L^r inequalities for Walsh series, $0<r<1$

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1. Introduction. Let $w_0, w_1, \ldots$ denote the *Walsh—Paley functions* (see [5]). Thus, for each integer $k \geqq 0$ and each point x belonging to the unit interval [0, 1], the identity

$$w_k(x) = \prod_{j=0}^{\infty} \exp(i\pi x_{j+1} k_j) \tag{1}$$

holds, where the numbers x_j and k_j are either 0 or 1 and come from the binary expansions of x and k:

$$x = \sum_{j=1}^{\infty} x_j 2^{-j}, \quad k = \sum_{j=0}^{\infty} k_j 2^j.$$

(When $x \in [0, 1)$ is a dyadic rational the finite binary expansion is used.)

Given any *Walsh series* $W = \sum_{k=1}^{\infty} a_k w_k$, denote its *n-th partial sums* by

$$W_n = \sum_{k=1}^{n-1} a_k w_k,$$

its *n-th partial Cesaro sums* by

$$\sigma_n = \sum_{k=1}^{n-1} \left(1 - \frac{k}{n}\right) a_k w_k,$$

and its *n-th layer* by

$$\Delta_n = \sum_{k=2^{n-1}}^{2^n - 1} a_k w_k,$$

for $n = 1, 2, \ldots$. Notice that the Walsh series W has no constant term, and thus that $W = \sum_{n=1}^{\infty} \Delta_n$. This has been done for convenience to avoid writing a separate constant term in each of the inequalities derived below. It does not affect the generality of our results.

In Section 2 a basic inequality is derived which is a Walsh series analogue for

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L^r norms, $0<r<1$, of a trigonometric result for L^p norms, $1<p<\infty$, due to MARCINKIEWICZ [7]. In Section 3 we shall apply this basic inequality to estimate the L^r norms of the following three series:

(2) $$S_1 = \left(\sum_{k=1}^{\infty} \frac{(W_k-\sigma_k)^2}{k}\right)^{1/2},$$

(3) $$S_2 = \left(\sum_{k=1}^{\infty} (W_{2^k}-\sigma_{2^k})^2\right)^{1/2},$$

and

(4) $$S_3 = \left(\sum_{n=1}^{\infty} \Delta_n^2\right)^{1/2}.$$

The results of Section 3 are summarized as follows.

Theorem. *Let* $0<r<1$. *There is an absolute constant* α_r *depending only on* r *such that given any Walsh series* W *the following three inequalities hold:*

(5) $$\|S_1\|_{L^r} \leqq \alpha_r \|S_2\|_{L^1},$$

(6) $$\|S_2\|_{L^r} \leqq \alpha_r \|S_3\|_{L^1},$$

and

(7) $$\|S_3\|_{L^r} \leqq \alpha_r \|S_1\|_{L^1}.$$

In the case that W is a trigonometric series and $1<r<\infty$, the theorem above was obtained by ZYGMUND [13]. SUNOUCHI [11] used Zygmund's techniques to show that for Walsh series, the L^p norms of the series S_1, S_2, and S_3 are equivalent, for each $1<p<\infty$.

In Section 4 we apply the theorem above to obtain some inequalities relating a Walsh series to its term by term dyadic derivative. The surprising thing is that under suitable hypotheses, there is a direct relationship between the H^r norm of a function f and the growth of the partial sums of the formal dyadic derivative of the Walsh series representing f.

It should be pointed out that if the series S_1, S_2, and S_3 are replaced by appropriate maximal functions, then equivalence in L^r norms, $0<r<1$, can be restored. In connection with this remark see BURKHOLDER and GUNDY [2], especially Section 5. We do not proceed in this manner because the maximal function form of the theorem above proves intractable for studying the term by term dyadic derivative.

