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On the stability and convergence of solutions of differential equations by Liapunov's direct method

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1. Introduction

By means of a modification of Liapunov's direct method we give sufficient conditions for the stability of solutions of ordinary differential equations and for the existence of finite limits of certain functions (specially, of a part of coordinates) along solutions as $t \to \infty$. For the study of this problem, T. A. BURTON [2], J. R. HADDOCK [5, 6] and L. HATVANI [8, 13] used modifications in which the derivative of the Liapunov function was estimated by the norm of a linear combination of components of the right-hand side of the system. T. A. BURTON [3] has extended this method for the estimate in which a power of a linear combination of the right-hand sides occurs. In this paper we investigate the case when the estimate contains a monotone function of a linear combination of the right-hand sides. We apply our results to studying the asymptotic behaviour of solutions of certain second order non-linear differential equations and the stability properties of motions of mechanical systems under the action of potential and dissipative forces depending also on the time.

2. The main results

Consider the differential system

$$\dot{x}(t) = X(t, x), \tag{2.1}$$

where $t \in R_+ = [0, \infty)$, x belongs to the n-dimensional Euclidean space R^n, $X \in C(R_+ \times \Gamma, R^n)$; $\Gamma \subset R^n$ is an open set.

Let us introduce some notations. Denote by (x, y) the scalar product of vectors $x, y \in R^n$. $\|x\| = (x, x)^{1/2}$ is the norm of the vector $x \in R^n$. Let B_H denote the set of elements $x \in R^n$ such that $\|x\| < H$ $(H > 0)$. The distance $\varrho(H_1, H_2)$

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between the sets $H_1, H_2 \subset R^n$ is defined by

$$\varrho(H_1, H_2) = \inf \{\|x-y\|: x \in H_1, y \in H_2\}.$$

$\bar{H}$ denotes the closure of the set H. Let K denote the class of increasing functions $a \in C(R_+, R_+)$ for which $a(0)=0$ and $a(s)>0$ for all $s>0$. Denote by L^+ the class of Lebesgue measurable functions $f: R_+ \to R_+ \cup \{\infty\}$, by L_p^+ $(0<p<\infty)$ and L_∞^+ the classes of the functions $f \in L^+$ with

$$\int_0^\infty f^p(s)\,ds < \infty, \quad \sup_{s \in R_+} \operatorname{ess} f(s) < \infty,$$

respectively. Let $u(t; t_0, u_0)$ be the maximal noncontinuable solution of the equation

$$\dot{u} = r(t, u) \tag{2.2}$$

through (t_0, u_0), where $r \in C(R_+ \times R_+, R_+)$.

Let us given a function $\omega \in C(R_+ \times R_+, R_+)$ with $\omega(t, \cdot) \in K$. In the sequel we shall often have to solve an inequality of type $\omega(t, f(t)) \leqq g(t)$ for the function f. This motivates the following notations:

$$\omega(t, \infty) = \lim_{u \to \infty} \omega(t, u) \quad (\leqq \infty),$$

$$\omega^{-1}(t, v) = \max \{u: \omega(t, u) \leqq v\},$$

$$\omega^{-1}(t, w(t, \infty)) = \infty.$$

The function $\omega^{-1}(t, v)$ is defined for $t \in R_+$, $0 \leqq v \leqq \omega(t, \infty)$, it is increasing in u, continuous on the right and satisfies the inequality

$$\omega^{-1}(t, \omega(t, u)) \geqq u \quad (t \in R_+, u \in R_+).$$

For every δ $(0<\delta\leqq\infty)$ denote by D_δ the set of functions $f \in L^+$ for which $f(t) \leqq \omega(t, \delta)$ $(t \in R_+)$, and define the map $\Omega_\delta: D_\delta \to L^+$ by

$$(\Omega_\delta f)(t) = \omega^{-1}(t, f(t)) \quad (t \in R_+, f \in D_\delta).$$

For a function $V \in C^1(R_+ \times \Gamma', R^k)$ $(\Gamma' \subset \Gamma)$ we define the derivative $\dot{V} \in C(R_+ \times \Gamma', R^k)$ of the function V with respect to (2.1) as follows

$$\dot{V}(t, x) = \frac{\partial V(t, x)}{\partial t} + \frac{\partial V(t, x)}{\partial x} X(t, x) \quad (t \in R_+, x \in \Gamma').$$

Obviously, if $x(t)$ is a solution of equation (2.1), then

$$\frac{d}{dt} V(t, x(t)) = \dot{V}(t, x(t)).$$

Let us given a function $W\in C^1(R_+\times\Gamma, R^k)$. In the sequel we examine the asymptotic behavior of W along solutions of (2.1), i.e. the asymptotic behavior of the function $W(t, x(t))$. In the following theorem we use the set $\bigcap_{t\geq 0}\overline{W([t,\infty),\Gamma)}$, which consists of all $w\in R^k$ for which there exist sequences $\{t_i\}$, $\{x_i\}$ with $x_i\in\Gamma$, $t_i\to\infty$, $W(t_i, x_i)\to w$ as $i\to\infty$.

