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On variation spaces of harmonic maps into spheres

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1. Introduction

Given a harmonic map $f: M \to S^n$ [3] of a compact oriented Riemannian manifold M into the Euclidean n-sphere S^n, $n \geqq 2$, a vector field v along f, i.e. a section of the pull-back bundle $\mathscr{F} = f^*(T(S^n))$, gives rise to a (one-parameter, geodesic) variation $f_t = \exp \circ (tv): M \to S^n$, $t \in \mathbf{R}$, where $\exp: T(S^n) \to S^n$ is the exponential map. The element $v \in C^\infty(\mathscr{F})$ is said to be a *harmonic variation* if f_t is harmonic for all $t \in \mathbf{R}$ and the set of all harmonic variations v (or the variation space) of f is denoted by $V(f) \subset C^\infty(\mathscr{F})$. Then [11] $v \in V(f)$ if and only if $\|v\| = \text{const.}$ and

(i) $\nabla^2 v = \operatorname{trace} R(f_*, v) f_*$ (i.e. v is a Jacobi field along f [3]),

(ii) $\operatorname{trace} \langle f_*, \nabla v \rangle = 0$,

where $\langle\, ,\rangle$ and ∇ are the induced metric and connection of the Riemannian-connected bundle $\mathscr{F} \otimes \Lambda^*(T^*(M))$, $\nabla^2 = \operatorname{trace} \nabla \circ \nabla$ [9], R is the curvature tensor of S^n and the differential f_* of f is considered as a section of $\mathscr{F} \otimes T^*(M)$. Denote by $K(f)$ the linear space of all vector fields v along f satisfying (i) and (ii). The equation (i) being (strongly) elliptic [9] $\dim K(f) < \infty$ and $V(f) = \{v \in K(f) \mid \|v\| = \text{const.}\} \subset K(f)$ is a subset with the obvious property $\mathbf{R} V_0(f) = V(f)$, where $V_0(f) = \{v \in K(f) \mid \|v\| = 1\}$.

The purpose of this paper is to give a geometric description of the variation space $V(i) \subset K(i)$ $(\cong \mathbf{R}^N)$ of the canonical inclusion $i: S^m \to S^n$, where $N = m(m+1)/2 + (n-m)(m+1)$. In Section 2 we collect the necessary tools from matrix theory used in the sequel, especially we describe the singular value decomposition of rectangular matrices (see e.g. [7]). In Section 3 the problem of determining $V_0(i)$ is reduced to the geometric characterization of an (algebraic) set of matrices. Then the singular value decomposition of these matrices are exploited to get a description of $V_0(i) \subset K(i)$ as a set of orbits (under a linear Lie group action) which contains a (twisted) simplex as a global section (Theorem 1). In particular, we prove that

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$V(\mathrm{id}_{S^{2r-1}})$, $r\in\mathbf{N}$, is the double cone over the irreducible Hermitian symmetric space $SO(2r)/U(r)$ $(=V_0(\mathrm{id}_{S^{2r-1}}))$. (Note that $V(\mathrm{id}_{S^{2r}})=0$ because $\chi(S^{2r})=2$ [11].) In Section 4 we first give an alternative description of the linear space $K(f)$. In particular, we obtain that there is a one-to-one correspondence between the elements of $V_0(f)$ and the orthogonal pairs $f, f^1: M\to S^n$ of harmonic maps with the same energy density $e(f)=e(f^1)$ [3]. Second, as an example, we determine $K(f)$ for the Veronese surface $f: S^2\to S^4$ and prove that $K(f)\cong K(\mathrm{id}_{S^4})$ and $V(f)=V(\mathrm{id}_{S^4})=0$ hold.

Throughout this paper all manifolds, maps, bundles, etc. will be smooth, i.e. of class C^∞. The report [3] is our general reference for harmonic maps though we adopt the sign conventions of [6].

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2. Preliminaries from matrix theory

First we fix some notations used in the sequel. Denote by $M(p, q)$ the linear space of $(p\times q)$ matrices and, as usual, let I_p and 0_p the unit and zero elements of $M(p,p)$. A matrix $A\in M(p,q)$ with entries a_{ij}, $i=1, \dots, p$, $j=1, \dots, q$, is said to be (rectangular) diagonal if

$$a_{ij} = \begin{cases} 0, & \text{if } i\neq j,\ i=1, \dots, p,\ j=1, \dots, q, \\ \sigma_i, & \text{if } i=j,\ i=1, \dots, \min(p,q) \end{cases}$$

holds. We write $A=\mathrm{diag}\,(\sigma_1, \dots, \sigma_d)^p_q$ with $d=\min(p,q)$ and, in case $p=q$, we omit the indices p and q.

The singular value decomposition of rectangular matrices is given in the following theorem. (For the proof, see [7].)

Theorem A. *For any matrix $B\in M(p,q)$ there exist orthogonal matrices $V\in O(p)$ and $U\in O(q)$ such that*

$$V^{\mathrm{T}}BU = \mathrm{diag}\,(\sigma_1, \dots, \sigma_d)^p_q$$

with $\sigma_i\geqq 0$, $i=1, \dots, d=\min(p,q)$. The matrices V, U and the values σ_i are determined by the relations:

(A_1) $V^{\mathrm{T}}BB^{\mathrm{T}}V = \mathrm{diag}\,(\sigma_1^2, \dots, \sigma_d^2, \dots, \sigma_p^2)$,

(A_2) $U^{\mathrm{T}}B^{\mathrm{T}}BU = \mathrm{diag}\,(\sigma_1^2, \dots, \sigma_d^2, \dots, \sigma_q^2)$,

(A_3) $BU = V\,\mathrm{diag}\,(\sigma_1, \dots, \sigma_d)^p_d$,

where $\sigma_i=0$ for $d<i\leqq\max(p,q)$.

Remark. The numbers $\sigma_i \geqq 0$, $i=1, \ldots, d$, are called the singular values of B. Clearly, V and U can always be chosen such that $\sigma_1 \geqq \sigma_2 \geqq \ldots \geqq \sigma_d$ holds.

Denote by $\Lambda_r \in so(2r)$ the skew-symmetric matrix

$$\Lambda_r = \operatorname{diag}\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right)$$

and put $\Lambda_1 = \Lambda$. In the next theorem we collect some properties of skew-symmetric matrices (cf. [8] pp. 151, 231).