2. The basic inequality. Given a Walsh series W, denote its *maximal function* by

$$W^* = \sup_{n>0} |W_{2^n}|.$$

BURKHOLDER and GUNDY [2] have shown that given $0<r<\infty$ there exist constants

a_r and A_r depending only on r such that

$$a_r \|W^*\|_{L^r} \leqq \left\| \left(\sum_{n=1}^{\infty} \Delta_n^2 \right)^{1/2} \right\|_{L^r} \leqq A_r \|W^*\|_{L^r} \tag{8}$$

holds for all Walsh series W.

Given any function f, integrable over the interval $[0, 1]$, denote its *Walsh—Fourier series* by $W[f]$. Denote the partial Cesaro sums of $W[f]$ by $\sigma_n[f]$ and the n-th layer of $W[f]$ by $\Delta_n[f]$ $n = 1, 2, \ldots$. It is well-known (see [5]) that if $\dotplus$ represents dyadic addition then

$$\Delta_n[f, x] = \int_0^1 f(t) \left(\sum_{k=2^{n-1}}^{2^n-1} w_k(x \dotplus t) \right) dt \tag{9}$$

for $x \in [0, 1]$ and $n = 1, 2, \ldots$.

Our main goal in this section is to sketch a proof of the following inequality.

Lemma. *Let $0<r<1$ and suppose that $p_1, p_2, \ldots$ is a sequence of integers which diverges to $+\infty$. There exists a constant β_r, depending only on r, such that*

$$\left\{ \int_0^1 \left(\sum_{n=1}^{\infty} W_{p_n}^2[f_n, x] \right)^{r/2} dx \right\}^{1/r} \leqq \beta_r \int_0^1 \left(\sum_{n=1}^{\infty} f_n^2(x) \right)^{1/2} dx$$

holds for any sequence $f_1, f_2, \ldots$ of functions which belong to $L^1[0, 1]$.

To prove this lemma set $\varphi(x) = \left(\sum_{n=1}^{\infty} f_n^2(x) \right)^{1/2}$, for $x \in [0, 1]$, and assume without loss of generality that $\varphi \in L^1[0, 1]$. Let $r_1, r_2, \ldots,$ denote the *Rademacher functions,* i.e., $r_n = w_{2^{n-1}}$ for $n = 1, 2, \ldots,$ and consider the series

$$F(x, y) = \sum_{n=1}^{\infty} r_n(y) f_n(x), \quad x, y \in [0, 1].$$

We claim that the assumption $\varphi \in L^1(0, 1]$ guarantees that for a.e. $y \in [0, 1]$ some subsequence of the series $F(x, y)$ converges in the $L^1(dx)$ norm. In fact, according to Khinchin's inequality there exist constants b_r and B_r, for $0<r<\infty$, such that

$$b_r \|\varphi\|_{L^r}^r \leqq \int_0^1 \int_0^1 |F(x, y)|^r \, dx \, dy \leqq B_r \|\varphi\|_{L^r}^r \tag{10}$$

In particular, for $r=1$ we have that

$$\int_0^1 \int_0^1 \left| \sum_{k=n}^{m} r_k(y) f_k(x) \right| dx \, dy \leqq B_1 \int_0^1 \left(\sum_{k=n}^{m} f_k^2(x) \right)^{1/2} dx. \tag{11}$$

By the Lebesgue dominated convergence theorem the left-hand-side of (11) converges to zero, as $n, m \to \infty$. Therefore $F(x, y)$ converges in the L^1 norm on the unit square $[0, 1]\times[0, 1]$. In particular, for a.e. $y\in[0, 1]$ there exists a subsequence $n_1<n_2<\dots$ such that $\sum_{n=1}^{n_j} r_n(y)f_n\to F(\cdot, y)$ in $L^1[0, 1]$ norm, as $j\to\infty$, and the claim is established. It follows from (9) that

$$\Delta_k(F(\cdot, y)) = \sum_{n=1}^{\infty} r_n(y)\Delta_k(f_n) \tag{12}$$

holds for a.e. $y\in[0, 1]$ and for all $k\geqq 1$.

