Approximation in $L^1$ by Kantorovich polynomials

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1.

This paper is a continuation of two earlier ones [11, 12]. Let

$$K_n(f; x) = \sum_{k=0}^{n} \left( (n+1)^{(k+1)/(n+1)} \int_{k/(n+1)}^{(k+1)/(n+1)} f(u) \, du \right) b_{n,k}(x), \quad b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$$

be the Kantorovich-variant of the Bernstein operator. A series of papers contains results for the approximation properties of $K_n(f)$ in integral metrics (for references see the survey article [3]). However, the analogue of the well-known equivalence theorem of BERENS and LORENTZ [5] or that of LORENTZ and SCHUMAKER [7] and DITZIAN [6] is not known for them. The problem is the characterization of $\|K_n(f) - f\|_{L^1(0,1)} = O(n^{-a})$ $(0 < a < 1)$ in terms of a certain modulus of smoothness, and the aim of this paper is to give this characterization.

For $f \in L^p(0, 1)$, $p > 1$ we proved in [12]

**Theorem A.** If $1 < p < \infty$, $0 < a < 1$ and $f \in L^p(0, 1)$ then

(i) $\|K_n(f) - f\|_{L^p} = O(n^{-a})$

and

(ii) $(\alpha) \|A^*_h f(\cdot; x)\|_{L^p(0,1-h)} = O(h^a),$

$(\beta) \|f(\cdot + h) - f(\cdot)\|_{L^p(0,1-h)} = O(h^a)$

are equivalent.

Here

$$A^*_h f(\cdot; x) = f(x-h) - 2f(x) + f(x+h)$$

(we deviate from the custom and write $\|f(x)\|_{L^p}$ instead of $\|f(\cdot)\|_{L^p}$ if the former is more suggestive).
For the saturation case \( \alpha = 1 \) we have (see \([9, 10, 4, 12]\))

**Theorem B.** If \( 1 < p < \infty \) and \( f \in L^p(0, 1) \) then the following are equivalent:

1. \( \|K_n(f) - f\|_{L^p} = O(n^{-1}) \),
2. \( f \) has an absolutely continuous derivative with \( x(1-x)f''(x) \in L^p(0, 1) \),
3. \( \left\| x(1-x)\Delta^*_n(F; x) \right\|_{L^p(b, 1-b)} = O(h^2) \),
4. \( \left\| x(1-x)\Delta^*_n(f; x) \right\|_{L^p(b, 1-b)} = O(h^2) \),
5. \( \|\Delta^*_n(x(1-x)F; x)\|_{L^p(b, 1-b)} = O(h^2) \).

Here \( F(x) = \int_0^x f(u) \, du \) and naturally (ii) means that "\( f \) coincides a.e. with a function which has absolutely continuous derivative".

Turning to \( L^1 \) let us mention the saturation result (see \([8, 2]\)):

**Theorem C.** For \( f \in L^1(0, 1) \) the following conditions are equivalent:

1. \( \|K_n(f) - f\|_{L^1} = O(n^{-1}) \),
2. \( f \) is absolutely continuous and \( x(1-x)f'(x) \) is of bounded variation,
3. \( \left\| x(1-x)\Delta^*_n(F; x) \right\|_{BV + L^\infty(b, 1-b)} = O(h^2) \).

Here \( BV + L^\infty \) denotes the sum of the two norms: total variation and ess. supremum.

Examples show that Theorem B does not hold for \( L^1 \), i.e., the \( BV \)-norm in Theorem C seems to be the appropriate one and we cannot hope in replacing it by an \( L^1 \)-norm.

The difference between Theorems B and C suggests also that we should exchange the \( L^p \)-norm in Theorem A for a \( BV \)-norm or something like that to obtain a correct result in \( L^1 \) (see also the conjecture in \([3]\)). Thus, it is rather surprising that Theorem A holds word for word when \( p = 1 \):

**Theorem 1.** If \( 0 < \alpha < 1 \) and \( f \in L^1(0, 1) \) then

1. \( \|K_n(f) - f\|_{L^1} = O(n^{-\alpha}) \) and
2. \( \|\Delta^*_n(x(1-x)F; x)\|_{L^1(b, 1-b)} = O(h^2) \),
3. \( \|\Delta^*_n(x(1-x)F; x)\|_{L^1(b, 1-b)} = O(h^2) \).

are equivalent.