Theorem B. *For any matrix* $\mathcal{J} \in so(p)$ *we have*

(B_1) rank $\mathcal{J} = 2r \leqq p$;

(B_2) *The* $2r$ *nonzero eigenvalues of* $\mathcal{J}$ *appear in pairs* $\lambda_{2i-1} = \lambda_{2i} = \pm\sqrt{-1}\sigma_i$ *with* $\sigma_i > 0$, $i=1, \ldots, r$, *while zero is an eigenvalue with multiplicity* $p-2r$;

(B_3) *There exists* $U \in O(p)$ *such that*

$$U^{\mathrm{T}} \mathcal{J} U = \operatorname{diag}(0_{p-2r}, \sigma_1 \Lambda, \ldots, \sigma_r \Lambda) \tag{1}$$

or equivalently

$$U^{\mathrm{T}} \mathcal{J} U = \begin{cases} \operatorname{diag}(\hat{\sigma}_1 \Lambda, \ldots, \hat{\sigma}_{p/2} \Lambda), & \text{if } p \text{ is even}, \\ \operatorname{diag}(0, \hat{\sigma}_1 \Lambda, \ldots, \hat{\sigma}_{[p/2]} \Lambda), & \text{if } p \text{ is odd}, \end{cases} \tag{1'}$$

where $\hat{\sigma}_1 = \ldots = \hat{\sigma}_{[(p-2r)/2]} = 0$ *and* $\hat{\sigma}_{[(p-2r)/2]+i} = \sigma_i$, $i = 1, \ldots, r$;

(B_4) *With the same matrix* $U \in O(p)$ *we have*

$$U^{\mathrm{T}}(-\mathcal{J}^2) U = \begin{cases} \operatorname{diag}(\hat{\sigma}_1^2 I_2, \ldots, \hat{\sigma}_{p/2}^2 I_2) & \text{if } p \text{ is even}, \\ \operatorname{diag}(0, \hat{\sigma}_1^2 I_2, \ldots, \hat{\sigma}_{[p/2]}^2 I_2), & \text{if } p \text{ is odd}, \end{cases} \tag{2}$$

in particular, the nonzero singular values of $\mathcal{J}$ *have even multiplicities.*

3. Variation space of the canonical inclusion $i: S^m \to S^n$

Let $i: S^m \to S^n$ be the canonical inclusion and let $W^1, \ldots, W^k$, $k=n-m$, denote the system of orthonormal parallel sections of the normal bundle of i defined by the standard base vectors $e_{m+2}, \ldots, e_{n+1} \in \mathbf{R}^{n+1}$.

According to a result of [11] $v \in K(i)$ if and only if the tangential part $\mathcal{J}$ of v is a Killing vector field on S^m and there exist vectors $b_1, \ldots, b_k \in \mathbf{R}^{m+1}$ such that the orthogonal decomposition

$$v_x = \mathcal{J}_x + \sum_{j=1}^{k} \langle b_j, x \rangle W_x^j, \quad x \in S^m,$$

is valid. Hence the linear map $\Psi: K(i) \to so(m+1) \times M(k, m+1)$ defined by $\Psi(v) = (\mathcal{J}, B)$, $v \in K(i)$, where $\mathcal{J}$ is the tangential part of v and $B \in M(k, m+1)$

consists of the row vectors $b_1, \ldots, b_k \in \mathbf{R}^{m+1}$ occurring in the decomposition of v above, is a linear isomorphism. In what follows, we identify $K(i)$ and $so(m+1)\times M(k, m+1)$ via Ψ. Further, $V(i)=\mathbf{R}V_0(i)\subset K(i)$, where $V_0(i)= =\{v\in K(i) \mid \|v\|=1\}$. Thus, for $v=(\mathscr{J}, B)\in V_0(i)$, we have

$$1 = \|v_x\|^2 = \|\mathscr{J}_x\|^2+\sum_{j=1}^{k}\langle b_j, x\rangle^2 = \langle -\mathscr{J}^2x, x\rangle+\langle B^{\mathrm{T}}Bx, x\rangle, \quad x\in S^m,$$

i.e.

$$V_0(i) = \{(\mathscr{J}, B)\in so(m+1)\times M(k, m+1) \mid -\mathscr{J}^2+B^{\mathrm{T}}B = I_{m+1}\}.$$

The objective of this section is to give a geometric description of the set $V_0(i)\subset \subset K(i)$. Before stating our main theorem we introduce some notations. For the given positive integers m and n, $m\leqq n$, set

$$t = \begin{cases} \min\left((m+1)/2, [k/2]\right), & \text{if } m+1 \text{ is even,} \\ \min\left(m/2, [(k-1)/2]\right), & \text{if } m+1 \text{ is odd,} \end{cases}$$

where $k=n-m$, and define

$$\Delta_t = \{(\sigma_1, \ldots, \sigma_t)\in\mathbf{R}^t \mid 1 \geqq \sigma_1 \geqq \ldots \geqq \sigma_t \geqq 0\}.$$

So $\Delta_t\subset\mathbf{R}^t$ is a (linear) simplex which reduces to a point if $t=0$. (Note that $t\geqq -1$ and equality holds if and only if $m=n$ is even, in which case $V_0(i)=\emptyset$ [11] and we put $\Delta_{-1}=\emptyset$.)

A linear representation of the Lie group $O(m+1)\times O(k)$ on the vector space $K(i)=so(m+1)\times M(k, m+1)$ is given by

$$(U, V)\cdot(\mathscr{J}, B) = (U\mathscr{J}U^{\mathrm{T}}, VBU^{\mathrm{T}}),$$

$(U, V)\in O(m+1)\times O(k)$, $(\mathscr{J}, B)\in so(m+1)\times M(k, m+1)$. Clearly, the subset $V_0(i)\subset \subset K(i)$ is invariant, i.e. $V_0(i)$ is the union of orbits crossing $V_0(i)$. Finally we introduce certain subgroups of $O(m+1)\times O(k)$ which will be the isotropy subgroups at points of $V_0(i)$. For given nonnegative integers $a_0, b_0, c_1, c_2, \ldots, c_{s+1}$ with $m+1=a_0+2c_1+\ldots+2c_{s+1}$ and $k=a_0+2c_1+\ldots+2c_s+b_0$ define the subgroups

$$\mathscr{G}(c_1, \ldots, c_{s+1}) = \{(A_0, C_1, \ldots, C_{s+1}; A_0, C_1, \ldots, C_s, B_0)\in O(m+1)\times O(k) \mid$$
$$A_0\in O(a_0), \quad B_0\in O(b_0), C_i\in U(c_i), \quad i = 1, \ldots, s+1\},$$

where $U(c_i)$ is considered as a subgroup of $SO(2c_i)$ via the canonical embedding $U(c_i)\to SO(2c_i)$, $i=1, \ldots, s+1$. The isotropy type i.e. the set of all conjugacy classes of a subgroup $\mathscr{G}\subset O(m+1)\times O(k)$ is denoted by $(\mathscr{G})$. The main result of this section is the following:

Theorem 1. *There exists an embedding $\Phi: \Delta_t\to K(i)$ such that $\Phi(\Delta_t)$ is a global section of the invariant subset $V_0(i)$ (i.e. $\Phi(\Delta_t)\subset V_0(i)$ and any orbit on*

$V_0(i)$ cuts $\Phi(\Delta_t)$ at exactly one point). Moreover, for $\sigma=(\sigma_0, \dots, \sigma_0, \sigma_1, \dots, \sigma_1, \dots, \sigma_{s+1}, \dots, \sigma_{s+1})\in\Delta_t$, where $1=\sigma_0>\sigma_1>\dots>\sigma_s>\sigma_{s+1}=0$ and σ_i occurs c_i times in σ, $i=0, \dots, s+1$, the isotropy type of the orbit through $\Phi(\sigma)$ is $(\mathscr{G}(c_1, \dots, c_s, c_{s+1}+([(m+1)/2]-t)^+)$ ($^+=$ positive part) or equivalently this orbit has the form

$$(O(m+1)\times O(k))/\mathscr{G}(c_1, \dots, c_s, c_{s+1}+([m+1)/2]]-t)^+).$$

In particular, for each open face Δ of the simplex Δ_t the orbits through $\Phi(\Delta)$ have the same type.