Theorem 2.1. *Suppose that for each* $w_1, w_2\in\bigcap_{t\geq 0}\overline{W([t,\infty),\Gamma)}$ *there exist functions* $V\in C^1(R_+\times\Gamma, R_+)$, $r, r_1, \omega\in C(R_+\times R_+, R_+)$, *open sets* $H_1, H_2\subset R^k$ *and a constant* $T>0$ *satisfying the following conditions:*

(A) $w_1\in H_1, w_2\in H_2, \varrho(H_1, H_2)>0$;

(B) $r(t, u)$ *is increasing in* u *and the solutions of equation* (2.2) *are bounded;*

(C) $r_1(t, u)$ *is increasing in* u *and* $r_1(\cdot, u)\in L_1^+$ $(u\in R_+)$;

(D) $\omega(t,\cdot)\in K$ $(t\in R_+)$ *and* Ω_∞ *maps* $D_\infty\cap L_1^+$ *into* L_1^+;

(E) $\dot V(t,x)\leq r(t, V(t,x))$ $(t\in R_+, x\in\Gamma)$;

(F) $\dot V(t,x)\leq -\omega(t, \|\dot W(t,x)\|)+r_1(t, V(t,x))$

for all (t,x) *such that* $t\geq T$, $x\in\Gamma$, $W(t,x)\notin\bar H_1\cup\bar H_2$.

Then for every solution $x(t)$ *of* (2.1) *defined on* $[t_0,\infty)$ *either* $\|W(t,x(t))\|\to\infty$ *or* $W(t,x(t))\to$const. *as* $t\to\infty$.

Proof. First of all, observe that

$$r(\cdot, u_0)\in L_1^+ \quad (u_0\in R_+). \tag{2.3}$$

Indeed, let $u_0\in R_+$. By virtue of the monotonicity of $r(t,u)$ in u we have

$$\dot u(t; t_0, u_0) = r(t, u(t; t_0, u_0)) \geq r(t, u_0);$$

therefore, assertion (2.3) holds.

Now, consider a solution $x\colon [t_0,\infty)\to R^n$ of (2.1) and put $w(t)=W(t,x(t))$. Suppose that the assertion of the theorem is not true, i.e., there exist two distinct elements w_1, w_2 of the set $\bigcap_{t\geq t_0}\overline{w([t,\infty))}$. Consider some sets H_1, H_2, functions V, r, r_1, ω and some constant T corresponding to w_1, w_2 in the sense of the assumptions of the theorem.

By the basic theorem on differential inequalities, from assumptions (B) and (E) we obtain the estimate

$$V(t,x(t))\leq u(t; t_0, V(t_0, x_0))\leq C=\text{const} \quad (t\in[t_0,\infty)).$$

So,

$$\frac{d}{dt}\Big(V(t,x(t))+\int_t^\infty r(s,C)\,ds\Big) = \dot V(t,x(t))-r(t,C)\leq 0,$$

consequently,

$$f(t)=r(t,C)-\dot V(t,x(t))\in L_1^+.$$

Since $w_1, w_2 \in \bigcap_{t \geqq t_0} \overline{w([t, \infty))}$, there exist two sequences $\{t_i\}$, $\{t_i^*\}$ such that

$$T \leqq t_i < t_i^* < t_{i+1} \quad (i = 1, 2, \ldots), \quad \lim_{i \to \infty} t_i = \infty; \tag{2.3}$$

$$w(t_i) \in \bar{H}_1, \; w(t_i^*) \in \bar{H}_2 \quad (i = 1, 2, \ldots),$$

$$w(t) \notin \bar{H}_1 \cup \bar{H}_2 \quad \left(t \in \bigcup_{i=1}^{\infty} (t_i, t_i^*)\right).$$

Introduce the notation

$$g(t) = \max\left(0, \min\left(\omega(t, \infty), r_1(t, C) - \dot{V}(t, x(t))\right)\right).$$

Then by condition (F) we have

$$g(t) \geqq \omega\left(t, \|\dot{w}(t)\|\right) \quad \left(t \in \bigcup_{i=1}^{\infty} (t_i, t_i^*)\right).$$

So,

$$\|\dot{w}(t)\| \leqq \omega^{-1}\left(t, g(t)\right) \quad \left(t \in \bigcup_{i=1}^{\infty} (t_i, t_i^*)\right).$$

Therefore,

$$N\varrho(H_1, H_2) \leqq \sum_{i=1}^{N} \|w(t_i) - w(t_i^*)\| =$$

$$= \sum_{i=1}^{N} \left\| \int_{t_i}^{t_i^*} \dot{w}(t)\, dt \right\| \leqq \sum_{i=1}^{N} \int_{t_i}^{t_i^*} \omega^{-1}\left(t, g(t)\right) dt.$$

This means that $\omega^{-1}(\cdot, g(\cdot)) \notin L_1$. Consequently, by condition (D), $g \notin L_1^+$.

On the other hand, we have

$$g(t) \leqq r_1(t, C) - \dot{V}\left(t, x(t)\right) \leqq f(t) + r_1(t, C)$$

for all t such that $r_1(t, C) - \dot{V}\left(t, x(t)\right) \geqq 0$. By virtue of $f(t) \geqq 0$, $r_1(t, C) \geqq 0$ we have

$$g(t) \leqq f(t) + r_1(t, C) \quad (t \in R_+),$$

which contradicts $f, r_1(\cdot, C) \in L_1^+$. The theorem is proved.