Let us mention that although (ii)\(\Rightarrow\) (i) holds also for \( \alpha = 1 \), neither (ii) (a), nor (ii) (β) is necessary for (i) in the case \( \alpha = 1 \). This is shown by the function \( f(x) = \log x \) (\( x \in (0, 1) \)).

The first result with the modulus of smoothness \( \sup_{0 < h < \delta} \|\Delta^*_n(x(1-x); f, x)\|_{L^p(b, 1-b)} \) (more precisely, with its analogue) was proved in \([11]\) for the Szász—Kantorovich operators:

\[
M_n(f; x) = \sum_{k=0}^{\infty} \left( n \int_{k/n}^{(k+1)/n} f(u) \, du \right) p_{n,k}(x), \quad p_{n,k}(x) = e^{-nx} \frac{\left( nx \right)^k}{k!}, \quad x \geq 0.
\]
Theorem D. For $1 < p < \infty$, $0 < \alpha < 1$ and $f \in L^p(0, \infty)$ the following conditions are equivalent:

(i) $\|M_n(f) - f\|_{L^p(0, \infty)} = O(n^{-\alpha})$,

(ii) $\|A_n^{\alpha} f(x)\|_{L^p(h, \infty)} = O(h^{2\alpha})$,

(\beta) $\|f(\cdot + h) - f(\cdot)\|_{L^p(0, \infty)} = O(h^\alpha)$.

This is true just as well for $p = 1$:

Theorem 2. Theorem D holds also when $p = 1$.

We shall prove only Theorem 2, but our method works also for $K_n$ (the technical details are somewhat easier for $M_n$); we refer to [12] for the necessary changes in the proof (observe that [12] relates to [11] about as Theorem 1 relates to Theorem 2). The only point in our proof which might not be obvious for $K_n$ is the delicate formula (2.5) but the analogue of this was given in [12, (4.5)].

Although Theorems A and 1 (D and 2) have the same form, here we have to use a different method since in the case $p > 1$ the proof rested heavily on the maximal inequality. Nevertheless, the roots of the proofs of the inverse parts are the same: the so called elementary method of inverse results developed by BERENS and LORENTZ [5], and BECKER and NESSEL [1].

2. Proof of Theorem 2

1. Proof of (ii) $\Rightarrow$ (i). First we derive from (ii) three further inequalities.

Inequality 1.

$$\int_{h/2}^h \int_0^h |f(x) - f(y)| \, dx \, dy = 2 \int_0^h de \int_0^{h-e} |f(x+e) - f(x)| \, dx \equiv K \int_0^h e^\alpha \, de \equiv K h^{\alpha + 1}.$$

Inequality 2.

$$A(f, h) \stackrel{\text{def}}{=} \frac{1}{h} \int_0^h |f(x + \tau) - f(x)| \, d\tau \quad \equiv K h^{2\alpha} \quad (h \equiv 0).$$

Proof. For any $f \in L^1(0, \infty)$,

$$\int_{h/2}^h \int_0^{h/2} |f(x + \tau)| \, d\tau \, dx \equiv K h^{-2} \int_0^{2h^2} |f(x + u)| \, du \, dx +$$

$$+ K \int_0^{\infty} |f(u)| \frac{h \sqrt{u}}{u} \, du \equiv K \|f\|_{L^1}.$$
and if $f$ is absolutely continuous with $f' \in L_1$ then

$$A(f, h) = \int_{h}^{\infty} \frac{1}{x} \int_{0}^{h} \left| f'(x+u) \right| du \, dx = \int_{h}^{\infty} \frac{1}{x} \int_{0}^{h} \left| f'(x+u) \right| du \, dx \equiv K \| f' \|_{L^1}$$