Remarks 1. Each orbit consists of 1, 2 or 4 components. More precisely, the subgroups $\mathscr{G}(c_1, \dots, c_{s+1})\subset SO(m+1)\times SO(k)$ being connected, the orbit $(O(m+1)\times O(k))/\mathscr{G}(c_1, \dots, c_s, c_{s+1}+([(m+1)/2]-t)^+)$ has N components, where

$$N=\begin{cases}1, & \text{if } k>0 \text{ and } a_0b_0>0,\\ 2, & \text{if } k>0,\ a_0b_0=0 \text{ and } a_0+b_0>0 \text{ or if } k=0,\\ 4, & \text{if } k>0 \text{ and } a_0=b_0=0.\end{cases}$$

2. By a result of [13] for any locally rigid harmonic embedding $f: M\to S^n$ we have $V(f)=V(i)$, where $i: S^m\to S^n$ is the inclusion and m is the dimension of the least totally geodesic submanifold of S^n containing the image of f. Thus Theorem 1 gives a description of the variation space of all locally rigid harmonic embeddings.

The proof of Theorem 1 is broken up into a few lemmas. Let $(\mathscr{J}, B)\in V_0(i)$ be fixed. Then, by Theorem B, there exists $U\in O(m+1)$ such that $U^{\mathrm{T}}\mathscr{J}U$ and $U^{\mathrm{T}}(-\mathscr{J}^2)U$ have the form (1′) and (2), resp., with

$$0\leqq\hat{\sigma}_1\leqq\dots\leqq\hat{\sigma}_{[(m+1)/2]}.$$

Thus, by $B^{\mathrm{T}}B=I_{m+1}+\mathscr{J}^2$, we obtain

$$U^{\mathrm{T}}B^{\mathrm{T}}BU=\begin{cases}\operatorname{diag}(\sigma_1^2I_2, \dots, \sigma_{[(m+1)/2]}^2I_2), & \text{if } m+1 \text{ is even},\\ \operatorname{diag}(1, \sigma_1^2I_2, \dots, \sigma_{[(m+1)/2]}^2I_2), & \text{if } m+1 \text{ is odd},\end{cases}$$

where $\sigma_i^2=1-\hat{\sigma}_i^2$, $i=1, \dots, [(m+1)/2]$. Clearly, $1\geqq\sigma_1^2\geqq\dots\geqq\sigma_{[(m+1)/2]}^2\geqq0$ is satisfied. Then the values σ_i^2, $i=1, \dots, [(m+1)/2]$, occurring twice in $B^{\mathrm{T}}B$, are the eigenvalues of the positive semidefinite matrix $B^{\mathrm{T}}B$. The nonzero eigenvalues of $B^{\mathrm{T}}B$ and BB^{T} being the same, the system of eigenvalues of $BB^{\mathrm{T}}\in M(k, k)$ can be obtained from that of $B^{\mathrm{T}}B\in M(m+1, m+1)$ by supplementing or omitting $|k-(m+1)|$ zeros according as $k\geqq m+1$ or $k<m+1$. In the latter case, for some index $t_0\leqq[k/2]$, $\sigma_i=0$, $i>t_0$, must be valid. The determination of the minimal value of t_0 can be done by making distinction according to the parity of k. Hence we have

$$U^{\mathrm{T}}B^{\mathrm{T}}BU =$$

$$=\begin{cases}\operatorname{diag}(\sigma_1^2 I_2, \ldots, \sigma_{(m+1)/2}^2 I_2) & \text{for } k \geqq m+1,\ m+1 \text{ even},\\ \operatorname{diag}(\sigma_1^2 I_2, \ldots, \sigma_{k/2}^2 I_2, 0_{m+1-k}) & \text{for } k \text{ even},\ k < m+1,\ m+1 \text{ even},\\ \operatorname{diag}(\sigma_1^2 I_2, \ldots, \sigma_{[k/2]}^2 I_2, 0_{m+1-2[k/2]}) & \text{for } k \text{ odd},\ k < m+1,\ m+1 \text{ even},\\ \operatorname{diag}(1, \sigma_1^2 I_2, \ldots, \sigma_{[(m+1)/2]}^2 I_2) & \text{for } k \geqq m+1,\ m+1 \text{ odd},\\ \operatorname{diag}(1, \sigma_1^2 I_2, \ldots, \sigma_{(k-1)/2}^2 I_2, 0_{m+1-k}) & \text{for } k \text{ odd},\ k < m+1,\ m+1 \text{ odd},\\ \operatorname{diag}(1, \sigma_1^2 I_2, \ldots, \sigma_{[(k-1)/2]}^2 I_2, 0_{m-2[(k-1)/2]}) & \text{for } k \text{ even},\ k < m+1,\ m+1 \text{ odd}.\end{cases}$$

A case-by-case verification shows that the minimal value of t_0 is the number t defined before Theorem 1. Thus we obtain

$$U^{\mathrm{T}}B^{\mathrm{T}}BU = \begin{cases}\operatorname{diag}(\sigma_1^2 I_2, \ldots, \sigma_t^2 I_2, 0_{m+1-2t}), & \text{if } m+1 \text{ is even},\\ \operatorname{diag}(1, \sigma_1^2 I_2, \ldots, \sigma_t^2 I_2, 0_{m-2t}), & \text{if } m+1 \text{ is odd},\end{cases}$$

and consequently (1′) has the form

$$U^{\mathrm{T}}\mathscr{J}U = \begin{cases}\operatorname{diag}(\hat{\sigma}_1 \Lambda, \ldots, \hat{\sigma}_t \Lambda, \Lambda_{(m+1-2t)/2}), & \text{if } m+1 \text{ is even},\\ \operatorname{diag}(0, \hat{\sigma}_1 \Lambda, \ldots, \hat{\sigma}_t \Lambda, \Lambda_{(m-2t)/2}), & \text{if } m+1 \text{ is odd}.\end{cases}$$

Lemma 1. *Let $(\mathscr{J}, B)\in K(i)$. Then $(\mathscr{J}, B)\in V_0(i)$ if and only if there exists $(U, V)\in O(m+1)\times O(k)$ such that $(\mathscr{J}, B)=(U\mathscr{J}(\hat{\sigma})U^{\mathrm{T}}, VB(\sigma)U^{\mathrm{T}}$, where*

$$\mathscr{J}(\hat{\sigma}) = \begin{cases}\operatorname{diag}(\hat{\sigma}_1 \Lambda, \ldots, \hat{\sigma}_t \Lambda, \Lambda_{(m+1-2t)/2}) & \text{if } m+1, \text{ is even}\\ \operatorname{diag}(0, \hat{\sigma}_1 \Lambda, \ldots, \hat{\sigma}_t \Lambda, \Lambda_{(m-2t)/2}), & \text{if } m+1, \text{ is odd},\end{cases}$$

$$B(\sigma) = \begin{cases}\operatorname{diag}(\sigma_1 I_2, \ldots, \sigma_t I_2, 0_{d-2t})_k^{m+1}, & \text{if } m+1, \text{ is even},\\ \operatorname{diag}(1, \sigma_1 I_2, \ldots, \sigma_t I_2, 0_{d-1-2t})_k^{m+1}, & \text{if } m+1, \text{ is odd},\end{cases}$$

with $\sigma\in\Delta_t$, $\hat{\sigma}_i=\sqrt{1+\sigma_i^2}$, $i=1, \ldots, t$, and $d=\min(m+1, k)$.