Theorem 2.2. *Suppose that there exist functions* $V \in C^1(R_+ \times \Gamma, R_+)$, $r, \omega \in C(R_+ \times R_+, R_+)$ *such that assumptions* (B), (D) *and*

$$\dot{V}(t, x) \leqq -\omega\left(t, \|\dot{W}(t, x)\|\right) + r\left(t, V(t, x)\right) \quad (t \in R_+,\; x \in \Gamma) \tag{F$_1$}$$

are fulfilled. Then $W\left(t, x(t)\right) \to \text{const.}$ *as* $t \to \infty$ *for every solution* $x(t)$ *of* (2.1) *defined on* $[t_0, \infty)$.

Proof. By Theorem 2.1, it is sufficient to show that $w(t) = W\left(t, x(t)\right)$ is bounded for every solution of (2.1) defined on $[t_0, \infty)$.

Suppose the contrary. Then there exist two sequences $\{t_i\}$, $\{t_i^*\}$ and a natural number $M>0$ such that

$$T \leqq t_i < t_i^* \leqq t_{i+1} \quad (i = 1, 2, \ldots), \quad \lim_{i\to\infty} t_i = \infty,$$

$$\|w(t_i)\| = i, \ \|w(t_i^*)\| = i+1 \quad (i = M, M+1, \ldots),$$

$$i < \|w(t)\| < i+1, \ t \in (t_i, t_i^*) \quad (i = M, M+1, \ldots),$$

are fulfilled. So

$$N \leqq \sum_{i=M}^{N+M} (\|w(t_i^*)\| - \|w(t_i)\|) =$$

$$= \sum_{i=M}^{M+N} \int_{t_i}^{t_i^*} \frac{d}{dt} \|w(t)\|\, dt \leqq \sum_{i=M}^{M+N} \int_{t_i}^{t_i^*} \|\dot{w}(t)\|\, dt.$$

Hence, by virtue of (F_1) we have

$$N \leqq \sum_{i=M}^{M+N} \int_{t_i}^{t_i^*} \omega^{-1}(t, g_1(t))\, dt \leqq \int_{t_M}^{t_{M+N}^*} \omega^{-1}(t, g_1(t))\, dt,$$

where

$$g_1(t) = \min\big(\omega(t, \infty),\ r\big(t, \sup_{t \geqq T} V(t, x(t))\big) - \dot{V}(t, x(t))\big).$$

This inequality contradicts $g_1 \in L_1^+$, which concludes the proof.

Theorem 2.3. *Let* $0 \in \Gamma$ *and* $X(t, 0) \equiv 0$ *for all* $t \in R_+$. *Suppose there exist functions* $a, b \in K$, $V \in C^1(R_+ \times B_H, R_+)$ $(B_H \subset \Gamma)$, $\omega, r \in C(R_+ \times R_+, R_+)$ *such that*

(B_1) $r(t, 0) = 0$ *for all* $t \in R_+$, $r(\cdot, u) \in L_1^+$ *for all* $u > 0$, $r(t, u)$ *is increasing in* u *and the zero solution of equation* (2.2) *is unique;*

(D_1) $\omega(t, \cdot) \in K$ $(t \in R_+)$ *and the map* $\Omega_\infty\colon D_\infty \cap L_1^+ \to L_1^+$ *is continuous at* $u(t) \equiv 0$ *in* L_1*-norm;*

(F_2) $\dot{V}(t, x) \leqq -a(\|W(t, x)\|)\omega(t, \|\dot{W}(t, x)\|) + r(t, V(t, x))$ *for all* $t \in R_+, x \in B_H$;

(G) $V(t, 0) = 0, W(t, 0) = 0$ *for all* $t \in R_+$ *and* $b(\|x\|) \leqq V(t, x) + \|W(t, x)\|$ $(t \in R_+, x \in B_H)$.

Then the zero solution of equation (2.1) *is stable, and for every solution* $x(t)$ *of* (2.1) *with sufficiently small* $\|x(t_0)\|$ *the function* $W(t, x(t))$ *has a finite limit as* $t \to \infty$.

Proof. We first prove that the zero solution of equation (2.2) is stable. Suppose the contrary. Then there exist a number $\varepsilon_0 > 0$, sequences $\{u_i\}$, $\{t_i\}$ of solutions of (2.2) and positive numbers, respectively, such that

$$u_i(0) \to 0 \quad \text{as} \quad i \to \infty,$$

$$u_i(t_i) = \varepsilon_0, \quad u_i(t) < \varepsilon_0 \quad (0 \leqq t < t_i, \ i = 1, 2, \ldots).$$

Define

$$r_i(t) = \begin{cases} r(t, u_i(t)) & 0 \leqq t \leqq t_i, \\ 0 & t_i \leqq t. \end{cases}$$

By virtue of (B_1) we have

$$0 \leqq r_i(t) \leqq r(t, \varepsilon_0) \quad (i = 1, 2, \ldots), \quad r_i(t) \to 0 \quad \text{as} \quad i \to \infty \quad (t \in R_+).$$

Applying Lebesgue's dominated convergence theorem we obtain

$$\int_0^{t_i} r(t, u_i(t))\, dt = \int_0^{\infty} r_i(t)\, dt \to 0 \quad \text{as} \quad i \to \infty.$$

By integration of (2.2) it follows that

$$\varepsilon_0 - u_i(0) = \int_0^{t_i} r(t, u_i(t))\, dt.$$

Hence, if $i \to \infty$, we get $\varepsilon_0 = 0$, which is a contradiction. Consequently, the zero solution of (2.2) is stable.