Let now $f \in L^1$ be arbitrary for which (ii) (β) holds, and let

$$g_h(x) = \frac{1}{h^2} \int_{0}^{h} f(x+\tau) \, d\tau.$$ 

For this

$$\| f - g_h \|_{L^1} \leq h^{-2} \int_{0}^{h} \| f(\cdot + \tau) - f(\cdot) \|_{L^1} \, d\tau \leq K h^{-2} \int_{0}^{h} \tau^2 \, d\tau = K h^2 \| f \|_{L^1}$$

and

$$\| g_h' \|_{L^1} = h^{-2} \| f(\cdot + h^2) - f(\cdot) \|_{L^1} \leq K h^{2a-2}$$

by which

$$A(f, h) \equiv A(f - g_h, h) + A(g_h, h) \equiv K \| f - g_h \|_{L^1} + h^2 \| g_h' \|_{L^1} \equiv K h^{2a}.$$ 

**Inequality 3.**

$$\left\| \frac{1}{h^{\frac{1}{a}}} \int_{0}^{h} |A^*_u(f; x)| \, d\tau \right\|_{L^1(h^a, \infty)} = \left\| \frac{1}{h} \int_{0}^{h} |A^*_u(f; x)| \, du \right\|_{L^1(h^a, \infty)} \equiv \frac{1}{h} \int_{0}^{h} \| A^*_u(f; x) \|_{L^1(h^a, \infty)} \, du \leq K \frac{1}{h} \int_{0}^{h} u^{2a} \, du \leq K h^{2a}.$$ 

Now the analogous inequalities for $L^p$ were the only tools used at the proof of (ii)⇒(i) in [11, Theorem 1], and this proof equally holds, using Inequalities 1—3, for $p = 1$. For the details see [11].

**II. Proof of (i)⇒(ii) (β).** Let

$$v(f; \delta) = v(f) = \sup_{0 \leq h \leq \delta} \| f(\cdot + h) - f(\cdot) \|_{L^1(0, \infty)}.$$ 

It is sufficient to prove that for $0 < h \leq 1$, $n \geq 1$,

$$v(h) \leq K \left( n^{-a} + nhv \left( \frac{1}{n} \right) \right);$$

see [1, Lemma 2.1].

But

$$v(f; h) \equiv v(f - M_n(f); h) + v(M_n(f); h)$$
and here, by (i),

\[ v(f - M_n(f); h) \leq 2 \| f - M_n(f) \|_{L^1} \equiv Kn^{-\alpha}. \]

By

\[ M_n'(f; x) = n \sum_{k=0}^{\infty} \left( n \int_0^{1/n} \left( f \left( \frac{k}{n} + u \right) - f \left( \frac{k+1}{n} + u \right) \right) du \right) p_{n,k}(x) \]

we have

\[ v(M_n(f); h) \leq \int_0^\infty dx \int_0^h |M_n'(f; x+u)| du \equiv \int |M_n'(f)|_{L^1} du \equiv \]

\[ \equiv hn \sum_{k=0}^{\infty} \int_0^{1/n} \left| f \left( \frac{k}{n} + u \right) - f \left( \frac{k+1}{n} + u \right) \right| du \int_0^{\infty} n p_{n,k}(x) dx = \]

\[ = hn \sum_{k=0}^{\infty} \int_0^{1/n} \left| f \left( \frac{k}{n} + u \right) - f \left( \frac{k+1}{n} + u \right) \right| du = hn \left\| f \left( \cdot + \frac{1}{n} \right) - f(\cdot) \right\|_{L^1} \equiv hn\nu \left( \frac{1}{n} \right), \]

and the proof is complete.

