Acta Sci. Math. 46 (1983), 203—209

On the absolute Riesz summability of orthogonal series

L. LEINDLER

1. Let Σa_n be a given infinite series and s_n denote its nth partial sum. If $\{p_n\}$ is a sequence of positive numbers, and

$$P_n = \sum_{k=0}^{n} p_k \to \infty \quad \text{as} \quad n \to \infty,$$

then the nth Riesz mean R_n of Σa_n is defined by

$$R_n = \frac{1}{P_n} \sum_{k=0}^{n} p_k s_k. \tag{1.1}$$

If the series

$$\sum_{n=1}^{\infty} |R_n - R_{n-1}| \tag{1.2}$$

converges, then the series Σa_n is said to be summable $|R, P_n, 1|$. It is clear that if $p_k=1$ then (1.1) reduces to the classical $(C, 1)$-mean, and $|R, n+1, 1|$ means that the series Σa_n is absolute $(C, 1)$-summable.

Let $\{\varphi_n(x)\}$ be an orthonormal system defined on the finite interval (a, b). We consider the orthogonal series

$$\sum_{k=0}^{\infty} c_k \varphi_k(x) \quad \text{with} \quad \sum_{k=0}^{\infty} c_k^2 < \infty. \tag{1.3}$$

Furthermore let $P(x)$ be a strictly increasing function such that $P(n)=P_n$ and linear between n and $n+1$. We denote the inverse function of $P(x)$ by $\Lambda(x)$ and put $\nu_m=[\Lambda(2^m)]$, where $[x]$ denotes the integral part of x.

K. Tandori [5] proved that *the condition*

$$\sum_{m=0}^{\infty} \left\{ \sum_{n=2^m+1}^{2^{m+1}} c_n^2 \right\}^{1/2} < \infty \tag{1.4}$$

Received December 5, 1981.

is necessary and sufficient that series (1.3) *for every orthonormal system* $\{\varphi_n(x)\}$ *should be absolute* $(C, 1)$*-summable, or summable* $|R, n+1, 1|$ *almost everywhere in* (a, b).

We ([1]) showed that *condition* (1.4) *is also necessary and sufficient that series* (1.3) *for every orthonormal system* $\{\varphi_n(x)\}$ *be absolute* (C, α)*-summable with* $\alpha > 1/2$ *almost everywhere.* In [1] we also gave conditions implying the absolute $(C, 1/2)$- and (C, α)-summability with $-1 < \alpha < 1/2$.

The result of Tandori was generalized by F. Móricz [3] to the absolute Riesz summability as follows.

Theorem A. *Orthogonal series* (1.3) *for every orthonormal system* $\{\varphi_n(x)\}$ *is summable* $|R, P_n, 1|$ *almost everywhere if and only if*

$$\sum_{m=0}^{\infty} \left\{ \sum_{n=\nu_m+1}^{\nu_{m+1}} c_n^2 \right\}^{1/2} < \infty, \tag{1.5}$$

where $C_m = \left\{ \sum_{n=\nu_m+1}^{\nu_{m+1}} c_n^2 \right\}^{1/2} = 0$ *if* $\nu_{m+1} = \nu_m$.

Recently Y. Okuyama and T. Tsuchikura [4] gave a condition which is equivalent to (1.5) and it does not use the concept of $\Lambda(x)$.

More precisely they proved

Theorem B. *Condition* (1.5) *is equivalent to*

$$\sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}} \left\{ \sum_{k=1}^{n} P_{k-1}^2 c_k^2 \right\}^{1/2} < \infty. \tag{1.6}$$

Using these theorems and some lemmas the authors of [4] also proved the following

Theorem C. *If the series*

$$\sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}} \left\{ \sum_{k=1}^{n} P_{k-1}^2 (a_k^2 + b_k^2) \right\}^{1/2} \tag{1.7}$$

converges, then almost all series of

$$\sum_{n=1}^{\infty} \pm (a_n \cos nx + b_n \sin nx) \tag{1.8}$$

are summable $|R, P_n, 1|$ *almost everywhere, and if series* (1.7) *diverges, then almost all series of* (1.8) *are non-summable* $|R, P_n, 1|$ *almost everywhere.*

2. In the present note we prove certain symmetrical analogues of Theorems B and C.

Theorem 1. *Condition* (1.5) *is equivalent to*

$$\sum_{n=0}^{\infty} p_n \left\{ \sum_{k=n}^{\infty} P_k^{-2} c_k^2 \right\}^{1/2} < \infty. \tag{2.1}$$