Let us denote by $\delta_1(\varepsilon)$, $\delta_2(\varepsilon)$ the numbers corresponding to ε in the definition of stability of the zero solution of (2.2) and in the definition of continuity of Ω_∞, respectively. Let $0 < \varepsilon < H$, $t_0 \in R_+$ be fixed arbitrarily. Choose ε_1 so that

$$\text{(2.4)} \qquad \varepsilon_1 < b(\varepsilon), \quad \int_0^{\infty} r(t, \varepsilon_1)\, dt + \varepsilon_1 < \delta_2\left(\frac{b(\varepsilon) - \varepsilon_1}{2}\right) a\left(\frac{b(\varepsilon) - \varepsilon_1}{2}\right)$$

and define $\delta = \delta(\varepsilon, t_0)$ such that $0 < \delta < \frac{\varepsilon}{2}$ and $\|x_0\| < \delta$ imply

$$\text{(2.5)} \qquad V(t_0, x_0) < \delta_1(\varepsilon_1), \quad \|W(t_0, x_0)\| < (b(\varepsilon) - \varepsilon_1)/2.$$

Consider a solution $x(t)$ of (2.1) with $\|x(t_0)\| < \delta$. Denote by $[t_0, A)$ the maximal interval to the right in which $\|x(t)\| < H$ is true. By assumption (F_2) we have

$$\dot V(t, x(t)) \leqq r(t, V(t, x(t))) \quad (t \in [t_0, A)),$$

hence and from (2.5) it follows

$$V(t, x(t)) \leqq u(t, V(t_0, x(t_0))) \leqq \varepsilon_1 \quad (t \in [t_0, A)).$$

We show that the inequality $\|x(t)\| < \varepsilon$ also is satisfied for $t \in [t_0, A)$. Otherwise there exists a $T \in (t_0, A)$ such that $\|x(T)\| = \varepsilon$. Consequently,

$$\|W(T, x(T))\| \geqq b(\|x(T)\|) - V(T, x(T)) \geqq b(\varepsilon) - \varepsilon_1.$$

So, by (2.5) there are $t_1, t_2 \in (t_0, A)$ such that the function $w(t) = W(t, x(t))$ satisfies

$$\|w(t_1)\| = (b(\varepsilon) - \varepsilon_1)/2, \quad \|w(t_2)\| = b(\varepsilon) - \varepsilon_1,$$

$$(b(\varepsilon) - \varepsilon_1)/2 < \|w(t)\| < b(\varepsilon) - \varepsilon_1 \quad (t \in (t_1, t_2)).$$

Using assumption (F_2), we obtain

$$\|\dot{w}(t)\| \leqq \omega^{-1}(t, u(t)) \quad (t \in (t_1, t_2))$$

where

$$u(t) = \min\left(\omega(t, \infty), \frac{r(t, \varepsilon_1) - \dot{V}(t, x(t))}{a((b(\varepsilon)-\varepsilon_1)/2)}\right).$$

By integration over (t_1, t_2) this implies that

$$\int_{t_1}^{t_2} \omega^{-1}(t, u(t))\, dt \geqq (b(\varepsilon)-\varepsilon_1)/2. \tag{2.6}$$

On the other hand, from (2.4) it follows that

$$\int_{t_1}^{t_2} u(t)\, dt \leqq \left(\int_{t_1}^{t_2} r(t, \varepsilon_1)\, dt + V(t_1, x(t_1)) - V(t_2, x(t_2))\right) \frac{1}{a((b(\varepsilon)-\varepsilon_1)/2)} \leqq$$

$$\leqq \frac{1}{a((b(\varepsilon)-\varepsilon_1)/2)} \left(\int_{t_1}^{\infty} r(t, \varepsilon_1)\, dt + V(t_1, x(t_1))\right) < \delta_2((b(\varepsilon)-\varepsilon_1)/2),$$

which contradicts (2.6). This means that $\|x(t)\| < \varepsilon$ is satisfied for all $t \in [t_0, A)$. Therefore, $A = \infty$ and the zero solution is stable.

The other statements of the theorem follows from Theorem 2.1.

Remark 2.1. If we put $W(t, x) = (x_1, \ldots, x_k)$ $(1 \leqq k \leqq n)$, where $x_1, \ldots, x_k, \ldots, x_n$ are the components of the vector x, then our theorems with

$$\|\dot{W}(t, x)\| = \left(\sum_{i=1}^{k} X_i^2(t, x)\right)^{1/2}$$

yield conditions on the convergence of the components $x_1, \ldots, x_k$ along solutions.

Remark 2.2. If

$$V(t, x) + \|W(t, x)\| \to \infty \quad \text{as,} \quad x \to R^n \setminus \Gamma \quad \text{or} \quad \|x\| \to \infty$$

for every $t \in R_+$, then under the assumptions of Theorem 2.2 every solution of equation (2.1) can be continued to $[t_0, \infty)$.