For later application let us prove also the inequality

\[ (2.1) \quad I(f; \delta) = \left| \frac{\delta}{\sqrt{\pi}} \left( f(x + \delta \sqrt{x}) - f(x - \delta \sqrt{x}) \right) \right|_{L^1(\delta^2, \infty)} \equiv K\delta^{2\alpha}. \]

In fact, for the function

\[ g_\delta(x) = \frac{1}{\delta^2} \int_0^{\delta^2} f(x + u) du \]

we have proved above

\[ \| f - g_\delta \|_{L^1} \equiv \delta^{-2} \int_0^{\delta^2} \| f(\cdot + u) - f(\cdot) \|_{L^1} du \equiv K\delta^{2\alpha} \]

and

\[ \| g_\delta' \|_{L^1} \equiv \delta^{-2} \| f(\cdot + \delta^2) - f(\cdot) \|_{L^1} \equiv K\delta^{2\alpha-2}, \]

by which

\[ I(f; \delta) \equiv I(f - g_\delta; \delta) + I(g_\delta; \delta) \equiv \]

\[ \equiv \| f - g_\delta(x + \delta \sqrt{x}) \|_{L^1(\delta^2, \infty)} + \| (f - g_\delta)(x - \delta \sqrt{x}) \|_{L^1(\delta^2, \infty)} + \]

\[ + \int_{\delta}^{\delta^2} \frac{\delta}{\sqrt{\pi}} \left( \frac{\sqrt{x}}{\delta^2} \right) \int_{\delta^2}^{\delta^2} |g_\delta'(x + u)| du dx \equiv K\delta^{2\alpha} + \int_{-\delta}^{\delta} \delta \int_{-\delta}^{\delta} |g_\delta'(x + u)\sqrt{x}| du dx \equiv \]

\[ \equiv K\delta^{2\alpha} + K\delta \int_{-\delta}^{\delta} \| g_\delta' \|_{L^1} \equiv K(\delta^{2\alpha} + \delta^2 \| g_\delta \|_{L^1}) \equiv K\delta^{2\alpha}. \]

III. Proof of (i)⇒(ii) (α). First let us prove the following
Lemma. Let $0 < h \leq 1$, $h^2 \leq n^{-1} \leq h$, $k = 0, 1, 2, \ldots$. Then there is an absolute constant $K$ for which

$$
(1) \int_{h^2}^{\infty} dx \int_{-h^2}^{h^2} p_{n,k}(x+u+v) \, du \, dv \leq K \frac{h^2(k+1)}{n^2},
$$

$$
(2) \int_{h^2}^{\infty} dx \int_{-h^2}^{h^2} \frac{k}{(x+u+v)^2} p_{n,k}(x+u+v) \, du \, dv \leq K h^2,
$$

$$
(3) \int_{h^2}^{\infty} dx \int_{-h^2}^{h^2} \frac{1}{(x+u+v)^2} p_{n,k}(x+u+v) \, du \, dv \leq K \frac{h^2}{n^3}.
$$

Proof. $p_{n,k}(x)$ increases on $(0, k/n)$ and decreases on $(k/n, \infty)$, hence

$$
\int_{-h^2}^{h^2} p_{n,k}(x+u+v) \, du \, dv \leq \begin{cases} h^2 x p_{n,k}(x+h \sqrt{x}) & \text{for } x \in (0, k/n-h \sqrt{k/n}), \\ h^2 x \max_y p_{n,k}(y) & \text{for } x \in (k/n-h \sqrt{k/n}, k/n+2h \sqrt{k/n}), \\ h^2 x p_{n,k}(x-h \sqrt{x}) & \text{for } x \in (k/n+2h \sqrt{k/n}, \infty). \end{cases}
$$

Since

$$
\int_{h^2}^{\infty} \left| g(x \pm h \sqrt{x}) \right| dx \leq 2 \int_{0}^{\infty} g(x) \, dx, \quad xp_{n,k}(x) = \frac{k+1}{n} p_{n,k+1}(x),
$$

and $\max_y p_{n,k}(y) = p_{n,k}(k/n) \leq K/\sqrt{k+1}$ (use Stirling's formula), we obtain easily

$$
\int_{h^2}^{\infty} dx \int_{-h^2}^{h^2} p_{n,k}(x+u+v) \, du \, dv \leq K \left( \frac{(k+1)h^2}{n} \right) \|p_{n,k+1}\|_{L^1} + h^2 \frac{k}{n} \left( \frac{1}{\sqrt{k+1}} \right) \leq K \left( \frac{h^2(k+1)}{n^2} \right). \quad (k = 0, 1, 2, \ldots).
$$

