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Meromorphic functions of operators

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Let T be a bounded operator on a separable Hilbert space. Combining previous results of HALMOS [4] and FILLMORE [3] concerning operator identities of the forms $0=f(T^*)$ and $T=f(T^*)$ with f entire, MOORE [6] proved the following general theorem:

Theorem A. [6] *Suppose that p is a polynomial, f is an entire function, and $p(T)=f(T^*)$. Then there is a polynomial q (of the same degree as p when T is not algebraic) such that $p(T)=q(T^*)$.*

The proof of this theorem required a key replacement of the operator identity by a complex variable identity, followed by a version of the Jacobi polynomial expansion theorem, resultant arguments, and a theorem of Picard. In this paper we begin with the complex variable identity and generalize Theorem A utilizing a more geometric argument, motivated by FILLMORE [3] and based on the monodromy theorem and the Weierstrass preparation theorem. A good reference for the classical complex variable theorems is HILLE [5]. We prove:

Theorem 1. *Let r be a rational function, M a meromorphic function in the complex plane, and assume that $r(T)=M(T^*)$. (Thus the poles of r and M lie outside of $\sigma(T)$ and $\sigma(T^*)$, respectively.) Then there is a rational function q such that $r(T)=q(T^*)$. Moreover, when T is not algebraic, M itself must be rational and of the same order as r.*

Before beginning the proof we state the replacement theorem of MOORE [6] for convenience.

Theorem B. [6] *Let f and g be analytic in neighborhoods of $\sigma(T)$ and $\sigma(T^*)$, respectively, and suppose that $g(T)=f(T^*)$. Then for $z\in\sigma(T)$, $g(z)=f(\bar{z})$.*

Proof of Theorem 1. If $\sigma(T)$ is finite then $\sigma(r(T))$ is finite and $r(T)$ is normal, hence algebraic. Thus T and T^* are algebraic, and M may be replaced by a rational function.

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Otherwise $\sigma(T)$ is infinite and contains a limit point α. First note that if r has order less than one, $F(T^*)=0$ for some entire function F, so T^* and thus T is algebraic and the theorem holds. Hence we may assume that N, the order of r, satisfies $N\geqq 1$.

By Theorem B we know that $r(z)=M(\bar{z})$ for z in $\sigma(T)$. Clearly we may take $\alpha=0$ and $r(0)=0$, modifying T, M, and r, if necessary. Then $z^n r_1(z)=\bar{z}^m M_1(\bar{z})$ for z in $\sigma(T)$, where $r_1(0)\neq 0$, $M_1(0)\neq 0$ and n and m are positive integers. Taking the modulus and letting $z\in\sigma(T)\to 0$, we see that $n=m$. Thus we may write $r(z)=h(z)^m$ and $M(\bar{z})=k(\bar{z})^m$, where h and k are analytic and invertible in a neighborhood W of 0. Since $(h(z))^m=(k(\bar{z}))^m$ for z's in $\sigma(T)$ (which is an infinite set), we may assume that h and k are chosen so that $h(z_n)=$ $=k(\bar{z}_n)$ for some sequence $\{z_n\}\subset\sigma(T)$ with $z_n\to 0$.

Computing, $z_n=h^{-1}\circ k(\bar{z}_n)$, so $\bar{z}_n=\overline{h^{-1}}\circ\bar{k}(z_n)=\bar{h}^{-1}\circ\bar{k}(z_n)$, where $\bar{h}(z)$ is defined to be $\overline{h(\bar{z})}$.

Let $S=\{z\in W: \bar{z}=\bar{h}^{-1}\circ\bar{k}(z)\}$. Then $\sigma(T)\subset S$. Using a consequence of the Weierstrass preparation theorem [7], we conclude that S is the intersection of (real analytic) arcs with only a finite number in any compact set. Using the fact that S contains a limit point we conclude that S contains a real analytic arc γ. Choose a point z_0 in γ so that $r'(z_0)$ is not zero or infinity.