Remark 2.3. If there exists $d \in K$ such that

$$\|\dot{W}(t, x)\| \leqq d(\|x\|) \quad t \in R_+, \ x \in B_H)$$

then in Theorem 2.3 assumption (D_1) may be replaced by the following:

(D_2) $\omega(t, \cdot) \in K$ $(t \in R_+)$ and the map $\Omega_\delta \colon D_\delta \cap L_1^+ \to L_1^+$ is continuous at $u(t) \equiv 0$ in L_1-norm for some $\delta > 0$.

In the following we give realization of assumptions (D), (D_1), (D_2) in some important special cases. Let $N(u)$ be a continuous convex function which satisfies the following conditions:

$$N\in K,\quad \lim_{u\to\infty}\frac{N(u)}{u}=0,\quad \lim_{u\to\infty}\frac{N(u)}{u}=\infty.$$

Put

$$M(u)=\int_0^u \sup\left\{t\colon \frac{d}{dt}N(t)\leqq s\right\}ds.$$

If $s(t), r(t)$ are measurable on $[0, T]$ and

$$\int_0^T N(s(t))\,dt<\infty,\quad \int_0^T M(r(t))\,dt<\infty$$

then, by the generalized Hölder inequality (see [10], p. 222—233) the function $s(t)\cdot r(t)$ is integrable and

$$\int_0^T s(t)r(t)\,dt\leqq\left(1+\int_0^T N(s(t))\,dt\right)\left(1+\int_0^T M(r(t))\,dt\right). \tag{2.7}$$

Lemma 2.1. *Let a continuous function* $\lambda(t)\geqq 0$ *satisfy the inequality*

$$\int_0^\infty M\left(\frac{1}{\lambda(t)}\right)dt<\infty.$$

If $\omega(t,u)$ *is defined by* $\omega(t,u)=N(\lambda(t)u)$ $(t\in R_+,\ u\in R_+)$ *then* (D) *is satisfied.*

Proof. It is easy to see that $\lambda(t)>0$ almost everywhere, and

$$\omega(t,\infty)=\begin{cases}\infty, & \lambda(t)>0,\\ 0, & \lambda(t)=0,\end{cases}$$

$$\omega^{-1}(t,u)=\frac{N^{-1}(u)}{\lambda(t)}\quad (\lambda(t)>0,\ u\in R_+)$$

Let $u\in L_1^+\cap D_\infty$. Applying inequality (2.7) we have

$$\int_0^T \omega^{-1}(t,u(t))\,dt=\int_{\substack{\lambda(t)>0\\ t<T}}\frac{N^{-1}(u(t))}{\lambda(t)}\,dt\leqq$$

$$\leqq\left(1+\int_0^T N(N^{-1}(u(t)))\,dt\right)\left(1+\int_0^T M\left(\frac{1}{\lambda(t)}\right)dt\right)\leqq$$

$$\leqq\left(1+\int_0^\infty u(t)\,dt\right)\left(1+\int_0^\infty M\left(\frac{1}{\lambda(t)}\right)dt\right)<\infty$$

for all $T>0$. So, $\int_0^\infty \omega^{-1}(t,u(t))\,dt<\infty$ which was to be proved.

Remark 2.4. If $\omega(t, u)=\mu(t)u^{\alpha}$ $(t\in R_+, u\in R_+)$, where $1<\alpha=\text{const.}$, $\mu\in C(R_+, R_+)$, $1/\mu\in L^+_{1/(\alpha-1)}$ then assumptions (D), (D_1) are satisfied.

This assertion follows from the ordinary Hölder inequality. T. A. Burton [3] considered this case studying the boundedness and the existence of the limit of solutions.

Obviously, if $\omega(t, u)=\mu(t)u$ where $\mu\in C(R_+; [c, \infty))$ and $0<c=\text{const.}$, then (D), (D_1) are satisfied. This case was studied in [2, 5, 6, 7, 13].

Lemma 2.2. *Let g be a continuous strictly increasing function such that*

$$\lim_{u\to\infty} g(u) = \infty, \quad g(u) \geqq cu^{\nu} \quad (0 \leqq u \leqq u_0),$$

where $c>0$, $\nu\geqq 1$ are some constants. Let us choose a continuous function $\lambda(t)$ such that $1/\lambda\in L^+_{1/(\nu-1)}\cap L^+_{\infty}$ and put $\omega(t, u)=\lambda(t)g(u)$. Then ($D_2$) is satisfied. Moreover, if

$$0 < \liminf_{u\to\infty} \frac{g(u)}{u} \tag{2.8}$$

then (D_1) is also true.