For $k \geq 2$ inequality (2) follows from (1), since $k x^{-\sigma} p_{n,k}(x) = (n^2(k-1)) p_{n,k-2}(x)$. For $k = 1$ we have

$$
\int_{h^2}^{\infty} dx \int_{-h^2}^{h^2} n(x+u+v)^{-1} e^{-n(x+u+v)} \, du \, dv = n \int_{h^2}^{\infty} dx \int_{-h^2}^{h^2} h \sqrt{x-|\tau|} e^{-n(x+\tau)} \, d\tau \leq \int_{-h^2}^{h^2} e^{-n(x-v)} \, d\tau \int_{-h^2}^{h^2} h \sqrt{x-|\tau|} \, d\tau \equiv 2n \int_{h^2}^{\infty} \left( \frac{h}{\sqrt{x}} \right) \int_{-h^2}^{h^2} e^{-n(x-\tau)} \, d\tau \, dx \equiv K h^2.
$$
Finally, (3) follows from (1) for \( k=0 \), and for \( k \geq 1 \) we have

\[
\int_{\mathbb{R}^2} dx \int_{-h \sqrt{n}}^{h \sqrt{n}} \frac{(k/n - (x+u+v))^2}{(x+u+v)^2} p_{n,k}(x+u+v) \, du \, dv \leq K \int_{\mathbb{R}^2} dx \int_{-h \sqrt{n}}^{h \sqrt{n}} \left( \frac{k+1}{n(x+u+v)} \right)^2 p_{n,k}(x+u+v) \, du \, dv +
\]

\[
+ K \int_{\mathbb{R}^2} dx \int_{-h \sqrt{n}}^{h \sqrt{n}} \left( \frac{k}{n} - (x+u+v) \right)^2 p_{n,k}(x+u+v) \, du \, dv.
\]

Here the first term is at most \( Kh^2/n^2 \) for \( k=1 \) (see (2)) and

\[
K \int_{\mathbb{R}^2} dx \int_{-h \sqrt{n}}^{h \sqrt{n}} p_{n,k-2}(x+u+v) \, du \, dv \equiv Kh^6 \equiv Kh^2/n^2
\]

for \( k \geq 2 \).

The second term can be estimated as we have done in inequality (1) (use that \((k/n - x)^2 p_{n,k}(x)\) increases on \((0, (k+1)/n - \sqrt{2k+1}/n)\) and decreases on \(((k+1)/n + \sqrt{2k+1}/n, \infty)\) together with the facts

\[
\int_{0}^{\infty} n \left( \frac{k}{n} - x \right)^2 p_{n,k}(x) \, dx = \frac{1}{n},
\]

\[
\max \left( \frac{k+1}{n} + \sqrt{2k+1}/n + 4h \sqrt{k+1}/n \right) \int_{\mathbb{R}^2} dx \int_{-h \sqrt{n}}^{h \sqrt{n}} \left( \frac{k}{n} - (x+u+v) \right)^2 p_{n,k}(x+u+v) \, du \, dv \equiv
\]

\[
K \frac{\sqrt{k}}{n} \left( \frac{k}{n} \right)^{-2} h^2 \frac{k}{n} \left( \sqrt{k}/n \right)^2 \frac{1}{\sqrt{k}} \equiv K \frac{h^2}{n^2}.
\]

Let us turn back to (ii) (a), and let

\[
\omega(f; \delta) = \omega(\delta) = \sup_{0 < h \leq \delta} \int_{\mathbb{R}^2} |A^x_{n}(f; x)| \, dx.
\]

