(i) *X is lightly compact.*

(ii) *Every countable regular-open cover of X contains a finite subfamily whose union is dense in X.*

(iii) *For any countable family of regular-open sets $\{G_n \mid =1, 2, ...\}$ with the finite intersection property, $\bigcap_{n=1}^{\infty} \overline{G}_n\neq\emptyset$.*

(iv) *For any countable family of regular-closed sets $\{F_n \mid n=1, 2, ...\}$ such that $\bigcap_{n=1}^{\infty} F_n=\emptyset$, there exists a finite subfamily $\{F_1, F_2, ..., F_m\}$ such that $\bigcap_{i=1}^{m} \mathring{F}_i=\emptyset$.*

Proof. It is straightforward.

We next give a sufficient condition for a space X to be lightly compact.

Theorem 3. *Let X be any topological space. If every point of X is contained in only finitely many open sets, then X is lightly compact.*

Proof. Suppose X is not lightly compact. Then there exists a locally finite family Ψ of open sets which is not finite. Let $x\in X$ and let N_x be an open neighbourhood of x meeting only finitely many $W\in\Psi$, say $N_x\cap W_{\alpha_i}\neq\emptyset$ $(i=1, 2, ..., n)$ and $N_x\cap W_\alpha=\emptyset$ for all $\alpha\neq\alpha_1, \alpha_2, ..., \alpha_n$. This implies that $x\notin\overline{W}_\alpha$ if $\alpha\neq\alpha_1, \alpha_2, ..., \alpha_n$. By Lemma 1, there are infinitely many $\overline{W}_\alpha$ and $x\in X-\overline{W}_\alpha$, $\alpha\neq\alpha_1, \alpha_2, ..., \alpha_n$. That is, x is contained in infinitely many open sets. This is a contradiction which completes the proof.

Theorem 4. *A space X is lightly compact whenever a dense subset of it is lightly compact.*

Proof. Let A be a lightly compact dense subset of X. If $\{G_n \mid n=1, 2, ...\}$ is a countable open filter base in X, then $\{G_n\cap A \mid n=1, 2, ...\}$ is a countable open filter base in A. Since $\bigcap_{n=1}^{\infty} \overline{(G_n\cap A)}^A\neq\emptyset$, then $\bigcap_{n=1}^{\infty} \overline{G}_n^X\neq\emptyset$. Hence X is lightly compact.

We know that in a first countable Hausdorff space every countably compact subset is closed. The following theorem shows that a similar result can also be obtained for lightly compact spaces.

Theorem 5. *Every lightly compact subset of a first countable Hausdorff space is closed.*

Proof. Let Y be a lightly compact subset of a first countable Hausdorff space X. Suppose Y is not closed in X. Take $y \in \bar{Y} - Y$. Let $\{G_n \mid n=1, 2, \ldots\}$ be a countable open neighbourhood base at y. Then $\{G_n \cap Y \mid n=1, 2, \ldots\}$ is a countable open filter base in Y which has no adherent point because

$$\bigcap_{n=1}^{\infty} \overline{(G_n \cap Y)}^Y \subseteqq \bigcap_{n=1}^{\infty} (\bar{G}_n^X \cap Y) = \{y\} \cap Y = \emptyset.$$

This is a contradiction.

It is known that a continuous image of a lightly compact space is lightly compact. For a weakly continuous function we have the following theorem. First recall that a function $f: X \to Y$ is *weakly continuous* [3] if for each $x \in X$ and each open set V containing $f(x)$, there exists an open set U containing x such that $f(U) \subseteqq \bar{V}$. Equivalently, $f: X \to Y$ is weakly continuous iff for each open set V in Y, we have $f^{-1}(V) \subseteqq [f^{-1}(\bar{V})]^\circ$ ([3], Theorem 1).

Theorem 6. *A weakly continuous image of a countably compact space is lightly compact.*

Proof. Let X be a countably compact space and $f: X \to Y$ be a weakly continuous onto function. If $\{G_n \mid n=1, 2, \ldots\}$ is a countable open cover of Y, then $\bigcup_{n=1}^{\infty} f^{-1}(G_n) = X$. Since f is weakly continuous, $f^{-1}(G_n) \subseteqq [f^{-1}(\bar{G}_n)]^\circ$ for $n=1, 2, \ldots$. Hence $\{[f^{-1}(\bar{G}_n)]^\circ \mid n=1, 2, \ldots\}$ is a countable open cover of X. Since X is countably compact, there exists a finite subfamily $\{G_1, G_2, \ldots, G_n\}$ such that $\bigcup_{i=1}^{n} [f^{-1}(\bar{G}_i)]^\circ = X$. Take any $y \in Y$. Since f is onto, there exists an $x \in X$ such that $f(x) = y$. Suppose $x \in [f^{-1}(\bar{G}_j)]^\circ$, $1 \leqq j \leqq n$. So $x \in f^{-1}(\bar{G}_j)$, that is $f(x) = y \in \bar{G}_j$, $1 \leqq j \leqq n$. Hence $\bigcup_{i=1}^{n} \bar{G}_i = Y$. Thus Y is lightly compact.