By Theorem A and Theorem 1 we immediately obtain

Corollary 1. *Condition* (2.1) *is necessary and sufficient that series* (1.3) *for any orthonormal system* $\{\varphi_n(x)\}$ *should be summable* $|R, P_n, 1|$ *almost everywhere.*

Hence we get

Corollary 2. *If*

$$\sum_{n=0}^{\infty} \frac{p_n}{P_n} \left\{ \sum_{k=n}^{\infty} c_k^2 \right\}^{1/2} < \infty \tag{2.2}$$

then series (1.3) *for every orthonormal system* $\{\varphi_n(x)\}$ *is summable* $|R, P_n, 1|$ *almost everywhere.*

It is well known, by the Riesz—Fischer theorem, that series (1.3) converges in L^2 to a square-integrable function f; and if $E_n^{(2)}(f)$ denotes the best approximation to f in the metric of L^2 by means of polynomials of $\varphi_0, \ldots, \varphi_{n-1}$, then

$$E_n^{(2)}(f) = \left\{ \sum_{k=n}^{\infty} c_k^2 \right\}^{1/2}.$$

Thus, by Corollary 2, *condition*

$$\sum_{n=0}^{\infty} \frac{p_n}{P_n} E_n^{(2)}(f) < \infty \tag{2.3}$$

also implies the $|R, P_n, 1|$ *summability of* (1.3) *for every orthonormal system* $\{\varphi_n\}$ *almost everywhere.*

If $\{\varphi_n\}$ is the trigonometric system, i.e., if we consider the following orthogonal series

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x), \tag{2.4}$$

then using Corollary 2 and the following estimation (see [2], Hilfssatz II)

$$E_n^{(2)}(f) \leqq w_n^{(2)}\left(\frac{1}{n}, f\right),$$

$$w_2^{(2)}(\delta, f) := \left\{ \frac{1}{\delta} \int_0^{\delta} \left(\int_0^{2\pi} [f(x+2t) + f(x-2t) - 2f(x)]^2 \, dx \right) dt \right\}^{1/2},$$

we also have a further

Corollary 3. *If*

$$\sum_{n=0}^{\infty} \frac{p_n}{P_n} w_2^{(2)}\left(\frac{1}{n}, f\right) < \infty \tag{2.5}$$

then series (2.4) *is summable* $|R, P_n, 1|$ *almost everywhere.*

The next theorem is the analogue of Theorem C.

Theorem 2. *If the series*

$$\sum_{n=1}^{\infty} p_n \left\{ \sum_{k=n}^{\infty} (a_k^2 + b_k^2) P_k^{-2} \right\}^{1/2} \tag{2.6}$$

converges, then all series of (1.8) *are summable* $|R, P_n, 1|$ *almost everywhere, and if series* (2.6) *diverges, then almost all series of* (1.8) *are non-summable* $|R, P_n, 1|$ *almost everywhere.*

3. In order to prove our theorems we require the following lemmas.

Lemma 1 ([3]). *Suppose that the set of points for which the Rademacher series* $\sum_{n=0}^{\infty} c_n r_n(x)$ *is summable* $|R, P_n, 1|$ *is of positive measure, then condition* (1.5) *holds.*

Lemma 2. *Let*

$$A_n(x) = \varrho_n \cos(nx + Q_n) \quad \text{with} \quad \varrho_n = (a_n^2 + b_n^2)^{1/2}.$$

If the series

$$\sum_{n=1}^{\infty} p_n \left\{ \sum_{k=n}^{\infty} A_k^2(x) P_k^{-2} \right\}^{1/2} \tag{3.1}$$

converges on a set E_0 *of positive measure, then the series*

$$\sum_{n=1}^{\infty} p_n \left\{ \sum_{k=n}^{\infty} \varrho_k^2 P_k^{-2} \right\}^{1/2} \tag{3.2}$$

converges. Conversely, the convergence of (3.2) *implies that of* (3.1) *for every* x.

The proof of Lemma 2 follows the same line as that of an analogous lemma of Y. Okuyama and T. Tsuchikura [4].

