Acta Sci. Math. 46 (1983), 121—125

A note on multifunctions

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1. Introduction

A function $F: X \to p(Y) - \{\emptyset\}$ is called a multifunction from X to Y and is usually denoted by $F: X \to Y$, where $p(Y)$ is the power set of Y. The *graph of* F is the subset $\{(x, y) \mid x \in X$ and $y \in F(x)\}$ of $X \times Y$. We will denote the graph of F by $G(F)$. If X and Y are topological spaces and $F: X \to Y$ is a multifunction we will say that F has a *closed graph* if $G(F)$ is a closed subset of $X \times Y$. The graph $G(F)$ is closed iff for each point $(x, y) \notin G(F)$, there exist open sets $U \subset X$ and $V \subset Y$ containing x and y, respectively, such that $F(U) \cap V = \emptyset$. The graph $G(F)$ is said to be *strongly closed* [4] if for each point $(x, y) \notin G(F)$, there exist open sets $U \subset X$ and $V \subset X$ containing x and y respectively, such that $F(U) \cap \bar{V} = \emptyset$, where $\bar{V}$ denotes the closure of V. A multifunction $F: X \to Y$ is called *upper semicontinuous (weakly upper semicontinuous)* if for each $x \in X$ and each open set $V \subset Y$ containing $F(x)$, there exists an open set $U \subset X$ containing x such that $F(U) \subset V$ $(F(U) \subset \bar{V})$. It is not difficult to see that F is upper semicontinuous iff $F^{-1}(K) = \{x \in X \mid F(x) \cap K \neq \emptyset\}$ is closed in X whenever K is closed in Y. We will say that a multifunction $F: X \to Y$ is *point closed (point compact)* if $F(x)$ is closed (compact) in Y for each $x \in X$. The definition of an open or closed multifunction is analogous to the definition of an open or closed single valued mapping.

A multifunction $F: X \to Y$ is said to be *almost upper semicontinuous* if for each point $x \in X$ and each open set $V \subset Y$ containing $F(x)$, there exists an open set $U \subset X$ containing x such that $F(U) \subset \mathring{\bar{V}}$, where $\mathring{\bar{V}}$ denotes the interior of the closure of V.

A subset K of a topological space X is called *quasi H-closed relative to* X if for each open cover $\{G_\alpha \mid \alpha \in \Delta\}$ of K, there exists a finite subfamily $\{G_{\alpha_i} \mid i = 1, 2, \ldots, n\}$ such that $K \subset \bigcup_{i=1}^{n} \bar{G}_{\alpha_i}$. If X is quasi H-closed relative to X, then it is

Received June 16, 1982, and in revised form December 14, 1982.

called quasi H-closed. When X is Hausdorff, the word "quasi" is omitted in these two definitions.

A Hausdorff space X is said to be locally *H-closed* [4] if every point of X has a neighbourhood which is H-closed. A space X is called *c-compact* [3] if every closed set of X is quasi H-closed relative to X.

Let X be a topological space and $A\subset X$. If D is a directed set and $\Phi: D\to A$ is a net, then we say it *r-accumulates* [3] to $x\in A$ if for each open set $V\subset X$ containing x and every $b\in D$, $\Phi(T_b)\cap\overline{V}\neq\emptyset$, where $T_b=\{c\in D \mid c\geqq b\}$. A space X is c-compact iff for each closed set $A\subset X$ and each net $\{x_\alpha\}$ in A, there exists a point $x\in A$ such that $\{x_\alpha\}$ r-accumulates to x [3, Th. 3].

2. c-compact, H-closed spaces and multifunctions with strongly closed graph

Theorem 2.1. *Let $F: X\to Y$ be a multifunction and Y be a c-compact space. If F has strongly closed graph, then F is upper semicontinuous.*

Proof. Suppose there exists a closed subset K in Y such that $F^{-1}(K)$ is not closed in X. Take $x_0\in\overline{F^{-1}(K)}-F^{-1}(K)$. Hence there exists a net $\{x_\alpha\}_{\alpha\in A}$ in $F^{-1}(K)$ such that $x_\alpha\to x_0$. Now let $\{y_\alpha\}_{\alpha\in A}$ be a net in K such that $y_\alpha\in F(x_\alpha)\cap K$ for each α. Since K is closed and Y is c-compact, there exists a point $y_0\in K$ such that the net $\{y_\alpha\}_{\alpha\in A}$ r-accumulates to y_0. Since $y_0\notin F(x_0)$, then $(x_0, y_0)\notin G(F)$ and since $G(F)$ is strongly closed, there are open sets $U\subset X$ and $V\subset Y$ containing x_0 and y_0, respectively, such that $(U\times\overline{V})\cap G(F)=\emptyset$. But $x_\alpha\to x_0$ implies there exists an $\alpha_0\in A$ such that for every $\alpha\in A$ and $\alpha\geqq\alpha_0$, $x_\alpha\in U$, and $\{y_\alpha\}_{\alpha\in A}$ r-accumulates to y_0 implies there exists some $\alpha_1\in A$ and $\alpha_1\geqq\alpha_0$ such that $y_{\alpha_1}\in\overline{V}$. From this it follows that $(x_{\alpha_1}, y_{\alpha_1})\in(U\times\overline{V})\cap G(F)$ which is a contradiction. Hence F is upper semicontinuous.