It is sufficient to prove that for \( 0 < h^2 \leq 1/n \leq h \leq 1 \) we have

\[
\omega(h) \equiv K \left( n^{-\alpha} + h^2 n \left( n^{-\alpha} + \omega \left( \frac{1}{n} \right) \right) \right),
\]
see [1, Lemma 2.1]. Since (i) yields
\[ \|A^*_h f - M_n(f); x\|_{L^1(h^*, \infty)} \leq K \|f - M_n(f)\|_{L^1} \leq Kn^{-a}, \]
an easy consideration shows that it is enough to prove
\[ \|A^*_h f - M_n(f); x\|_{L^1(h^*, \infty)} \leq K n^{h^2} \left( n^{-a} + \omega \left( \frac{1}{\sqrt{n}} \right) \right) \left( h^2 \leq \frac{1}{n} \leq h \right). \]

Let
\[ \mathcal{M}_n(f; x) = \sum_{k=1}^{\infty} \left( \int_{k/n}^{(k+1)/n} f(u) \, du \right) p_{n,k}(x) = M_n(f; x) - ne^{-nx} \int_{0}^{1/n} f(u) \, du. \]

a) (ii) (\beta) (which we have proved above) gives
\[ \|\left( \int_{0}^{1/n} f(u) \, du \right) A^*_h (e^{-at}; x)\|_{L^1(h^*, \infty)} \leq K n^{h^2} \left( \int_{0}^{1/n} f(u) \, du \right) \leq K n^{h^2} n^{-a}. \]

b) Let \( F_1(x) = \int_{0}^{x} f(t) \, dt, \) \( F_2(x) = \int_{0}^{x} F_1(t) \, dt \) and \( f_\delta(x) = \frac{1}{\delta^2} \int_{-\delta}^{\delta} du \int_{-u}^{u} f(x + v \sqrt{\delta}) \, dv \)

\[ f_\delta(x) = \frac{1}{\delta^2} \int_{-\delta}^{\delta} du \int_{-u}^{u} f(x + v \sqrt{\delta}) + f(x - v \sqrt{\delta}) \, dv = \frac{1}{\delta^2} A^*_h f_\delta(x). \]

We have
\[ \|f - f_\delta\|_{L^1(\delta^2, \infty)} \leq \frac{1}{\delta^2} \int_{0}^{\delta} du \int_{0}^{u} \|A^*_h f; x\|_{L^1(\delta^2, \infty)} \, dv \leq \omega(\delta) \]
and
\[ f_\delta(x) = \frac{2}{\delta^2 x^3} A^*_h f_\delta(x) - \frac{2}{\delta^2 x^2} A^*_h (F_2; x) - \frac{5}{4\delta^2 x^{3/2}} (F_1(x + \delta \sqrt{x}) - F_1(x - \delta \sqrt{x})) + \frac{1}{\delta^2 x} A^*_h (f; x) \]
\[ + \frac{1}{\delta^2 x} (f(x + \delta \sqrt{x}) - f(x - \delta \sqrt{x}) + \frac{1}{4x^2} (f(x + \delta \sqrt{x}) + f(x - \delta \sqrt{x})). \]
and the key point in our theorem is that the latter is equal to

\[
\frac{f''(x)}{x^2} = \frac{2}{x^2} (f'(x) - f(x)) - \frac{5}{4x^2} \frac{1}{\delta} \int_0^\delta A^*(f; x) dt +
\]

\[
(2.5) \quad + \frac{1}{4x^2} A^*_\delta y_x(f; x) + \frac{1}{\delta^2 x} A^*_\delta y_x(f; x) + \frac{1}{\delta x^{3/2}} (f(x + \delta \sqrt{x}) - f(x - \delta \sqrt{x})) -
\]

\[
- \frac{2}{\delta x^{3/2}} \frac{1}{\delta} \int_0^\delta (f(x + t \sqrt{x}) - f(x - t \sqrt{x})) dt.
\]