Proof. The assumptions imply

$$\lambda(t) \geqq c_1 = \text{const.} > 0 \quad (t\in R_+), \quad \omega(t, \infty) = \infty \quad (t\in R_+),$$

$$\omega^{-1}(t, v) = g^{-1}(v/\lambda(t)) \ (v\in R_+, t\in R_+), \quad g^{-1}(v) \leqq (v/c)^{1/\nu} \ (0 \leqq v \leqq g(u_0)).$$

Let $u\in L_1^+\cap D_\delta$. Then, for $\nu>1$ by means of Hölder inequality we obtain

$$\int\limits_{u(t)\leqq c_1 g(u_0)} \omega^{-1}(t, u(t))\, dt \leqq \frac{1}{c^{1/\nu}} \int\limits_{u(t)\leqq c_1 g(u_0)} \left(\frac{u(t)}{\lambda(t)}\right)^{1/\nu} dt \leqq$$

$$\leqq \frac{1}{c^{1/\nu}} \Big(\int_0^\infty u(t)\, dt\Big)^{1/\nu} \Big(\int_0^\infty (\lambda(t))^{1/(1-\nu)} dt\Big)^{\nu/(\nu-1)}$$

and

$$\int\limits_{c_1 g(u_0)\leqq u(t)} \omega^{-1}(t, u(t))\, dt \leqq \int\limits_{c_1 g(u_0)\leqq u(t)} g^{-1}(g(\delta))\, dt \leqq \frac{\delta}{c_1 g(u_0)} \int_0^\infty u(t)\, dt.$$

Consequently,

$$\int_0^\infty \omega^{-1}(t, u(t))\, dt \leqq c_2 \Big(\int_0^\infty u(t)\, dt\Big)^{1/\nu} + c_3 \int_0^\infty u(t)\, dt$$

for some $c_2, c_3>0$. This inequality is obvious for $\nu=1$, therefore (D_2) is satisfied, indeed.

By (2.8) there exist positive constants K and u_1 such that $g^{-1}(u) \leqq Ku$ $(u_1 \leqq u)$. If $u \in L_1^+$, then

$$\int_{u_1 \leqq u(t)} \omega^{-1}(t, u(t))\, dt \leqq \frac{K}{c_1} \int_0^\infty u(t)\, dt,$$

$$\int_{u_0 \leqq u(t) \leqq u_1} \omega^{-1}(t, u(t))\, dt \leqq \frac{g^{-1}(u_1/c)}{u_0} \int_0^\infty u(t)\, dt,$$

so, using the preceding argument, it is easy to verify assumption (D_1).

Example 2.1. Let us define

$$\omega(t, u) = \begin{cases} \lambda(t) \exp[\log^3 u], & u > 0 \\ 0, & u = 0 \end{cases} \quad (t, u \in R_+),$$

where $\lambda(t)$ is continuous, $\lambda(t) \geqq c = \text{const.} > 0$ and

$$\int_0^\infty \exp\left[\log^{1/3} \frac{ce}{\lambda(t)}\right] dt < \infty$$

(e.g., $\lambda(t) = \exp[t^3]$ or $\exp[\delta \log^3(1+t)]$, where $\delta > 1$). Then (D) is satisfied. Indeed,

$$\omega^{-1}(t, u) = \exp\left[\log^{1/3} \frac{u}{\lambda(t)}\right] \quad (t \in R_+,\ u \in R_+),$$

and if $u \in L_1^+$, then

$$\int_0^\infty \omega^{-1}(t, u(t))\, dt \leqq \int_{u(t) \leqq ce} \exp\left[\log^{1/3} \frac{ce}{\lambda(t)}\right] dt + \int_{u(t) \geqq ce} \exp\left[\log^{1/3} \frac{u(t)}{\lambda(t)}\right] dt$$

$$\leqq \int_0^\infty \exp\left[\log^{1/3} \frac{ce}{\lambda(t)}\right] dt + \frac{1}{c} \int_0^\infty u(t)\, dt < \infty.$$

3. Applications to second order differential equations and mechanical system

I. Consider the differential equation

$$(p(t)\dot{x})^{\cdot} + q(t) g(x) = 0, \tag{3.1}$$

where $p, q \in C^1(R_+, R_+)$, $g \in C(R, R)$, $p(t) > 0$, $q(t) > 0$ $(t \in R_+)$, $\int_0^x g(u)\, du \geqq 0$ $(x \in R)$. Attractivity and asymptotic stability of the trivial solution $x = \dot{x} = 0$ have been studied by many authors under the assumption that $x = 0$ is an isolated solution of the equation $g(x) = 0$ [8, 9, 12]. Now we are going to apply Theorem 2.2, 2.3

to get sufficient conditions for the existence of $\lim_{x\to\infty} x(t)$ in the case when $x=0$ is, possibly, a non-isolated solution of $g(x)=0$.

By introducing the variable $y=p(t)\dot{x}$, equation (3.1) can be written in the form

$$\dot{x} = y/p(t), \quad \dot{y} = -q(t)g(x). \tag{3.2}$$

For this equation let us choose the Liapunov function

$$V(t, x, y) = \frac{\varrho(t)}{2p(t)} y^2 + \varrho(t)\, q(t) \int_0^x g(u)\, du,$$

where $\varrho \in C^1(R_+, R_+ \setminus \{0\})$. The derivative of V with respect to (3.2) reads as follows:

$$\dot{V}(t, x, y) = \left(\frac{\varrho(t)}{p(t)}\right)^{\cdot} \frac{y^2}{2} + (\varrho(t) q(t))^{\cdot} \int_0^x g(u)\, du.$$

Let the functions W, r, ω be defined by

$$W(t, x, y) = x, \quad r(t, u) = \frac{[(\varrho(t) q(t))^{\cdot}]_+}{\varrho(t) q(t)} u, \quad \omega(t, u) = -\frac{p^2(t)}{2}\left(\frac{\varrho(t)}{p(t)}\right)^{\cdot} u^2.$$