Proof. If $(\mathscr{J}, B)\in V_0(i)$ then there exists $U\in O(m+1)$ such that $U^{\mathrm{T}}B^{\mathrm{T}}BU= =B(\sigma)^{\mathrm{T}}B(\sigma)$ and $U^{\mathrm{T}}\mathscr{J}U=\mathscr{J}(\hat{\sigma})$ with $0\leqq\hat{\sigma}_1\leqq\ldots\leqq\hat{\sigma}_{[(m+1)/2]}$. The diagonal entries of $U^{\mathrm{T}}B^{\mathrm{T}}BU$ are the eigenvalues of $B^{\mathrm{T}}B$ and hence, by Theorem A, there exists $V\in O(k)$ such that the pair (U, V) perform the singular value decomposition of B, i.e. we have $V^{\mathrm{T}}BU=B(\sigma)$. Thus, $(U^{\mathrm{T}}\mathscr{J}U, V^{\mathrm{T}}BU)=(\mathscr{J}(\hat{\sigma}), B(\sigma))$, $\sigma\in\Delta_t$. The converse being obvious the proof is finished.

By the lemma above the map $\Phi\colon \Delta_t\to K(i)$, $\Phi(\sigma)=(\mathscr{J}(\hat{\sigma}), B(\sigma))$, $\sigma\in\Delta_t$, is an embedding with $(O(m+1)\times O(k))\cdot\Phi(\Delta_t)=V_0(i)$. Moreover, the eigenvalues of $\mathscr{J}$ and the singular values of B are invariants characterizing the orbit through $(\mathscr{J}, B)$ uniquely. Thus $\Phi(\Delta_t)$ is a global section on $V_0(i)$ which accomplishes the proof of the first statement of Theorem 1.

Let $\sigma=(\sigma_0, \ldots, \sigma_0, \sigma_1, \ldots, \sigma_1, \ldots, \sigma_{s+1}, \ldots, \sigma_{s+1})\in\Delta_t$ be fixed with $1=\sigma_0> >\sigma_1>\ldots>\sigma_s>\sigma_{s+1}=0$ and σ_i occurs c_i times in σ, $i=0, \ldots, s+1$. It remains to compute the isotropy type of the orbit through $\Phi(\sigma)$. The isotropy subgroup at $\Phi(\sigma)$ consists of pairs (U, V) such that $U\mathscr{J}(\hat{\sigma})=\mathscr{J}(\hat{\sigma})U$ and $VB(\sigma)=B(\sigma)U$. First we study the second relation. Consider $B(\sigma)\in M(k, m+1)$ as a matrix

$$B(\sigma)=\begin{bmatrix}\Sigma & 0\\ 0 & 0\end{bmatrix},$$

where $\Sigma=\operatorname{diag}(\sigma_0 I_{a_0}\, \sigma_1 I_{2c_1}, \ldots, \sigma_s I_{2c_s})\in M(r, r)$, $r=a_0+2\sum_{i=1}^{s} c_i$,

$$a_0=\begin{cases}2c_0, & \text{if } m+1 \text{ is even,}\\ 2c_0+1, & \text{if } m+1 \text{ is odd,}\end{cases}$$

and 0 on the right lower corner is of size $(k-r)\times(m+1-r)$.

Lemma 2. *Let $(U, V)\in O(m+1)\times O(k)$ such that $VB(\sigma)=B(\sigma)U$ holds. Then we have $V=\operatorname{diag}(A_0, C_1, \ldots, C_s, B_0)$ and $U=\operatorname{diag}(A_0, C_1, \ldots, C_s, C_{s+1})$, where $A_0\in O(a_0)$, $B_0\in O(k-r)$, $C_i\in O(2c_i)$, $i=1, \ldots, s$, $C_{s+1}\in O(m+1-r)$.*

Proof. Let $V\in O(k)$ and $U\in O(m+1)$ have the partitioned forms (conformal to that of $B(\sigma)$ above):

$$V=\begin{bmatrix}V_0 & R\\ S & B_0\end{bmatrix} \quad\text{and}\quad U=\begin{bmatrix}U_0 & P\\ Q & C_{s+1}\end{bmatrix},$$

where $V_0, U_0\in M(r, r)$, $B_0\in M(k-r, k-r)$, $C_{s+1}\in M(m+1-r, m+1-r)$. (The size of C_{s+1} can be expressed as $m+1-r=2c_{s+1}+2([(m+1)/2]-t)^+$). Substituting these into the equations $VB(\sigma)=B(\sigma)U$, $VV^{\mathrm{T}}=I_k$, $UU^{\mathrm{T}}=I_{m+1}$ we obtain $R=0$, $S=0$, $V_0\in O(r)$, $B_0\in O(k-r)$ and $P=0$, $Q=0$, $U_0\in O(r)$, $C_{s+1}\in O(m+1-r)$. Thus the first equation reduces to $V_0\Sigma=\Sigma U_0$, i.e. by $\det\Sigma=\sigma_1^{2c_1}\ldots\sigma_s^{2c_s}>0$, $V_0=\Sigma U_0\Sigma^{-1}$. Substituting this into the orthogonality relation $V_0^T V_0=I_r$ we get $U_0\Sigma^2=\Sigma^2 U_0$ which gives for $U_0=(C_{ij})$, $C_{00}\in M(a_0, a_0)$, $C_{i0}\in M(2c_i, a_0)$, $C_{0j}\in M(a_0, 2c_j)$, $C_{ij}\in \in M(2c_i, 2c_j)$, $i, j=1, \ldots, s$, the relations $C_{ij}=0$, if $i\neq j$. Hence, using the notations $C_{00}=A_0$ and $C_{ii}=C_i$, $i=1, \ldots, s$, we obtain $U_0=\operatorname{diag}(A_0, C_1, \ldots, C_s)$ with $A_0\in O(a_0)$, $C_i\in O(2c_i)$, $i=1, \ldots, s$. As U_0 and Σ commute we have $V_0=U_0$ which accomplishes the proof.

Consider now the second equation $U\mathscr{J}(\hat{\sigma})=\mathscr{J}(\hat{\sigma})U$, where U has the form given in Lemma 2. Clearly, this equation is satisfied if and only if $C_i\in Z(\Lambda_{c_i})$, $i=1, \ldots, s$, $C_{s+1}\in Z(\Lambda_{(m+1-r)/2})$, where $Z(\Lambda_p)$ denotes the centralizer of Λ_p in $O(2p)$.

Lemma 3. *The centralizer* $Z(\Lambda_p)\subset O(2p)$ *is connected and there exists* $U_0\in O(2p)$ *such that* $\mathrm{Ad}\,(U_0)Z(\Lambda_p)=U(p)\subset SO(2p)$, *where* Ad *denotes the adjoint representation of* $O(2p)$.