Theorem 2.2. *Let $F: X\to Y$ be a point compact multifunction and Y a locally H-closed (H-closed) space. If for each subset K, H-closed in Y, $F^{-1}(K)$ is closed in X then F has strongly closed graph.*

Proof. Suppose Y is locally H-closed. Take any point $(x, y)\notin G(F)$. Then $y\notin F(x)$. Since Y is Hausdorff, $F(x)$ is compact and $y\notin F(x)$, there are disjoint open sets V_1 and W in Y such that $y\in V_1$ and $F(x)\subset W$ [1, p. 225]. $V_1\cap W=\emptyset$ implies $\overline{V}_1\cap W=\emptyset$. On the other hand, there exists a neighbourhood V_2 of y which is H-closed. Put $V=V_1\cap\mathring{V}_2$. Then V is an open set containing y and $W\cap\overline{V}=\emptyset$. Since Y is Hausdorff and V_2 is H-closed in Y, then V_2 is closed in Y. Thus $\overline{V}\subset V_2$. $\overline{V}$ is a regularly closed subset in the H-closed set V_2. Therefore $\overline{V}$ is H-closed in V_2, so $\overline{V}$ is H-closed in Y. According to our assumption, $F^{-1}(\overline{V})$

is closed in X. Put $U=X-F^{-1}(\overline{V})$. Then U is an open set in X containing x and $F(U)\cap\overline{V}=\emptyset$. This shows that $G(F)$ is strongly closed.

Theorem 2.3. *Let $F:X\to Y$ be an almost upper semicontinuous point compact multifunction and Y Hausdorff. Then F has a strongly closed graph.*

Proof. Let $(x,y)\notin G(F)$. Since $F(x)$ is compact, $y\notin F(x)$ and Y is Hausdorff, there are disjoint open sets V and W containing y and $F(x)$, respectively. We can write $\overline{V}\cap\overset{\circ}{\overline{W}}=\emptyset$. Since F is almost upper semicontinuous there is an open set U in X containing x such that $F(U)\subset\overset{\circ}{\overline{W}}$. Now we have $F(U)\cap\overline{V}=\emptyset$. That is, $G(F)$ is strongly closed.

Corollary. *Let $F:X\to Y$ be a point compact multifunction and Y an H-closed space. The following are equivalent:*

(i) *F is almost upper semicontinuous,*
(ii) *F has strongly closed graph,*
(iii) *For each subset K, H-closed relative to Y, $F^{-1}(K)$ is closed in X,*
(iv) *For each H-closed subset K of Y, $F^{-1}(K)$ is closed in X.*

Proof. According to Theorem 2.3, (i) implies (ii). (ii) implies (iii), by Theorem 4.15 [4]. Since an H-closed subset of Y is H-closed relative to Y (the converse need not be true), the implication (iii)$\Rightarrow$(iv) is obvious.

Let us prove that (iv) implies (i). For any $x\in X$, let W be an open set containing $F(x)$. $\overset{\circ}{\overline{W}}$ is a regularly open set containing $F(x)$. $Y-\overset{\circ}{\overline{W}}$ is a regularly closed set. Since Y is H-closed then $Y-\overset{\circ}{\overline{W}}$ is H-closed. Hence by (iv), $F^{-1}(Y-\overset{\circ}{\overline{W}})$ is closed in X and $x\notin F^{-1}(Y-\overset{\circ}{\overline{W}})$. Thus there exists an open set U containing x such that $U\cap F^{-1}(Y-\overset{\circ}{\overline{W}})=\emptyset$. This implies that $F(U)\subset\overset{\circ}{\overline{W}}$, that is, F is almost upper semicontinuous.

Our next result is a generalization of Theorem 11 in [3], which was proved for a single valued mapping.

Theorem 2.4. *If $F:X\to Y$ is an open and closed multifunction from a regular space X into a c-compact space Y, and if $F^{-1}(y)$ is closed for each $y\in Y$, then F is upper semicontinuous.*

Proof. According to Theorem 3.4, Corollary 3.5 [5] F has closed graph. For an open multifunction the condition closed graph and strongly closed graph are identical. Hence $F:X\to Y$ has a strongly closed graph and Y is c-compact, so by Theorem 2.1, F is upper semicontinuous.