N. Levine [4] has introduced the concept of strongly continuous function. A function $f: X \to Y$ is said to be *strongly continuous* iff $f(\bar{A}) \subseteqq f(A)$ for every subset A of X. For a strongly continuous function we have:

Theorem 7. *A strongly continuous image of a lightly compact space is countably compact.*

Proof. Let X be a lightly compact space and $f: X \to Y$ be a strongly continuous and onto function. If $\{G_n \mid n=1, 2, \dots\}$ is a countable open cover of Y then $\bigcup_{n=1}^{\infty} f^{-1}(G_n)=X$. Since f is strongly continuous, hence continuous, $\{f^{-1}(G_n) \mid n=1, 2, \dots\}$ is a countable open cover of X. Since X is lightly compact there exists a finite subfamily $\{f^{-1}(G_1), f^{-1}(G_2), \dots, f^{-1}(G_m)\}$ such that $\bigcup_{i=1}^{m} \overline{f^{-1}(G_i)}=X$. This implies that

$$f(X) = Y = f\left(\bigcup_{i=1}^{m} \overline{f^{-1}(G_i)}\right) = \bigcup_{i=1}^{m} f(\overline{f^{-1}(G_i)}) \subseteq \bigcup_{i=1}^{m} f(f^{-1}(G_i)) = \bigcup_{i=1}^{m} G_i.$$

That is, Y is countably compact.

Theorem 8. *A one-to-one continuous map from a regular lightly compact space X onto a first countable Hausdorff space Y is a homeomorphism.*

Proof. Let $f: X \to Y$ be a continuous one-to-one and onto map. Let F be a closed subset of X. It can be shown that F can be written as an intersection of regular-closed subsets of the regular space X. Say $F=\bigcap_{\alpha \in \Delta} C_\alpha$, where all C_α are regular-closed subsets. Since X is lightly compact, for all $\alpha \in \Delta$, C_α is a lightly compact subset of X [1]. Hence for all $\alpha \in \Delta$, $f(C_\alpha)$ is a lightly compact subset of Y. By Theorem 5, for all $\alpha \in \Delta$, $f(C_\alpha)$ is a closed subset of Y. Since f is one-to-one, therefore

$$f(F) = \bigcap_{\alpha \in \Delta} f(C_\alpha).$$

That is, $f(F)$ is closed in Y. Thus f is a closed map, and hence it is a homeomorphism.

Recall that a space (X, τ) is called *first countable and Hausdorff minimal* if τ is first countable and Hausdorff, and if no first countable Hausdorff topology on X is strictly weaker than τ.

Corollary. [6. 2. 6. Theorem (vii)] *A first countable, regular, lightly compact Hausdorff space is first countable and Hausdorff minimal.*

Singal [5] has introduced the concept of nearly compact space. A space X is called *nearly compact* if every open cover of X has a finite subfamily such that the interiors of closures of sets in this family covers X. It can be shown that a space is nearly compact iff the intersection of a family of regular-closed sets with finite intersection property is not empty.

It is known that the product of a lightly compact space and a compact space is lightly compact. The next theorem gives a generalization of this result.

Theorem 9. *The product of a lightly compact space and a nearly compact space is lightly compact.*

Proof. Let X be a nearly compact space and Y be a lightly compact space. To show that the product space $X\times Y$ is lightly compact, it is enough to prove that every countable open filter base has an adherent point in $X\times Y$. Let $\mathscr{E}=\{G_n \mid n=1,2,3,\ldots\}$ be a countable open filter base in $X\times Y$. Then $\Pi_2(\mathscr{E})=\{\Pi_2(G_n) \mid n=1,2,3,\ldots\}$ is a countable open filter base in Y, where Π_2 is the second projection. Since Y is lightly compact, $\Pi_2(\mathscr{E})$ has an adherent point, that is $\bigcap_{n=1}^{\infty} \overline{\Pi_2(G_n)}\neq\emptyset$. Take $y\in\bigcap_{n=1}^{\infty}\overline{\Pi_2(G_n)}$. If V is an open set containing y, then for all n, $V\cap\Pi_2(G_n)\neq\emptyset$. Hence for all n, $\Pi_2^{-1}(V)\cap G_n\neq\emptyset$. Let $\Pi_1(\Pi_2^{-1}(V)\cap G_n)=U_{V,n}$. All $U_{V,n}$ are open sets in X. Now the family

$$\{U_{V,n} \mid V \text{ is open in } Y \text{ and } y\in V,\ n=1,2,3,\ldots\}$$

has the finite intersection property in X. In fact,

$$U_{V_1,n_1}\cap U_{V_2,n_2}=\Pi_1(\Pi_2^{-1}(V_1)\cap G_{n_1})\cap\Pi_1(\Pi_2^{-1}(V_2)\cap G_{n_2})\supseteqq$$
$$\supseteqq \Pi_1\{[\Pi_2^{-1}(V_1)\cap G_{n_1}]\cap[\Pi_2^{-1}(V_2)\cap G_{n_2}]\}=\Pi_1[\Pi_2^{-1}(V_1\cap V_2)\cap(G_{n_1}\cap G_{n_2})]\neq\emptyset.$$

Hence the family $\{\overline{U}_{V,n} \mid V$ is open in Y and $y\in V,\ n=1,2,3,\ldots\}$ is a collection of regular-closed sets with the finite intersection property. Since X is nearly-compact, $\bigcap_{V,n}\overline{U}_{V,n}\neq\emptyset$. Let $x\in\bigcap_{V,n}\overline{U}_{V,n}$. If we show that (x,y) is an adherent point of the filter base $\mathscr{E}$ in $X\times Y$, then the proof will be completed. Suppose $M\times N$ is a basic open set containing (x,y) in $X\times Y$. It is clear that $M\cap U_{N,n}\neq\emptyset$ for all n. Thus $M\cap\Pi_1(\Pi_2^{-1}(N)\cap G_n)\neq\emptyset$ for $n=1,2,\ldots$. Consequently $(M\times N)\cap\cap G_n\neq\emptyset$ for all n, that is $(x,y)\in\bigcap_{n=1}^{\infty}\overline{G}_n$. So $X\times Y$ is lightly compact.

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