Proof. It is well-known that $Z(\Lambda_p)\subset SO(2p)$ (cf. [8], Ch. IV. § 29, p. 248). First we prove that $Z(\Lambda_p)\subset SO(2p)$ is connected. Clearly, $\exp\big((\pi/2)\Lambda_p\big)=\Lambda_p$, where $\exp\colon so(2p)\to SO(2p)$ is the exponential map. Hence $T=\overline{\exp(\mathbf{R}\Lambda_p)}\subset SO(2p)$ is a toroidal subgroup which contains Λ_p, i.e. its centralizer $Z(T)$ is contained in $Z(\Lambda_p)$. On the other hand, if $U\in Z(\Lambda_p)$ then the geodesics $s\mapsto\exp\big((\pi/2)s\Lambda_p\big)\cdot U$, $s\mapsto U\cdot\exp\big((\pi/2)s\Lambda_p\big)$, $s\in\mathbf{R}$, (with respect to a biinvariant metric on $SO(2p)$) have common tangent vector at $s=0$, i.e. $\exp\big((\pi/2)s\Lambda_p\big)U=U\exp\big((\pi/2)s\Lambda_p\big)$ which implies that $U\in Z(T)$. Thus $Z(\Lambda_p)=Z(T)$ and hence connected (cf. [4], Cor. 2.8. p. 287). Finally, let

$$\mathscr{J}_p=\begin{bmatrix}0_p & I_p\\ -I_p & 0_p\end{bmatrix}$$

and choose $U_0\in O(2p)$ with $\mathrm{Ad}\,(U_0)\Lambda_p=\mathscr{J}_p$. Then $\mathrm{Ad}\,(U_0)Z(\Lambda_p)=Z\big(\mathrm{Ad}\,(U_0)\Lambda_p\big)=$ $=Z(\mathscr{J}_p)$ and the fixed point set of the automorphism $\mathrm{Ad}\,(\mathscr{J}_p)$ of $SO(2p)$ is $Z(\mathscr{J}_p)$. It is known that $Z(\mathscr{J}_p)=U(p)\subset SO(2p)$ ([4], p. 453—454) which accomplishes the proof.

By Lemmas 1—3, (U, V) belongs to the isotropy subgroup at $\Phi(\sigma)$ if and only if $(U, V)\in O(m+1)\times O(k)$ is conjugate to an element of $\mathscr{G}\big(c_1, \dots, c_s, c_{s+1}+$ $+([(m+1)/2]-t)^+\big)$ (under a conjugation which does not depend on (U, V)) which completes the proof of Theorem 1.

Example (Variation space of the identity of odd spheres). Consider the special case when $m=n=2r-1$ odd. Then $t=0$ and $V_0(\mathrm{id}_{S^{2r-1}})$ reduces to a single orbit through $\Lambda_r\in so(2r)$ under the adjoint representation of $O(2r)$ on $so(2r)$. We claim that this orbit is a disjoint union

$$\mathrm{Ad}\,\big(SO(2r)\big)\Lambda_r\cup\mathrm{Ad}\,\big(SO(2r)\big)\Lambda_r^-,$$

where $\Lambda_r^-=\mathrm{diag}\,(\Lambda, \dots, \Lambda, -\Lambda)\in so(2r)$. Indeed, denoting $R=\mathrm{diag}\,(1, \dots, 1, -1)\in$ $\in O(2r)$, we have $R\Lambda_rR=\Lambda_r^-$ and hence if $U\in O(2r)$ such that $\mathrm{Ad}\,(U)\Lambda_r=\Lambda_r^-$ then $\mathrm{Ad}\,(RU)\Lambda_r=\Lambda_r$ which implies $RU\in SO(2r)$, i.e. $\det U=-1$.

The Killing form of $so(2r)$ is a negative definite Ad-invariant scalar product on $so(2r)$ and so it follows easily that any ray in $so(2r)$ starting at the origin cuts the orbit $\mathrm{Ad}\,\big(SO(2r)\big)\Lambda_r$ (or $\mathrm{Ad}\,\big(SO(2r)\big)\Lambda_r^-$) at most once.

Case I: r is even. Then $\mathrm{Ad}\,(U_0)\Lambda_r=-\Lambda_r$ with $U_0=\mathrm{diag}\,(1, -1, 1, -1, \dots,$ $\dots, 1, -1)\in SO(2r)$, i.e. the orbit $\mathrm{Ad}\big(SO(2r)\big)\Lambda_r$ (and $\mathrm{Ad}\big(SO(2r)\big)\Lambda_r^-$) is central symmetric to the origin. Thus $V(\mathrm{id}_{S^{2r-1}})=\mathbf{R}\cdot V_0(\mathrm{id}_{S^{2r-1}})$ is a double cone over $\mathrm{Ad}\,\big(SO(2r)\big)\Lambda_r=SO(2r)/U(r)$.

Case II: r is odd. It follows easily that any line through the origin cuts $V_0(\mathrm{id}_{S^{2r-1}})$ twice and that the components $\mathrm{Ad}(SO(2r))\Lambda_r$ and $\mathrm{Ad}(SO(2r))\Lambda_r^-$ are central symmetric to each other, i.e. $V(\mathrm{id}_{S^{2r-1}})$ is again a double cone over $SO(2r)/U(r)$.

Remark. In the special case $r=2$ the space $V_0(\mathrm{id}_{S^3})$ is the disjoint union of two samples of $S^2(=SO(4)/U(2))$ which was already noticed in [13].

4. The Veronese surface

Let M be a compact oriented Riemannian manifold and consider a harmonic map $f: M\to S^n$. By the inclusion $j: S^n\to \mathbf{R}^{n+1}$ the map f becomes a vector-valued function $f: M\to\mathbf{R}^{n+1}$. Moreover, translating vectors tangent to $S^n\subset\mathbf{R}^{n+1}$ to the origin, a vector field v along $f: M\to S^n$ gives rise to a map $\hat{v}: M\to\mathbf{R}^{n+1}$ with the property $\langle f, \hat{v}\rangle=0$. The following lemma characterizes the elements of $K(f)$ in terms of the induced functions $\hat{v}$.

Lemma 4. *Let v be a vector field along $f: M\to S^n$. Then $v\in K(f)$ if and only if $\Delta^M\hat{v}=2e(f)\hat{v}$ holds, where $e(f)=\|f_*\|^2/2$ denotes the energy density of f.*

Proof. The covariant differentiation on S^n can be obtained from that of $\mathbf{R}^{n+1}$ by performing the orthogonal projection to the corresponding tangent space of S^n and thus, for $X\in\mathfrak{X}(M)$, we have

$$(\nabla_X v)^\wedge = X(\hat{v})-\langle X(\hat{v}), f\rangle f,$$

where X acts on $\hat{v}$ componentwise. An easy computation shows that

$$(\nabla_Y\nabla_X v)^\wedge = YX(\hat{v})-\langle YX(\hat{v}), f\rangle f-\langle X(\hat{v}), f\rangle Y(f), \quad X, Y\in\mathfrak{X}(M),$$

i.e.