Proof. First we prove the implication (3.1)⇒(3.2). By the assumption there exists a set $E \subset E_0$ of positive measure such that

$$I \equiv \sum_{n=1}^{\infty} p_n \int_E \left\{ \sum_{k=n}^{\infty} P_k^{-2} \varrho_k^2 \cos^2(kx + Q_k) \right\}^{1/2} dx \leqq K\mu(E), \tag{3.3}$$

where K denotes a positive constant and $\mu(E)$ denotes the Lebesgue measure of

E. Using the Minkowski inequality with $p=1/2$, we obtain that

$$(3.4)\quad \begin{aligned} I &\geqq \sum_{n=1}^{\infty} p_n \Big\{\sum_{k=n}^{\infty}\Big(\int_E P_k^{-1}\varrho_k\,|\cos(kx+Q_k)|\,dx\Big)^2\Big\}^{1/2} = \\ &= \sum_{n=1}^{\infty} p_n \Big\{\sum_{k=n}^{\infty} P_k^{-2}\varrho_k^2\Big(\int_E |\cos(kx+Q_k)|\,dx\Big)^2\Big\}^{1/2}. \end{aligned}$$

Using the Riemann—Lebesgue theorem and the following estimation

$$\int_E |\cos(kx+Q_k)|\,dx \geqq \int_E \cos^2(kx+Q_k)\,dx = \frac{1}{2}\int_E (1+\cos 2(kx+Q_k))\,dx =$$

$$= \frac{1}{2}\mu(E)+\frac{1}{2}\int_E \cos 2(kx+Q_k)\,dx$$

we obtain that for sufficiently large $k\geqq k_0$

$$(3.5)\quad \int_E |\cos(kx+Q_k)|\,dx \geqq \frac{1}{4}\mu(E) \equiv A.$$

Thus, by (3.4) and (3.5), we have that

$$(3.6)\quad I \geqq A \sum_{n=k_0}^{\infty} p_n \Big\{\sum_{k=n}^{\infty} P_k^{-2}\varrho_k^2\Big\}^{1/2},$$

whence

$$\sum_{k=1}^{\infty} P_k^{-2}\varrho_k^2 < \infty$$

follows obviously, and this implies that

$$(3.7)\quad \sum_{n=0}^{k_0-1} p_n \Big\{\sum_{k=n}^{\infty} P_k^{-2}\varrho_k^2\Big\}^{1/2} < \infty.$$

Summing up, by (3.3), (3.6) and (3.7), the implication (3.1)⇒(3.2) is proved.

Since $A_n^2(x)\leqq\varrho_n^2$, the implication (3.2)⇒(3.1) is trivial. Thus the proof is completed.

4. Now we can start the proofs of the theorems.

Proof of Theorem 1. First we prove that condition (1.5) implies (2.1). An elementary calculation shows that

$$(4.1)\quad \begin{aligned} &\sum_{k=\nu_0+1}^{\infty} p_k \Big\{\sum_{n=k}^{\infty} P_n^{-2}c_n^2\Big\}^{1/2} \leqq \sum_{m=0}^{\infty}\sum_{k=\nu_m+1}^{\nu_{m+1}} p_k \Big\{\sum_{n=k}^{\infty} P_n^{-2}c_n^2\Big\}^{1/2} \leqq \\ &\leqq \sum_{m=0}^{\infty}\sum_{k=\nu_m+1}^{\nu_{m+1}} p_k \Big\{\sum_{n=\nu_m+1}^{\infty} P_n^{-2}c_n^2\Big\}^{1/2} \leqq \sum_{m=0}^{\infty}\sum_{k=\nu_m+1}^{\nu_{m+1}} p_k \sum_{i=m}^{\infty} P_{\nu_i+1}^{-1}\Big\{\sum_{n=\nu_i+1}^{\nu_{i+1}} c_n^2\Big\}^{1/2} \equiv \sum\nolimits_1. \end{aligned}$$

Since

$$P_{\nu_i+1} = P(\nu_i+1) \geqq P(\Lambda(2^i)) = 2^i, \tag{4.2}$$

thus

$$\sum\nolimits_1 \leqq \sum_{m=0}^{\infty} \sum_{k=\nu_m+1}^{\nu_{m+1}} p_k \sum_{i=m}^{\infty} 2^{-i} C_i = \sum_{i=0}^{\infty} 2^{-i} C_i \sum_{m=0}^{i} \sum_{k=\nu_m+1}^{\nu_{m+1}} p_k \leqq$$

$$\leqq \sum_{i=0}^{\infty} 2^{-i} C_i \sum_{k=0}^{\nu_{i+1}} p_k \leqq \sum_{i=0}^{\infty} 2^{-i} C_i P(\Lambda(2^{i+1})) \leqq 2 \sum_{i=0}^{\infty} C_i. \tag{4.3}$$

By (4.1) and (4.3) the implication (1.5)⇒(2.1) is proved.