Next, we show that there exist constants c_r and C_r, depending only on r, such that

$$c_r\|\varphi\|_{L^r} \leqq \left\{\int_0^1 \left(\sum_{n,k=1}^{\infty} \Delta_k^2[f_n, x]\right)^{r/2} dx\right\}^{1/r} \leqq C_r\|\varphi\|_{L^1} \tag{13}$$

holds for $0<r<1$. Toward this let I denote the middle term of (13) and apply the two-dimensional version of Khinchin's inequality (see p. 84 of [7]) to I^r. Follow up by applying Khinchin's inequality to the inner-most integral of the resulting triple integral. What eventuates is that there exist constants d_r and D_r, depending only on r, such that

$$d_r \int_0^1\int_0^1 \left(\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty} r_n(y)\Delta_k[f_n, x]\right)^2\right)^{r/2} dy\,dx \leqq$$

$$\leqq I^r \leqq D_r \int_0^1\int_0^1 \left(\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty} r_n(y)\Delta_k[f_n, x]\right)^2\right)^{r/2} dy\,dx.$$

Continuing, we apply (12) and the Burkholder—Gundy inequality (8) to conclude that

$$a_r d_r \int_0^1\int_0^1 |F^*(x, y)|^r\,dx\,dy \leqq I^r \leqq A_r D_r \int_0^1\int_0^1 |F^*(x, y)|^r\,dx\,dy, \tag{14}$$

where $F^*(x, y)$ represents the maximal function $\sup_{n>0} |W_{2^n}[F(\cdot, y), x]|$ for each $x, y\in[0, 1]$. However, since for a.e. y the function $F(\cdot, y)$ is integrable, it is easy to see that

$$\int_0^1\int_0^1 |F(x, y)|^r\,dx\,dy \leqq \int_0^1\int_0^1 |F^*(x, y)|^r\,dx\,dy \leqq \gamma_r \int_0^1\int_0^1 |F(x, y)|\,dx\,dy$$

for $0<r<1$. (The constant γ_r either follows from known martingale inequalities or from a weak type (1, 1) estimate of Yano [12]. In connection with this see the comment on p. 734 in [1].) Consequently, inequality (13) follows from (14) and (10) with $c_r=a_r b_r d_r$ and $C_r=\gamma_r A_r B_r D_r$.

To complete the proof of the lemma, observe by SUNOUCHI [11] (pp. 7—8) that corresponding to each p_n there are numbers $\varepsilon_j^{(n)}\in\{0, 1\}$ $(j, n=1, 2, \ldots)$ such that

$$(15)\qquad w_{p_n}W_{p_n}[f_n] = \sum_{j=1}^{\infty} \varepsilon_j^n \Delta_j[w_{p_n}f_n].$$

It follows from (13), then, that

$$\int_0^1 \left(\sum_{n=1}^{\infty} W_{p_n}^2[f_n, x]\right)^{r/2} dx \equiv \int_0^1 \left(\sum_{n=1}^{\infty}\left(\sum_{j=1}^{\infty} \varepsilon_j^n \Delta_j[w_{p_n}f_n, x]\right)^2\right)^{r/2} dx \leqq$$

$$\leqq c_r^{-r}\int_0^1 \left(\sum_{n=1}^{\infty}\sum_{j=1}^{\infty} \varepsilon_j^n \Delta_j^2[w_{p_n}f_n, x]\right)^{r/2} dx.$$

In particular, another application of (13) results in the following inequality:

$$\int_0^1 \left(\sum_{n=1}^{\infty} W_{p_n}^2[f_n, x]\right)^{r/2} dx \leqq c_r^{-r}C_r^{+r}\left(\int_0^1\left(\sum_{n=1}^{\infty}|w_{p_n}(x)f_n(x)|^2\right)^{1/2} dx\right)^r.$$

The proof of the lemma is now complete with $\beta_r=C_r/c_r$ since $|w_{p_n}|\equiv 1$ for all integers n.