Then we have

$$\dot{V} \leqq -\omega(t, |\dot{W}|) + r(t, V), \quad \dot{W} = y/p(t).$$

We note that in this case the solutions of equation (2.2) are bounded provided that the inequality

$$\int_0^\infty \frac{[(\varrho(t) q(t))^{\cdot}]_+}{\varrho(t) q(t)}\, dt < \infty \tag{3.3}$$

is fulfilled. By virtue of Remark 2.4

$$\int_0^\infty \frac{ds}{p^2(s)(\varrho(s)/p(s))^{\cdot}} > -\infty, \quad (\varrho(t)/p(t))^{\cdot} < 0 \quad (t \in R_+) \tag{3.4}$$

imply assumption (D). Consequently, from Theorem 2.2 it follows the following:

Corollary 3.1. *If there exists a function* $\varrho \in C^1(R_+, R_+)$ *such that* (3.3) *and* (3.4) *are true, then the limit of every solution* $x(t)$ *of* (3.1) *defined on* $[t_0, \infty)$ *exists as* $t \to \infty$.

Suppose that

$$\frac{\varrho(t)}{p(t)} \geqq c = \text{const.} > 0 \quad (t \in R_+). \tag{3.5}$$

Then $V(t, x, y) + |W(t, x, y)| \geqq (y^2/2)c + |x|$. Using Remark 2.2 and Theorem 2.3, and taking into consideration the fact that the function $V(t, x, y)$ is non-increasing along the solutions of (3.2), provided that $\lim_{t\to\infty} \varrho(t) q(t)$ exists, we get

Corollary 3.2. *Suppose, that* (3.3)—(3.5) *are fulfilled. Then the zero solution of system* (3.2) *is stable. For every solution* $x(t)$ *of equation* (3.1), $\lim_{t\to\infty} x(t)$ *exists. Moreover, if* $\lim_{t\to\infty} q(t)\varrho(t)$ *exists, then* $\lim_{t\to\infty} \varrho(t)\dot{x}(t)$ *exists, too.*

It is worth noticing that these corollaries work in case $\int_0^\infty \frac{ds}{p(s)} < \infty$, whose interest consists in the fact that it cannot be reduced to an equation of type $\ddot{x}+a(t)g(x)=0$.

On the other hand, one can easily see that if

$$p(t)q(t) \geqq c = \text{const.} > 0, \quad (p(t)q(t))^{\cdot}\left[\varepsilon + \int_t^\infty \frac{ds}{p(s)}\right] \leqq q(t)$$

for t sufficiently large with some $\varepsilon > 0$, and $\int_0^\infty \frac{ds}{p(s)} < \infty$, then the fuctinon

$$\varrho(t) = p(t)\left[\varepsilon + \int_t^\infty \frac{ds}{p(s)}\right]$$

satisfies the conditions of Corollary 3.2.

II. Consider the differential equation

$$\ddot{x} + f(t, x, \dot{x})|\dot{x}|^\alpha \dot{x} + g(x) = 0, \tag{3.6}$$

where $f \in C(R_+ \times R \times R, R_+)$, $0 \leqq \alpha = \text{const.}$, $g \in C(R, R)$. A great number of papers have been devoted to the study of the conditions of the asymptotic stability and attractivity of the zero solution $x = \dot{x} = 0$. In these papers it is assumed that f is either bounded above or tends to infinity sufficiently slowly as $t \to \infty$ [1, 7, 8]. R. J. Ballieu and K. Peiffer [1] analyzed whether this condition is necessary. They proved for the case $\alpha = 0$ $f(t, x, \dot{x}) = \vartheta(t)$, $\limsup_{x\to 0} g(x)/x < \infty$ the following assertions: a) If $\vartheta(t)$ is increasing and $\int_0^\infty \frac{dt}{\vartheta(t)} = \infty$, then the zero solution of (3.3) is asymptotically stable. b) If $\vartheta(t)$ is increasing and $\int_0^\infty \frac{dt}{\vartheta(t)} < \infty$, then the zero solution of (3.3) is not attractive. Applying Theorem 2.3 we obtain that in the latter case the zero solution of (3.3) is stable, and every solution has a finite limit as $t \to \infty$.

Corollary 3.3. *Suppose that*

$$\int_0^x g(u)\,du \geqq 0 \quad (|x| \leqq c = \text{const.}),$$

$$f(t, x, \dot{x}) \geqq \vartheta(t)\, b(|x|) \quad (t \in R_+,\ |x|,\ |\dot{x}| \leqq c),$$

where $b \in K$ *and* $1/\vartheta \in L^1_{1/(1+\alpha)}$, $\vartheta(t)$ *is continuous. Then the zero solution of* (3.6) *is stable and for every solution* $x(t)$ *of* (3.6), $x(t) \to \text{const.}$, $\dot{x}(t) \to 0$ *as* $t \to \infty$ *provided that* $x^2(t_0) + \dot{x}^2(t_0)$ *is sufficiently small.*