$$(\nabla^2 v)^\wedge = -\Delta^M\hat{v}+\langle\Delta^M\hat{v}, f\rangle f-\mathrm{trace}\,\langle d\hat{v}, f\rangle\, df$$

holds. On the other hand, we have

$$(\mathrm{trace}\, R(f_*, v)f_*)^\wedge=(\mathrm{trace}\,\langle f_*, v\rangle f_*)^\wedge-2e(f)\hat{v}=$$

$$=\mathrm{trace}\,\langle df, \hat{v}\rangle df-2e(f)\hat{v}=-\mathrm{trace}\,\langle f, d\hat{v}\rangle\, df-2e(f)\hat{v}.$$

The identities yield that v is a Jacobi vector field along f if and only if

$$\Delta^M\hat{v}-\langle\Delta^M\hat{v}, f\rangle f = 2e(f)\hat{v} \tag{1}$$

is satisfied. Moreover, we have

$$\mathrm{trace}\,\langle f_*, \nabla v\rangle=\mathrm{trace}\,\langle df, d\hat{v}\rangle-\mathrm{trace}\,\langle d\hat{v}, f\rangle\langle df, f\rangle.$$

By $\|f\|^2=1$ the second term vanishes and so equation (ii) of Section 1 is equivalent to the following

(2) $$\operatorname{trace}\langle df, d\hat{v}\rangle = 0.$$

Further, harmonicity of f means that $\Delta^M f = 2e(f)f$ is valid and hence we get

$$\langle \Delta^M \hat{v}, f\rangle = -\langle \nabla^2 \hat{v}, f\rangle = -\operatorname{trace}\nabla\langle d\hat{v}, f\rangle + \operatorname{trace}\langle d\hat{v}, df\rangle =$$
$$= \operatorname{trace}\nabla\langle \hat{v}, df\rangle + \operatorname{trace}\langle d\hat{v}, df\rangle = 2\operatorname{trace}\langle d\hat{v}, df\rangle + \langle \hat{v}, \Delta^M f\rangle = 2\operatorname{trace}\langle d\hat{v}, df\rangle.$$

Assuming $v\in K(i)$ we obtain that $\langle \Delta^M \hat{v}, f\rangle = 0$ and hence (1) reduces to the equation given in the lemma. Conversely, multiplying this equation with f we get $\langle \Delta^M \hat{v}, f\rangle = 0$ and hence (1) and (2) are satisfied which accomplishes the proof.

Corollary. *Let $f, f'\colon M\to S^n$ be orthogonal harmonic maps with $e(f)=e(f')$. Then the (unique) vector field v along f with $\|v\|=1$ and* $\exp\circ((\pi/2)v)=f'$ *is a harmonic variation.*

Proof. By hypothesis $\hat{v}=f_{\pi/2}=f'$ and harmonicity of f' yields $\Delta^M \hat{v} = 2e(f')\hat{v} = 2e(f)\hat{v}$. Applying the lemma above we obtain that $v\in K(f)$ which accomplishes the proof.

Remark. According to a result of [11] a vector field v along f is a harmonic variation if and only if v is a Jacobi field along f and $e(f_t)=e(f)$ holds for all $t\in\mathbf{R}$. Hence there is a one-to-one correspondence between the harmonic variations of $V_0(f)$ and the orthogonal pairs of harmonic maps $f, f'\colon M\to S^n$ with $e(f)=e(f')$.

Now we turn to the variation space of the Veronese surface. Consider the eigenspace $\mathcal{H}_2$ of the Laplacian $\Delta=\Delta^{S^2}$ of the Euclidean sphere S^2 corresponding to the (second) eigenvalue $\lambda_2=6$ [1]. An element of $\mathcal{H}_2$ is the restriction (to S^2) of a homogeneous polynomial $p\colon \mathbf{R}^3\to\mathbf{R}$ of degree 2 which has the form

$$p = \sum_{k=1}^{3} a_k\varphi_k + 2\sum_{i<j} b_{ij}\varphi_{ij},$$

where $a_k, b_{ij}\in\mathbf{R}$ with $\sum_{k=1}^{3} a_k = 0$ and φ_k, φ_{ij}, $k=1,2,3$, $1\leq i<j\leq 3$, are scalars on S^2 defined by $\varphi_k(x)=x_k^2$, $\varphi_{ij}(x)=x_i x_j$, $x=(x_1,x_2,x_3)\in S^2$. (cf. [1] p. 176), in particular $\dim\mathcal{H}_2=5$.

Integration over S^2 defines a Euclidean scalar product on $\mathcal{H}_2$. Denoting $I=\|\varphi_k\|^2$ and $J=\|\varphi_{ij}\|^2$, the Veronese surface $f\colon S^2\to S^4$ is defined by

$$f(x_1,x_2,x_3) = \frac{N}{I-J}\sum_{k=1}^{3}\left(x_k^2-\frac{1}{3}\right)\varphi_k + \frac{2N}{J}\sum_{i<j} x_i x_j \varphi_{ij}, \quad (x_1,x_2,x_3)\in S^2,$$

where $N>0$ is a normalizing factor given by the condition $\|f\|=1$. Then f is full and homothetic [1]. It is well-known [1] that f factors through the canonical projection $\pi\colon S^2\to\mathbf{R}P^2$ yielding an embedding of $\mathbf{R}P^2$ into S^4.

Lemma 5. *For the Veronese surface* $f: S^2 \to S^4$, *if* $v \in K(f)$ *then* $\hat{v}: S^2 \to \mathcal{H}_2$ *has the decomposition*

$$\hat{v} = \sum_{k=1}^{3} a_k \varphi_k + 2 \sum_{i<j} b_{ij} \varphi_{ij},$$

where a_k, b_{ij}, $k=1, 3$, $1 \leq i < j \leq 3$, *are scalars on* S^2 *determined by the formulas*

$$a_1(x) = -\varepsilon x_2^2 + \varepsilon x_3^2 + 2\alpha_1 x_1 x_2 + 2\beta_1 x_1 x_3 - 2(\alpha_2 + \beta_3) x_2 x_3,$$

$$a_2(x) = \varepsilon x_1^2 - \varepsilon x_3^2 + 2\beta_2 x_1 x_2 - 2(\beta_1 + \alpha_3) x_1 x_3 + 2\alpha_2 x_2 x_3,$$

$$a_3(x) = -\varepsilon x_1^2 + \varepsilon x_2^2 - 2(\alpha_1 + \beta_2) x_1 x_2 + 2\alpha_3 x_1 x_3 + 2\beta_3 x_2 x_3,$$

$$b_{12}(x) = -\frac{\alpha_1}{2} x_1^2 - \frac{\beta_2}{2} x_2^2 + \frac{\alpha_1 + \beta_2}{2} x_3^2 - 2\gamma_1 x_1 x_3 + 2\gamma_2 x_2 x_3,$$

$$b_{23}(x) = \frac{\alpha_2 + \beta_3}{2} x_1^2 - \frac{\alpha_2}{2} x_2^2 - \frac{\beta_3}{2} x_3^2 - 2\gamma_2 x_1 x_2 + 2\gamma_3 x_1 x_3,$$

$$b_{13}(x) = -\frac{\beta_1}{2} x_1^2 + \frac{\alpha_3 + \beta_1}{2} x_2^2 - \frac{\alpha_3}{2} x_3^2 + 2\gamma_1 x_1 x_2 - 2\gamma_3 x_2 x_3,$$

$x = (x_1, x_2, x_3) \in S^2$, $\varepsilon, \alpha_k, \beta_k, \gamma_k \in \mathbf{R}$, $k = 1, 2, 3$. *In particular,* $\dim K(f) = 10$.