Thus $\bar{z}=\bar{h}^{-1}\circ\bar{k}(z)$ for z in γ, so $h(z)=k\circ\bar{h}^{-1}\circ\bar{k}(z)$ for z in γ. Because all functions are analytic in W we conclude that $h(z)=k\circ\bar{h}^{-1}\circ\bar{k}(z)$ for all z in W.

By choice of z_0, r is invertible in a connected neighborhood of z_0 contained in W, Ω. Again, let $R(w)=r(w+z_0)-r(z_0)$, $\mathcal{M}(\bar{w})=M(\bar{w}+\bar{z}_0)-M(\bar{z}_0)$, $\Omega_0=\Omega-z_0$, and $\gamma_0=\gamma-z_0$. Hence $R(z)=\mathcal{M}(\bar{z})$ for $z\in\gamma_0$ and R is invertible in Ω_0. Then arguing as before $z=R^{-1}\circ\mathcal{M}(\bar{z})$ so $R(z)=M\circ\bar{R}^{-1}\circ\bar{\mathcal{M}}(z)$ for all z in Ω_0.

Denote the complex plane by $\mathbf{C}$. Define $Z(f, a)=\{z\in\mathbf{C}: f(z)=a\}$. Let $Z_1=$ $=\{z\in\mathbf{C}: \bar{\mathcal{M}}(z)\in Z(\bar{R}', 0)\cup Z(\bar{R}', \infty)\}$. Each of the sets $Z(R', a)$ contains at most $2N$ elements since R has order N. But $\bar{\mathcal{M}}$ is meromorphic in the complex plane, so the set of points with $\bar{\mathcal{M}}(z)=c$ for any fixed c has no finite limit points. Thus we may join each of the points of Z_1 by a simple curve γ_1 accumulating only at ∞, chosen so that if $\Omega_1=\mathbf{C}-\gamma_1$, then Ω_1 is connected and simply connected. $\bar{R}^{-1}$ is one branch of the inverse of $\bar{R}$ in $\bar{\Omega}_0$. By construction branches of the inverse of $\bar{R}$ exist at every point of $\mathbf{C}-\gamma_1$. By the monodromy theorem (see [1, p. 134]) we see that $\bar{R}^{-1}$ can be continued into Ω_1, defining a single-valued analytic function (again denoted by $\bar{R}^{-1}$) in Ω_1.

Recall that $R(z)=\mathcal{M}\circ\bar{R}^{-1}\circ\bar{\mathcal{M}}(z)$ for $z\in\Omega_0$. Thus by permanence of functional relations $R(z)=\mathcal{M}\circ\bar{R}^{-1}\circ\bar{\mathcal{M}}(z)$ for $z\in\Omega_1$.

Suppose that for some $c\in\Omega_1$, $|Z(\bar{\mathcal{M}}, c)|>N$. Let $d=\mathcal{M}\circ R^{-1}(c)$. Then

$$N\geqq|Z(R, d)|\geqq|Z(\mathcal{M}\circ\bar{\mathcal{R}}^{-1}\circ\bar{\mathcal{M}}, d)|\geqq|Z(\bar{\mathcal{M}}, c)|>N.$$

This contradiction shows that $\bar{\mathscr{M}}$ is at most N-valent in Ω_1. Since Ω_1 is open and dense in $\mathbf{C}$, the open mapping principle shows that $\bar{\mathscr{M}}$ is at most N-valent in $\mathbf{C}$. Applying the Casorati—Weierstrass theorem and the open mapping principle (or using the great Picard theorem), we see that ∞ is not an essential singularity of $\bar{\mathscr{M}}$. Thus M is a rational function of order less than or equal to N. A symmetric argument shows that the order of M equals the order of r.

Note that in the case when both r and M are entire, the conclusion that M has order N means that M is a polynomial of degree N.

Remarks. (a) Letting T be a unitary operator shows that taking r to be a polynomial with M meromorphic does not allow us to conclude that M is itself a polynomial.

(b) Theorem 1 covers the case that $g(T^*)p(T)=f(T^*)q(T)$, where f and g are entire, p and q are polynomials, and $q(T)$ and $g(T^*)$ are invertible. We do not know how to handle more general identities with T and T^* appearing on both sides.