Now

\[
\|A^*_\delta y_x(M_n(f); x)\|_{L^1(h^3, \infty)} \leq
\]

\[
\leq \|A^*_\delta y_x(M_n(f - f_{1/\sqrt{n}}); x)\|_{L^1(h^3, \infty)} + \|A^*_\delta y_x(M_n(f_{1/\sqrt{n}}); x)\|_{L^1(h^3, \infty)}
\]

and below we estimate the two terms on the right side separately.

c) Since

\[
(p_{n,k}(x))^\alpha = \frac{n^2}{x^2} \left[\left(\frac{k}{n} - x\right)^2 - \frac{k}{n^2}\right] p_{n,k}(x),
\]

we obtain by (2) and (3) from the Lemma, and by (2.4) that

\[
\int_{h^3} A^*_\delta y_x(M_n(f - f_{1/\sqrt{n}}); x) dx = \int_{h^3} \int_{-h^{y/2}}^{h^{y/2}} M_n(f - f_{1/\sqrt{n}}; x + u + v) du dv \leq
\]

\[
= \sum_{k=1}^{(k+1)/n} \left( n \int_{k/n}^{(k+1)/n} \left| f - f_1 \right| (u) du \right) \left( \int_{k/n}^{(k+1)/n} dx \int_{-h^{y/2}}^{h^{y/2}} M_n(f - f_{1/\sqrt{n}}; x + u + v) du dv \right) \leq
\]

\[
\times p_{n,k}(x + u + v) du dv + \int_{h^3} \int_{-h^{y/2}}^{h^{y/2}} \left( \frac{k}{n} - (x + u + v) \right)^2 \times
\]

\[
\times p_{n,k}(x + u + v) du dv \leq Kh^2 n \sum_{k=1}^{(k+1)/n} \left( f - f_{1/\sqrt{n}} \right) (u) du = Kh^2 n \left\| f - f_{1/\sqrt{n}} \right\|_{L^1(1/n, \infty)} \leq Kh^2 n \omega \left( \frac{1}{\sqrt{n}} \right).
\]

d) We have also

\[
(p_{n,k}(x))^\alpha = n^2(p_{n,k-2}(x) - 2p_{n,k-1}(x) + p_{n,k}(x)) \quad (k = 1, 2, ..., p_{n-1}(x) \equiv 0),
\]
Thus
\[
\int_{h^2}^{\infty} |A_{h} (f_{1} (x); \gamma_n) | \, dx =
\]
\[
n^2 \int_{h^2}^{\infty} dx \left\{ \sum_{k=1}^{\infty} \left[ n \int_{0}^{1/n} f\left( \frac{k}{n} + u + v + w \right) du \, dv \, dw \right] \right\} \int_{-h}^{h} \frac{\sqrt{2}}{\gamma_n} \int_{-h}^{h} p_{n,k} (x + s + t) \, ds \, dt +
\]
\[
+ \left( -2n \int_{1/n}^{2/n} f_{1} (u) du + n \int_{2/n}^{3/n} f_{1} (u) du \right) \int_{-h}^{h} \frac{\sqrt{2}}{\gamma_n} \int_{-h}^{h} p_{n,0} (x + s + t) \, ds \, dt \right\} \leq
\]
\[
\leq K \sum_{k=1}^{\infty} \left[ n \int_{0}^{1/n} f\left( \frac{k}{n} + u + v + w \right) du \, dv \, dw \right] \frac{h^2}{2} +
\]
\[
+ Knh^2 \left( \left| \int_{1/n}^{2/n} f_{1} (u) du \right| + \left| \int_{2/n}^{3/n} f_{1} (u) du \right| \right) \leq
\]
\[
\leq Knh^2 \sum_{k=1}^{\infty} k \int_{0}^{1/n} f\left( \frac{k}{n} + u + v + w \right) \, du \, dv \, dw +
\]
\[
+ Knh^2 \left( n^{-2} + \omega \left( \frac{1}{\sqrt{n}} \right) \right) = Knh^2 A + Knh^2 \left( n^{-2} + \omega \left( \frac{1}{\sqrt{n}} \right) \right),
\]
where
\[
A = \sum_{k=1}^{\infty} k \int_{0}^{1/n} f\left( \frac{k}{n} + u + v + w \right) \, du \, dv \, dw,
\]
and where we used that
\[
\left| \int_{1/n}^{2/n} f_{1} (u) du \right| + \left| \int_{2/n}^{3/n} f_{1} (u) du \right| \equiv \| f - f_{1} \|_{L^{1} \left( \frac{1}{n}, \infty \right)} +
\]
\[
+ \left| \int_{1/n}^{\infty} f(u) du - \int_{2/n}^{\infty} f(u) du \right| + \left| \int_{2/n}^{\infty} f(u) du - \int_{3/n}^{\infty} f(u) du \right| \equiv K \left( \omega \left( \frac{1}{\sqrt{n}} \right) + n^{-2} \right).
\]