Proof. As $\hat{v}$ maps into $\mathcal{H}_2$ we have the decomposition of $\hat{v}$ as above with $\sum_{k=1}^{3} a_k = 0$. On the other hand, Lemma 4 implies that

$$0 = \Delta \hat{v} - 6\hat{v} = \sum_{k=1}^{3} (\Delta a_k - 6a_k) \varphi_k + 2 \sum_{i<j} (\Delta b_{ij} - 6b_{ij}) \varphi_{ij}$$

and hence orthogonality of the polynomials φ_{ij}, $i<j$, and the relations $\langle \varphi_k, \varphi_{ij} \rangle = 0$, $\langle \varphi_k, \varphi_r \rangle = J + \delta_{kr}(I - J)$, $k, r = 1, 2, 3$, $i<j$, yield that the scalars a_k, b_{ij}, $k = 1, 2, 3$, $i<j$, belong to $\mathcal{H}_2$. Thus

$$a_r = \sum_{k=1}^{3} a_k^r \varphi_k + 2 \sum_{i<j} b_{ij}^r \varphi_{ij}, \quad r = 1, 2, 3,$$

and

$$b_{pq} = \sum_{k=1}^{3} a_k^{pq} \varphi_k + 2 \sum_{i<j} b_{ij}^{pq} \varphi_{ij}, \quad 1 \leq p < q \leq 3,$$

where $a_k^r, b_{ij}^r, a_k^{pq}, b_{ij}^{pq} \in \mathbf{R}$ such that

$$(\mathrm{C}_1) \quad \sum_{k=1}^{3} a_k^r = 0 \quad \text{and} \quad \sum_{k=1}^{3} a_k^{pq} = 0, \quad r = 1, 2, 3, \ 1 \leq p < q \leq 3,$$

hold. Moreover, from the equation $\sum_{k=1}^{3} a_k = 0$ we obtain

$$(\mathrm{C}_2) \quad \sum_{r=1}^{3} a_k^r = 0 \quad \text{and} \quad \sum_{r=1}^{3} b_{ij}^r = 0.$$

Finally, the orthogonality relations for φ_k and φ_{ij} above imply that the condition $\langle f, \hat{v}\rangle = 0$ is equivalent to the equation

$$\sum_{k=1}^{3} a_k x_k^2 + 4 \sum_{i<j} b_{ij} x_i x_j = 0, \quad (x_1, x_2, x_3) \in S^2.$$

Substituting the explicit expressions of a_k and b_{ij} we get

$$\sum_{k=1}^{3} \sum_{r=1}^{3} a_k^r \varphi_k \varphi_r + 2 \sum_{i<j} \sum_{r=1}^{3} (b_{ij}^r + 2a_r^{ij}) \varphi_r \varphi_{ij} + 8 \sum_{i<j} \sum_{p<q} b_{ij}^{pq} \varphi_{ij} \varphi_{pq} = 0.$$

A straightforward computation, determining the coefficients of the fourth order homogeneous polynomial on the left hand side, shows that this equation is satisfied if and only if the following relations hold:

(C_3) $a_k^k = 0$ for $k = 1, 2, 3$,

(C_4) $b_{12}^1 + 2a_1^{12} = b_{12}^2 + 2a_2^{12} = b_{13}^1 + 2a_1^{13} = b_{13}^3 + 2a_3^{13} = b_{23}^2 + 2a_2^{23} = b_{23}^3 + 2a_3^{23} = 0$,

(C_5) $a_i^j + a_j^i + 8b_{ij}^{ij} = 0$ for $1 \leqq i < j \leqq 3$,

(C_6) $b_{23}^1 + 2a_1^{23} + 4b_{12}^{13} + 4b_{13}^{12} = b_{13}^2 + 2a_2^{13} + 4b_{12}^{23} + 4b_{23}^{12} = b_{12}^3 + 2a_3^{12} + 4b_{13}^{23} + 4b_{23}^{13} = 0$.

Putting $\varepsilon = a_1^2$, the relations (C_1)—(C_2)—(C_3) imply that the matrix $A = (a_k^r) \in M(3, 3)$ has the form

$$A = \begin{bmatrix} 0 & \varepsilon & -\varepsilon \\ -\varepsilon & 0 & \varepsilon \\ \varepsilon & -\varepsilon & 0 \end{bmatrix}$$

and consequently, by (C_5), $b_{ij}^{ij} = 0$ for $i < j$. Introducing the new (independent) variables

$$\alpha_1 = b_{12}^1, \; \alpha_2 = b_{23}^2, \; \alpha_3 = b_{13}^3,$$
$$\beta_1 = b_{13}^1, \; \beta_2 = b_{12}^2, \; \beta_3 = b_{23}^3,$$
$$\gamma_1 = b_{12}^{13}, \; \gamma_2 = b_{23}^{12}, \; \gamma_3 = b_{13}^{23},$$

we see that all the remaining coefficients are expressible in terms of the variables $\{\varepsilon, \alpha_k, \beta_k, \gamma_k \mid k = 1, 2, 3\}$ and a straightforward computation leads to the coefficients given in Lemma 5.

Our last result asserts that the Veronese surface is rigid. More precisely, we have the following

Theorem 2. *For the Veronese surface $f: S^2 \to S^4$ the variation space $V(f)$ is zero.*

Proof. Using the notations of Lemma 5 we parametrize $K(f)$ with the variables $\{\varepsilon, \alpha_k, \beta_k, \gamma_k \mid k = 1, 2, 3\}$. Putting $v \in K(f)$ we have

$$\hat{v} = \sum_{k=1}^{3} a_k \varphi_k + 2 \sum_{i<j} b_{ij} \varphi_{ij},$$

where the coefficients a_k, b_{ij}, $k = 1, 2, 3$, $1 \leqq i < j \leqq 3$, are given in Lemma 5.