3. A proof of the theorem. To prove (5), set $p_k=k$ and

$$f_k(x) = k^{-3/2}\sum_{j=0}^{2^n-1} ja_jw_j(x),$$

for $2^{n-1}\leqq k<2^n$, $x\in[0, 1]$, and observe that $(W_k-\sigma_k)=k^{-1}\sum_{j=0}^{k-1} ja_jw_j$. It follows from the lemma proved in Section 2 that

$$\|S_1\|_{L^r} \equiv \left\{\int_0^1\left(\sum_{n=1}^{\infty}\sum_{k=2^{n-1}}^{2^n-1} k^{-3}\left(\sum_{j=0}^{k-1} ja_jw_j(x)\right)^2\right)^{r/2} dx\right\}^{1/r} \leqq \beta_r\int_0^1\left(\sum_{k=1}^{\infty} f_k^2(x)\right)^{1/2} dx.$$

Since f_k^2 is dominated by

$$(16)\qquad 8\cdot 2^{-3n}\left(\sum_{j=0}^{2^n-1} ja_jw_j\right)^2 \equiv 8\cdot 2^{-n}(W_{2^n}-\sigma_{2^n})^2,$$

for $2^{n-1}\leqq k<2^n$, it follows that (5) holds with $\alpha_r=\sqrt{8}\,\beta_r$.

To verify (6) begin by observing that $W_n-\sigma_n=n^{-1}\sum_{j=1}^{n-1}(W_n-W_j)$ holds for any integer $n\geqq 1$. It follows from the Schwarz inequality that

$$\sum_{n=1}^{\infty}(W_{2^n}-\sigma_{2^n})^2 \leqq \sum_{n=0}^{\infty} 2^{-n}\sum_{k=1}^{n}\sum_{j=2^{k-1}}^{2^k-1}(W_{2^n}-W_j)^2.$$

If we set

$$G = \sum_{n=0}^{\infty} 2^{-n} \sum_{k=1}^{n} 2^k (W_{2^n} - W_{2^{k-1}})^2$$

we have by (3) and the lemma that

$$\|S_2\|_{L^r} \leqq \beta_r \int_0^1 [G(x)]^{1/2}\, dx. \tag{17}$$

Here we have used the lemma on a connected block of terms of a Walsh series instead of partial sums of Walsh series. This application is justified since such blocks are differences of partial sums of Walsh series.

Continuing, observe that

$$|W_{2^n} - W_{2^{k-1}}| \leqq |\Delta_k| + |\Delta_{k+1}| + \ldots + |\Delta_n|$$

holds and write $\Delta_j = 2^{j/4} \cdot 2^{-j/4} \Delta_j$ for each $j \in [k, n]$. Hence another application of the Schwarz inequality followed by a routine calculation results in the inequalities:

$$G \leqq \sum_{n=0}^{\infty} 2^{-n} \sum_{k=1}^{n} 2^k \left(\sum_{j=k}^{n} 2^{-j/2} \right) \left(\sum_{j=k}^{n} 2^{j/2} \Delta_j^2 \right) \leqq 3\sqrt{2} \sum_{n=0}^{\infty} 2^{-n} \sum_{k=1}^{n} 2^{k/2} \left(\sum_{j=k}^{n} 2^{j/2} \Delta_j^2 \right).$$

Reverse the two inner-most sums, and sum $2^{k/2}$ from $k=1$ to $k=j$ to verify that

$$G \leqq 6 \sum_{n=0}^{\infty} 2^{-n} \sum_{j=1}^{n} 2^j \Delta_j^2.$$

Now, interchange the order of summation again, and sum 2^{-n} from $n=j$ to $n=\infty$ to conclude that

$$G \leqq 6 \sum_{j=0}^{\infty} \Delta_j^2.$$

Finally, combine this inequality with (17) to verify that (6) holds with $\alpha_r = 6\beta_r$.