To estimate $A$ we apply (2.5). Taking absolute value in (2.5) term by term we increase $|f_{1}'' (x)|$. Now the first term on the right of (2.5) contributes to $A$ at
most by
\[\sum_{k=1}^{\infty} k \int_0^{1/n} \int_0^{1/n} \frac{2}{\left(\frac{2}{n} + u + v + w\right)} \left| \int_0^{1/n} ds \int_0^{s} \frac{d}{t} \sqrt{f\left(\frac{k}{n} + u + v + w\right)} \right| dt \, du \, dv \, dw \leq\]
\[\equiv 2n \int_0^{1/n} ds \int_0^{s} dt \sum_{k=1}^{\infty} \frac{n^2}{k} \int_0^{1/n} \int_0^{s} \frac{d}{t} \sqrt{f\left(\frac{k}{n} + u + v + w\right)} \right| dt \, du \, dv \, dw \leq\]
\[\equiv Kn \int_0^{1/n} ds \int_0^{s} \left| \sqrt{A_t\sqrt{x}}(f; x)\right|_{L^1\left(\frac{1}{n}, \infty\right)} dt \leq Kn \left(\frac{1}{\sqrt{n}}\right).
\]
Quite similarly the contribution of the second, third and fourth terms to \(A\) is at most \(Kn \left(\frac{1}{\sqrt{n}}\right)\).

Using inequality (2.1), the fifth term contributes to \(A\) at most by
\[\sum_{k=1}^{\infty} k \int_0^{1/n} \int_0^{1/n} \sqrt{n} \left| f\left(\frac{k}{n} + u + v + w + \frac{1}{\sqrt{n}} \sqrt{\frac{k}{n} + u + v + w}\right) -
-f\left(\frac{k}{n} + u + v + w - \frac{1}{\sqrt{n}} \sqrt{\frac{k}{n} + u + v + w}\right) \right| du \, dv \, dw \leq\]
\[\equiv K \int_0^{1/n} \frac{1}{\sqrt{nx}} \left| f\left(x + \frac{1}{\sqrt{n}} \sqrt{x}\right) - f\left(x - \frac{1}{\sqrt{n}} \sqrt{x}\right) \right| dx \leq Kn^{-a},
and a similar estimate can be given for the contribution of the sixth term:
\[\sum_{k=1}^{\infty} k \int_0^{1/n} \int_0^{1/n} \frac{2}{\sqrt{n}} \left| \sqrt{n} \int_0^{1/n} \left| f\left(\frac{k}{n} + u + v + w + t \sqrt{\frac{k}{n} + u + v + w}\right) -
-f\left(\frac{k}{n} + u + v + w - t \sqrt{\frac{k}{n} + u + v + w}\right) \right| dt \right| du \, dv \, dw \leq\]
\[\equiv \sum_{k=1}^{1/n} \frac{1}{t} \left( \int_0^{1/n} \left| f\left(x + t \sqrt{x}\right) - f\left(x - t \sqrt{x}\right) \right| dx \right) dt \leq K \int_0^{1/n} t^{2a-1} dt \leq Kn^{-a}.
Collecting our estimates from a) to d) we obtain (2.2) by which the proof is complete.
References


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