Next we prove the converse implication. It is clear that

$$P_{\nu_m} \leqq P(\Lambda(2^m)) = 2^m, \tag{4.4}$$

thus, by (4.2) and (4.4), we have that

$$\left(\sum_{k=\nu_{m-1}+1}^{\nu_m+1} p_k\right) P_{\nu_{m+1}}^{-1} = (P_{\nu_m+1} - P_{\nu_{m-1}}) P_{\nu_{m+1}}^{-1} \geqq (2^m - 2^{m-1}) 2^{-m-1} = \frac{1}{4}.$$

Using this inequality we obtain that

$$\sum_{m=1}^{\infty} C_m \leqq 4 \sum_{m=1}^{\infty} \left(\sum_{k=\nu_{m-1}+1}^{\nu_m+1} p_k\right) P_{\nu_{m+1}}^{-1} C_m \leqq$$

$$\leqq 4 \sum_{m=1}^{\infty} \sum_{n=\nu_{m-1}+1}^{\nu_{m+1}} p_k \left\{\sum_{n=\nu_m+1}^{\nu_{m+1}} P_n^{-2} c_n^2\right\}^{1/2} \equiv \sum\nolimits_2, \tag{4.5}$$

where

$$C_m(p) := \left\{\sum_{n=\nu_m+1}^{\nu_{m+1}} P_n^{-2} c_n^2\right\}^{1/2}$$

means zero if $\nu_m = \nu_{m+1}$. Therefore

$$\sum\nolimits_2 = 4 \sum_m{}' \sum_{n=\nu_{m-1}+1}^{\nu_m+1} p_k C_m(p), \tag{4.6}$$

where $\sum_m{}'$ denotes that the summation runs just through such indices m which have the property $\nu_{m+1} \geqq \nu_m + 1$. Then

$$\sum_m{}' \sum_{k=\nu_{m-1}+1}^{\nu_m+1} p_k C_m(p) \leqq \sum_m{}' \sum_{k=\nu_{m-1}+1}^{\nu_m+1} p_k \left\{\sum_{n=k}^{\infty} P_n^{-2} c_n^2\right\}^{1/2} \leqq$$

$$\leqq \sum_m{}' \sum_{k=\nu_{m-1}+1}^{\nu_{m+1}} p_k \left\{\sum_{n=k}^{\infty} P_n^{-2} c_n^2\right\}^{1/2} \leqq \tag{4.7}$$

$$\leqq 2 \sum_{m=1}^{\infty} \sum_{k=\nu_{m-1}+1}^{\nu_m} p_k \left\{\sum_{n=k}^{\infty} P_n^{-2} c_n^2\right\}^{1/2} \leqq 2 \sum_{k=0}^{\infty} p_k \left\{\sum_{n=k}^{\infty} P_n^{-2} c_n^2\right\}^{1/2}.$$

By (4.5), (4.6) and (4.7) we have

$$\sum_{m=1}^{\infty} C_m \leqq 8 \sum_{k=0}^{\infty} p_k \left\{ \sum_{n=k}^{\infty} P_n^{-2} c_n^2 \right\}^{1/2},$$

which proves the implication (2.1)⇒(1.5), and this completes the proof of Theorem 1.

Proof of Theorem 2. The proof is the same as that of Theorem C, the only difference is that we use Theorem 1 and Lemma 2 instead of Theorem B and Lemma 2 of [4].

The sketch of the proof is the following: By Lemmas 1 and 2 and Theorem 1 we have to follow the Paley and Zygmund argument (cf. [6, p. 214]).

References

[1] L. Leindler, Über die absolute Summierbarkeit der Orthogonalreihen, *Acta Sci. Math.*, **22** (1961), 243—268.

[2] L. Leindler, Über verschiedene Konvergenzarten trigonometrischer Reihen, *Acta Sci. Math.*, **25** (1964), 233—249.

[3] F. Móricz, Über die Rieszsche Summation der Orthogonalreihen, *Acta Sci. Math.*, **23** (1962), 92—95.

[4] Y. Okuyama and T. Tsuchikura, On the absolute Riesz summability of orthogonal series, *Analysis Math.*, 7 (1981), 199—208.

[5] K. Tandori, Über die orthogonalen Funktionen. IX. Absolute Summation, *Acta Sci. Math.*, **21** (1960), 292—299.

[6] A. Zygmund, *Trigonometric series*. I, University Press (Cambridge, 1959).

ATTILA JÓZSEF UNIVERSITY
BOLYAI INSTITUTE
ARADI VÉRTANÚK TERE 1
6720 SZEGED, HUNGARY