Note that the parametrization of $K(f)$ is chosen in such a way as the cyclic permutation $\pi=(123)$ of the indices on the right hand sides will permute the scalars a_1, a_2, a_3 and b_{12}, b_{23}, b_{13} cyclically. Now suppose, on the contrary, that $V(f)\neq\{0\}$, i.e. we may choose $v\in V(f)$ with $\|v\|^2=4\mathscr{J}$. Then we have

$$4J=\|v\|^2=\sum_{k=1}^{3}\sum_{r=1}^{3}a_k a_r\langle\varphi_k,\varphi_r\rangle+4J\sum_{i<j}b_{ij}^2=(I-J)\sum_{k=1}^{3}a_k^2+4J\sum_{i<j}b_{ij}^2,$$

or equivalently

$$1=\frac{1}{2}\sum_{k=1}^{3}a_k^2+\sum_{i<j}b_{ij}^2 \tag{3}$$

on S^2, where we used the equality $\dfrac{I-J}{4J}=\dfrac{1}{2}$ which can be obtained by integrating the polynomials φ_3^2 and φ_{23}^2 on S^2. Thus

$$(x_1^2+x_2^2+x_3^2)^2=\frac{1}{2}\sum_{k=1}^{3}a_k(x_1,x_2,x_3)^2+\sum_{i<j}b_{ij}(x_1,x_2,x_3)^2$$

is satisfied for all $(x_1,x_2,x_3)\in\mathbf{R}^3$. By computing the coefficients of the fourth order homogeneous polynomial on the right hand side we obtain a system of 15 quadratic equations in which the first 5 are given as follows

(i) $4\varepsilon^2+\alpha_1^2+\beta_1^2+(\alpha_2+\beta_3)^2=4$,

(ii) $\varepsilon(\alpha_1+2\beta_2)-\beta_1\gamma_1-(\alpha_2+\beta_3)\gamma_2=0$,

(iii) $-\varepsilon(\beta_1+2\alpha_3)+\alpha_1\gamma_1+(\alpha_2+\beta_3)\gamma_3=0$,

(iv) $\varepsilon(\alpha_2-\beta_3)+2\big(\alpha_1\beta_1-\beta_2(\beta_1+\alpha_3)-\alpha_3(\alpha_1+\beta_2)\big)-\alpha_1\gamma_2+\beta_1\gamma_3-4\gamma_2\gamma_3=0$,

(v) $-2\varepsilon^2+4\big(\alpha_1^2+\beta_2^2+(\alpha_1+\beta_2)^2\big)+\alpha_1\beta_2-\beta_1(\beta_1+\alpha_3)-\alpha_2(\alpha_2+\beta_3)+8(\gamma_1^2+\gamma_2^2)=4$,

and, the equation (3) being invariant under the cyclic permutation $\pi=(123)$ of the indices, the last 10 equations are obtained from (i)—(v) by performing the index permutations π and π^2. Denote the equations of the permuted systems by $(\mathrm{i})_\pi$—$(\mathrm{v})_\pi$ and $(\mathrm{i})_{\pi^2}$—$(\mathrm{v})_{\pi^2}$, respectively. Our purpose is to show that these equations have no solution. To do this, first denote by s the symmetric polynomial given by $s(x,y)=x^2+xy+y^2$, $x,y\in\mathbf{R}$. Then (v) can be written as

$$-2\varepsilon^2+8s(\alpha_1,\beta_2)+(\alpha_1\beta_2-\beta_1^2-\beta_1\alpha_3-\alpha_2^2-\alpha_2\beta_3)+8(\gamma_1^2+\gamma_2^2)=4.$$

Performing the index permutations π and π^2 and adding these three equations we get

$$-6\varepsilon^2+7\big(s(\alpha_1,\beta_2)+s(\alpha_2,\beta_3)+s(\alpha_3,\beta_1)\big)+16(\gamma_1^2+\gamma_2^2+\gamma_3^2)=12.$$

In a similar way, from (i)—$(\mathrm{i})_\pi$—$(\mathrm{i})_{\pi^2}$ it follows that

$$12\varepsilon^2+2\big(s(\alpha_1,\beta_2)+s(\alpha_2,\beta_3)+s(\alpha_3,\beta_1)\big)=12,$$

i.e. eliminating the terms containing the polynomial s we have

(4) $$24(1-\varepsilon^2)+8(\gamma_1^2+\gamma_2^2+\gamma_3^2) = 9.$$

On the other hand, fixing γ_i, $i=1, 2, 3$, the equations (ii)—(ii)$_\pi$—(ii)$_{\pi^2}$ and (iii)—(iii)$_\pi$—(iii)$_{\pi^2}$ form a linear system for the variables α_i, β_i, $i=1, 2, 3$. Denoting by $M(\gamma_1, \gamma_2, \gamma_3)\in M(6, 6)$ its matrix, we compute $\det M(\gamma_1, \gamma_2, \gamma_3)$. For $\xi, \eta, \zeta\in \mathbf{R}$ define

$$S(\xi, \eta, \zeta) = \begin{bmatrix} \varepsilon & 2\varepsilon & 0 & -\xi & -\eta & -\eta \\ -2\varepsilon & -\varepsilon & \xi & \xi & \eta & 0 \\ -\xi & -\xi & \varepsilon & 2\varepsilon & 0 & -\zeta \\ \xi & 0 & -2\varepsilon & -\varepsilon & \zeta & \zeta \\ 0 & -\eta & -\zeta & -\zeta & \varepsilon & 2\varepsilon \\ \eta & \eta & \zeta & 0 & -2\varepsilon & -\varepsilon \end{bmatrix}.$$

Permuting the rows and the coloumns of $M(\gamma_1, \gamma_2, \gamma_3)$ by the permutation (25) we obtain $S(\gamma_1, \gamma_2, \gamma_3)$ and consequently $\det M(\gamma_1, \gamma_2, \gamma_3)=\det S(\gamma_1, \gamma_2, \gamma_3)$. Similarly, by performing (135462) and (132465) on the rows and coloumns of $M(\gamma_1, \gamma_2, \gamma_3)$ we get $S(\gamma_2, \gamma_3, \gamma_1)$ and $S(\gamma_3, \gamma_1, \gamma_2)$ i.e. $\det S(\gamma_1, \gamma_2, \gamma_3) = \det S(\gamma_2, \gamma_3, \gamma_1)=\det S(\gamma_3, \gamma_1, \gamma_2)$. Thus, it is enough to compute $\det S(\xi, \eta, \zeta)$. To do this, let $S(\xi, \eta, \xi)$ have the decomposition

$$S(\xi, \eta, \zeta) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where $A\in M(4, 4)$. The matrix A is centroskew and so by using a result of [2] a direct computation shows that $\det A=(3\varepsilon^2-\xi^2)^2$. Assuming $3\varepsilon^2\neq\xi^2$ we have [2]

$$\det S(\xi, \eta, \zeta) = \det A \det(D-CA^{-1}B) = 3\varepsilon^2(3\varepsilon^2-(\xi^2+\eta^2+\zeta^2))^2.$$

Suppose now that $\gamma_1^2=\gamma_2^2=\gamma_3^2=3\varepsilon^2$. Then equation (4) implies that $15+8\varepsilon^2=0$ which is impossible. Hence there exists $i\in\{1, 2, 3\}$ such that $\gamma_i\neq 3\varepsilon^2$. Then, by the above, $\det M(\gamma_1, \gamma_2, \gamma_3)=3\varepsilon^2(3\varepsilon^2-(\gamma_1^2+\gamma_2^2+\gamma_3^2))^2$. Further, $\det M(\gamma_1, \gamma_2, \gamma_3)\neq 0$ since otherwise $\gamma_1^2+\gamma_2^2+\gamma_3^2=3\varepsilon^2$ which contradicts to (4). Thus the linear system in question has only trivial solution $\alpha_1=\alpha_2=\alpha_3=\beta_1=\beta_2=\beta_3=0$. Then equations (iv)—(iv)$_\pi$—(iv)$_{\pi^2}$ imply that two of the numbers $\gamma_1, \gamma_2, \gamma_3$ vanish. By equations (v)—(v)$_\pi$—(v)$_{\pi^2}$ we obtain $\varepsilon=0$ which again contradicts to (4).

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