To establish (7) begin with the trivial identity

$$W_{2^n} - W_{2^{n-1}} = (W_{2^n} - \sigma_{2^n}) + (\sigma_{2^n} - \sigma_{2^{n-1}}) + (\sigma_{2^{n-1}} - W_{2^{n-1}})$$

which holds for $n = 1, 2, \ldots$, and apply the Schwarz inequality to conclude that

$$S_3 \leqq 2S_2 + \left(\sum_{n=1}^{\infty} |\sigma_{2^n} - \sigma_{2^{n-1}}|^2 \right)^{1/2}. \tag{18}$$

Let S_4 represent the second term on the right hand side of (18). Correcting a misprint which appears on p. 9 of [11], it is known that

$$|\sigma_{2^n} - \sigma_{2^{n-1}}|^2 \leqq 2 \sum_{k=2^{n-1}}^{2^n - 1} \frac{|W_k - \sigma_k|^2}{k}.$$

Indeed,

$$|\sigma_{2^n}-\sigma_{2^{n-1}}| \leqq \sum_{k=2^{n-1}}^{2^n-1} |\sigma_{k+1}-\sigma_k| \leqq \sqrt{\sum_{k=2^{n-1}}^{2^n-1} k(\sigma_{k+1}-\sigma_k)^2}\cdot\sqrt{\sum_{k=2^{n-1}}^{2^n-1} k^{-1}} \leqq$$

$$\leqq \sqrt{2}\sqrt{\sum_{k=2^{n-1}}^{2^n-1} k(\sigma_{k+1}-\sigma_k)^2} \leqq \sqrt{2}\sqrt{\sum_{k=2^{n-1}}^{2^n-1} \frac{|W_k-\sigma_k|^2}{k}}.$$

It follows that $S_4 \leqq \sqrt{2} S_1$. Moreover, by Jensen's inequality it is known that $\|S_4\|_{L^r}^r \leqq \|S_4\|_{L^1}^r$. In particular,

$$\|S_4\|_{L^r}^r \leqq 2^{r/2}\|S_1\|_{L^1}^r. \tag{19}$$

To estimate S_2, observe that $2^{-2n} \leqq \sum_{j=2^{n-1}}^{2^n-1} j^{-3}$ and therefore by (16) that

$$|W_{2^n}-\sigma_{2^n}|^2 \leqq \left(\sum_{k=2^{n-1}}^{2^n-1} j^{-3}\right)\left(\sum_{i=1}^{2^n-1} ia_i w_i\right).$$

A final application of the lemma proved in Section 2 yields the following inequality:

$$\|S_2\|_{L^r} \leqq \beta_r \int_0^1 \left(\sum_{n=1}^{\infty} \sum_{j=2^{n-1}}^{2^n-1} j^{-3}\left|\sum_{i=1}^{j} ia_i w_i(x)\right|^2\right)^{1/2} dx \equiv \beta_r \|S_1\|_{L^1}.$$

Hence by (18) and (19), we conclude that

$$\|S_3\|_{L^r}^r \leqq (2\beta_r^r + 2^{r/2})\|S_1\|_{L^1}^r.$$

Inequality (7) therefore holds with $\alpha_r = (2\beta_r^r + 2^{r/2})^{1/r}$.

4. An application. BUTZER and WAGNER [3] introduced the following definition. A function f defined at points $x, x \dotplus 2^{-k}$ $(k=1, 2, \dots)$ on the unit interval is said to have a *dyadic derivative* $\dot{d}f$ at x if the following limit exists:

$$\dot{d}f(x) = \lim_{n\to\infty} \sum_{k=0}^{n} 2^{k-1}[f(x)-f(x \dotplus 2^{-k})].$$

It is not difficult to prove that $\dot{d}w_k(x) = kw_k(x)$ for every $x \in [0, 1]$ and every integer $k \geqq 0$. Thus given a Walsh series $W = \sum_{k=1}^{\infty} a_k w_k$, its term by term dyadic derivative is given by

$$\dot{W}(x) = \sum_{k=1}^{\infty} ka_k w_k(x).$$

Notice that

$$\dot{W}_N \equiv \sum_{k=1}^{N} ka_k w_k = N(W_N - \sigma_N) \tag{20}$$

holds for all integers $N \geqq 1$.