Proof. Equation (3.6) may be written in the form

$$\dot{x} = y, \quad \dot{y} = -f(t, x, y)|y|^{\alpha} y - g(x). \tag{3.7}$$

Let the Liapunov function V be defined by

$$V(x, y) = y^2/2 + \int_0^y g(u)\,du.$$

Since

$$\dot{V}(t, x, y) = -f(t, x, y)|y|^{\alpha+2} \quad (x, y \in R,\ t \in R_+),$$

we have the estimate

$$\dot{V}(t, x, y) \leqq -\vartheta(t) b(|x|)|y|^{\alpha+2} \quad (t \in R_+,\ |x|, |y| \leqq c).$$

Therefore, by $\omega(t, u) = \vartheta(t)|u|^{\alpha+2}$, $W(t, x, y) = x$ we obtain

$$\dot{V}(t, x, y) \leqq -b(|x|)\,\omega(t, |\dot{W}(t, x, y)|) \quad (t \in R_+,\ |x|, |y| \leqq c).$$

Consequently, by Remark 2.4 the assumptions of Theorem 2.3 are fulfilled. So, $x = y = 0$ is stable and $\lim_{t \to \infty} x(t)$ exists if $x^2(t_0) + y^2(t_0)$ is small. On the other hand, $V(t, x, y)$ is nonincreasing along solutions. This implies the existence of the limit $\lim_{t \to \infty} y(t)$, which, obviously, cannot differ from zero.

III. Corollary 3 can be generalized to mechanical systems with friction if the damping is increasing sufficiently fast in the time.

Consider a holonomic, rheonomic mechanical system being under the action of conservative, gyroscopic and dissipative forces, which may depend also on the time. The equation of motions in Lagrange's form reads as follows:

$$\frac{d}{dt}\frac{\partial T(q, \dot{q})}{\partial \dot{q}} - \frac{\partial T(q, \dot{q})}{\partial q} = -\frac{\partial \Pi(t, q)}{\partial q} + Q(t, q, \dot{q}), \tag{3.8}$$

where $q \in \Gamma \subset R^n$, $\dot{q} \in R^n$ denote the vectors of the generalized coordinates and velocities, respectively; $T \in C^2(\Gamma \times R^n, R_+)$ is the kinetic energy, $\Pi \in C^1(R_+ \times \Gamma, R)$ is the potential energy of the system, $Q \in C(R_+ \times \Gamma \times R^n, R^n)$ denotes the resultant of the gyroscopic and dissipative forces. We assume that

$$T(q, \dot{q}) = \dot{q}^{\mathrm{T}} A(q) \dot{q}/2,$$

where $A(q)$ is a symmetric positive definite matrix for each $q \in \Gamma$. Suppose that $0 \in \Gamma$, $\partial \Pi(t, 0)/\partial q = 0$, $Q(t, q, 0) = 0$ $(t \in R_+, q \in \Gamma)$. Under these conditions the state $q = \dot{q} = 0$ is an equilibrium of (3.8).

Corollary 3.4. *Suppose*

$$\Pi(t,q)\geqq 0,\quad \partial\Pi(t,q)/\partial t\leqq r\big(t,\Pi(t,q)\big)\quad (t\in R_+,\ q\in B_H\subset\Gamma),$$
$$\big(Q(t,q,\dot q),\dot q\big)\leqq -\vartheta(t)a(\|q\|)g(\|\dot q\|)\quad (t\in R_+,\ q,\dot q\in B_H),$$

where $a\in K$, $r\in C^1(R_+\times R_+, R_+)$, $r(t,\cdot)\in K$, $\int_0^\infty r(\tau,u)\,d\tau<\infty$ $(t,u\in R_+)$; *furthermore, suppose there exists a natural number* μ *such that* $g\in K$, $g'(0)=\ldots=g^{(\mu-1)}(0)=0$, $g^{(\mu)}(0)\neq 0$, $1/\vartheta\in L^+_{1/\mu}$, ϑ *is continuous.*

Then the equilibrium $q=\dot q=0$ *is stable and* $q(t)\to\text{const.}\in R^n$ *as* $t\to\infty$ *provided that* $q^2(t_0)+\dot q^2(t_0)$ *is sufficiently small.*

Proof. $A(q)$ is positive definite, so, introducing the new variables $x=q, y=\dot q$ equation (3.8) can be written in the form

$$\dot x = y,\quad \dot y = F(t,x,y). \tag{3.9}$$

In the capacity of Liapunov function choose the total mechanical energy

$$V(t,x,y)=T(x,y)+\Pi(t,x).$$

As is known [4],

$$\dot V(t,x,y)=\big(Q(t,x,y),y\big)+\frac{\partial\Pi(t,x)}{\partial t}\quad (t\in R_+,\ x\in\Gamma,\ \dot x\in R).$$

Consequently, if we define $W(t,x,y)=x$, $\omega(t,u)=\vartheta(t)g(u)$ we obtain

$$\dot V(t,x,y)\leqq -a(\|x\|)\vartheta(t)g(\|y\|)+r\big(t,\Pi(t,x)\big)\leqq$$
$$\leqq -a(\|x\|)\omega\big(t,\|\dot W(t,x,y)\|\big)+r\big(t,V(t,x,y)\big)$$

for every $t\in R_+$, $x,y\in B_H$. Therefore, the assertion follows from Theorem 2.3., Lemma 2.2 and Remark 2.3.

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