Theorem 2.5. *If* $F: X \to Y$ *is an upper semicontinuous point compact multifunction, then* F *is compact preserving.*

Proof. Let K be a compact subset of X and suppose $\{W_\alpha \mid \alpha \in \Delta\}$ is an open cover of $F(K)$. Take any $x \in K$, then $F(x)$ is a compact subset of Y and $F(x) \subset \subset F(K)$. Thus $\{W_\alpha \mid \alpha \in \Delta\}$ is an open cover of $F(x)$. Hence there is a finite subcover, say $\{W_{\alpha_1}(x), \dots, W_{\alpha_n}(x)\}$. Now put $V(x) = \bigcup_{i=1}^{n} W_{\alpha_i}(x)$. $V(x)$ is an open set containing $F(x)$. Since F is upper semicontinuous, there exists an open set $U(x) \subset X$ containing x such that $F(U(x)) \subset V(x)$. Now $\{U(x) \mid x \in K\}$ is an open cover of K and K is a compact subset of X. Take $x_1, x_2, \dots, x_m \in K$ such that $\{U(x_i) \mid i = 1, \dots, m\}$ is a subcover. Let $V(x_1), V(x_2), \dots, V(x_m)$ be the open sets corresponding to $U(x_1), U(x_2), \dots, U(x_m)$, respectively. Thus

$$F(K) \subset F\left(\bigcup_{i=1}^{m} U(x_i)\right) = \bigcup_{i=1}^{m} F(U(x_i)) \subset \bigcup_{i=1}^{m} V(x_i) =$$

$$= \bigcup \{W_{\alpha_1}(x_1), \dots, W_{\alpha_n}(x_1), \dots, W_{\beta_1}(x_m), \dots, W_{\beta_s}(x_m)\}$$

That is, we have a finite subcover of $\{W_\alpha \mid \alpha \in \Delta\}$. Hence $F(K)$ is compact in Y.

Corollary. *Let* $F: X \to Y$ *be an onto closed multifunction. If* F *has compact point inverses, then for each compact subset* K *of* Y $F^{-1}(K)$ *is compact in* X.

Proof. Since $(F^{-1})^{-1} = F$, then $F^{-1}: Y \to X$ is an upper semicontinuous point compact multifunction, hence F^{-1} is compact preserving.

Theorem 2.6. *Let* $F: X \to Y$ *be a weakly upper semicontinuous point compact multifunction. Then* F *maps a compact subset* K *of* X *onto subset* $F(K)$ *quasi H-closed relative to* Y.

Proof. The proof is the same as in Theorem 2.5.

Let $F: X \to Y$ be a multifunction. We can define a new multifunction $\bar{F}: X \to Y$ by setting $\bar{F}(x) = \overline{F(x)}$ for all $x \in X$. If Y is normal and $F: X \to Y$ is upper semicontinuous then $\bar{F}: X \to Y$ is upper semicontinuous [2]. We have the following new result.

Theorem 2.7. *If* $F: X \to Y$ *is weakly upper semicontinuous, then* $\bar{F}: X \to Y$ *is weakly upper semicontinuous.*

Proof. Let $x \in X$ and W an open set in Y containing $\bar{F}(x)$. Since $F(x) \subset \subset \overline{F((x} = \bar{F}(x) \subset W$ and F is weakly upper semicontinuous there is an open set U in X containing x such that $F(U) \subset \overline{W}$. This implies that $\overline{F(U)} \subset \overline{W}$. On the other hand

$$\bar{F}(U)=\bigcup_{x\in U}\bar{F}(x)=\bigcup_{x\in U}\overline{F(x)}\subset\overline{F(U)}.$$

Hence $\bar{F}(U)\subset\bar{W}$, that is, $\bar{F}$ is weakly upper semicontinuous.

Theorem 2.8. *If $F:X\to Y$ is weakly upper semicontinuous and Y is regular, then the graph of $\bar{F}$ is closed in $X\times Y$.*

Proof. $\bar{F}:X\to Y$ is weakly upper semicontinuous, by Theorem 2.7. Now suppose $(x,y)\notin G(\bar{F})$. $y\notin\bar{F}(x)=\overline{F(x)}$. Since Y is regular, there are open sets V and W containing y and $\bar{F}(x)$, respectively, such that $V\cap W=\emptyset$. Hence $V\cap\bar{W}=\emptyset$. From the weakly upper semicontinuity of $\bar{F}$, we have an open set U in X containing x such that $\bar{F}(U)\subset\bar{W}$. Hence $\bar{F}(U)\cap V=\emptyset$. That is, $G(\bar{F})$ is closed in $X\times Y$.

Corollary. [5, Theorem 3.3] *If $F:X\to Y$ is a point closed upper semicontinuous multifunction into a regular space, then F has a closed graph.*

Acknowledgement. The author wishes to thank the referee whose comments improved the exposition of the paper.

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