Let $0<r\leqq 1$ and let W be a Walsh series. We shall use the following measurements of how rapidly $k^{-3/2}W_k$ and $2^{-2}W_{2^k}$ decay:

$$\|W\|_{\mathcal{U}_r}=\left\|\left(\sum_{k=1}^{\infty}|W_k^2/k^3|\right)^{1/2}\right\|_{L^r},\quad \|W\|_{\mathcal{W}_r}=\left\|\left(\sum_{k=1}^{\infty}|2^{-k}W_{2^k}|^2\right)^{1/2}\right\|_{L^r}.$$

In spite of the suggestive notation, neither of these measurements are norms; the triangle inequality fails to hold. Observe by Jensen's inequality that

$$\|W\|_{\mathcal{U}_r}^2\leqq\sum_{k=1}^{\infty}\|W_k^2/k^3\|_{L^1}\quad\text{and}\quad\|W\|_{\mathcal{W}_r}^2\leqq\sum_{k=1}^{\infty}\|2^{-2k}W_{2^k}^2\|_{L^1}.$$

Thus $\|W\|_{\mathcal{U}_r}$ and $\|W\|_{\mathcal{W}_r}$ are both finite when W is a Walsh—Fourier series.

Recall that given a Walsh series W, the partial sums $\{W_{2^n},\ n\geqq 0\}$ form a dyadic martingale. Hence if S_3 is given by (4), then the dyadic H_r norm of W is given by $\|W\|_{H_r}\equiv\|S_3\|_{L^r}$ (see [6], especially the remarks on p. 193). Moreover, by the Burkholder—Gundy inequality (8), it follows that W belongs to dyadic H_r if and only if $W^*\in L^r$. In particular, since

$$\|W\|_{\mathcal{W}_r}\leqq\|W^*\|_{L^r}\left(\sum_{k=1}^{\infty}2^{-2k}\right)^{1/2},$$

we have that $\|W\|_{\mathcal{W}_r}$ is finite when W belongs to dyadic H^r.

It is now easy to see that for $0<r<1$ there exists an absolute constant α_r (depending only on r) such that

$$\|\dot{W}\|_{\mathcal{U}_r}\leqq\alpha_r\|\dot{W}\|_{\mathcal{W}_1},\tag{21}$$

$$\|\dot{W}\|_{\mathcal{W}_r}\leqq\alpha_r\|W\|_{H^1},\tag{22}$$

and

$$\|W\|_{H^r}\leqq\alpha_r\|\dot{W}\|_{\mathcal{U}_1}.\tag{23}$$

Indeed, by (20) $\|\dot{W}\|_{\mathcal{U}_r}=\|S_1\|_{L^r}$ and $\|\dot{W}\|_{\mathcal{W}_r}=\|S_2\|_{L^r}$ so inequalities (21), (22), and (23) are restatements of inequalities (5), (6), and (7).

Inequalities (22) and (23) are most useful. According to inequality (22), if W is the Walsh—Fourier series of some function f belonging to dyadic H_1, then $\|\dot{W}\|_{\mathcal{W}_r}<\infty$ for all $0<r<1$. In the case when $\dot{d}f=\dot{W}$ (see [8], [9], or [10]) we have that $\dot{d}f$ can be represented by a convergent Walsh series whose partial sums are reasonably well-behaved. According to inequality (23), if the partial sums of $\dot{W}$ are suitably well-behaved, then the original Walsh series must belong to dyadic H_r. In particular, if $\sum_{k=1}^{\infty}\|\dot{W}_k^2/k^3\|_{L^1}<\infty$, then W belongs to dyadic H_r, $0<r<1$.

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