(c) There should be some "Riemann surface" version of Theorem 1 valid for r an algebraic function with appropriate hypotheses concerning M.

We briefly wish to consider what compact sets K can be the spectrum of an operator T satisfying

$$f(T) = F(T)^*, \tag{1}$$

where f and F are analytic in a neighborhood of K. Notice that if

$$f(z) = \overline{F(z)} \tag{2}$$

for z in K, then (1) can be solved for a normal operator T and in many cases nonnormal operator solutions can be constructed as well.

Denote the real and imaginary parts of f and F by u, v and U, V, respectively. We see that (2) is equivalent to

$$u-U=0 \quad \text{and} \quad v+V=0 \quad \text{for } z \text{ in } K. \tag{3}$$

On the otherhand, let P and Q be any real-valued harmonic functions in a neighborhood of K with single-valued conjugates (denoted by $\tilde{P}$ and $\tilde{Q}$, respectively) in a neighborhood of K. Then if

$$P=0 \quad \text{and} \quad Q=0 \quad \text{for } z \text{ in } K \tag{4}$$

we may write $P=u-U$ and $Q=v+V$ where $u=(P-\tilde{Q})/2$, $U=(-\tilde{Q}-P)/2$, $v=\tilde{u}$, and $V=\tilde{U}$. Thus letting $f=u+iv$ and $F=U+iV$ we have established

Theorem 2. *There exist analytic functions f and F in a neighborhood of K with $f(z)=\overline{F(z)}$ for z in K if and only if there exist real harmonic functions P, Q in a neighborhood of K with single-valued conjugates in a neighborhood of K and with $P(z)=0$ and $Q(z)=0$ for z in K.*

Corollary. *Suppose that* Q *is a real harmonic function with single-valued conjugate in a neighborhood of* K *and* $Q(z)=0$ *for* z *in* K. *Then* $-\tilde{Q}+iQ= =\overline{-\tilde{Q}+iQ}$ *for* z *in* K.

Proof. Take $P\equiv 0$ in Theorem 2.

Theorem 2 and the corollary are useful for constructing various examples.

By the corollary, to understand K we must look at the zero set of a harmonic function. We mention a few well-known facts. Simply because a harmonic function h is locally a real analytic function in x and y, the Weierstrass preparation theorem [6] shows that locally $Z(h, 0)$ is a finite union of analytic arcs. Moreover, if the gradient of h vanishes at some point s, then the derivative of $h+i\tilde{h}$ vanishes at s. Thus locally the number of arcs and the types of singularities of $Z(h, 0)$ are restricted. In the case when f and F are analytic in a simply connected set, the maximum principle says that $Z(h, 0)$ contains no closed curves.

It may be of interest to see how the paks revious remmarnd geoetric considerations lead to a proof of a special case of Theorem A. Let K be an infinite compact set. Suppose that $p(z)=\overline{q(z)}$ for z in K, where p and q are polynomials with $\max(\deg p, \deg q)=m$. Let $u_1=\operatorname{Re} p-\operatorname{Re} q$ and $u_2=\operatorname{Im} p+\operatorname{Im} q$. Then $u_1=0$ and $u_2=0$ for z in K, where u_1 and u_2 are real harmonic polynomials of degree m. Since u_1 and u_2 vanish at so many common points (see [2, Chapter 1]), it follows that u_1 and u_2 have a common polynomial factor, h, of degree greater than 0. Let $f=u_1+i\tilde{u}_1$. Then f is a polynomial in z of degree m. So, at ∞, $Z(u_1, 0)$ has $2m$ branches. However the degree of u_1/h is less than m, so $Z(h, 0)$ must contain some branch which extends to ∞. But then $p(z)=\overline{q(z)}$ holds for some sequence of z's approaching ∞. Since p and q are polynomials, the degrees of p and q are equal.

I do not know whether Theorem 1 or even Theorem A can be proved analogously to the above special case with a more thorough understanding of the zero sets involved.

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