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Ergodic sequences of integers

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1. Introduction. Let $S=\{k_1, k_2, \ldots\}$ be an increasing sequence of positive integers. We shall call S an ergodic sequence provided

$$\lim_{n\to\infty} \frac{1}{n} e^{ik_j\alpha} = 0 \quad \text{for} \quad 0<\alpha<2\pi. \tag{1.1}$$

The reason for the terminology is as follows. If U is any unitary operator on a Hilbert space H, then if (1.1) holds we have

$$\operatorname*{strong\,limit}_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} U^{k_j} = P, \tag{1.2}$$

where P is the orthogonal projection of U on $\{f\in H \mid Uf=f\}$. Moreover if (1.2) is to hold for every such U, then (1.1) is both necessary and sufficient. For details see e.g., [1].

Ergodic sequences of integers have been constructed in [1] and [2]. In [4] NIDERREITER gives a method of constructing ergodic sequences which have density zero. In this paper we use a result due to WIENER and WINTNER [5], to construct large classes of such sequences, both random sequences and nonrandom sequences. Here is what we mean by a random sequence. Let $(\Omega, \mathscr{F}, P)$ be a probability space and let τ be a measure preserving, ergodic transformation defined on Ω. Let $A\in\mathscr{F}$ such that $0<P(A)$. Then there exists a measurable set $\Omega_0\subseteqq\Omega$ with $P(\Omega_0)=1$ with the following property. Let $\omega\in\Omega$ and define the sequence $S(\omega, A)$ by

$$S(\omega, A)=\{k \mid \chi_A(\tau^k\omega)=1\} \tag{1.3}$$

where χ_A is the indicator function of A. Then for each $\omega\in\Omega_0$ we shall see that $S(\omega, A)$ is an ergodic sequence. In Section 2 we present the necessary background material. In Section 3 we consider nonrandom sequences and in Section 4 we construct

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the random sequences mentioned above. Finally in Section 5 we mention some possible generalizations and related matters.

2. Background material. In this section we state two results which we shall use subsequently. Let μ be a Borel measure on the circle group T with Fourier coefficients $\hat{\mu}(n)$, $n=0, \pm 1, \dots$. Then we have

Theorem 1. (i) *Let* $\tau \in T$. *Then* $\mu(\{\tau\}) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{N} \hat{\mu}(n) e^{in\tau}$, *and*

(ii) $\sum_{\tau} |\mu(\{\tau\})|^2 = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |\hat{\mu}(n)|^2$.

The proof may be found in Katznelson [3, p. 42].

We shall primarily consider measures for which the Fourier coefficients are real so that in (i) we will have $\mu(\{\tau\}) = \lim_{N\to\infty} \frac{1}{N} \sum_{1}^{N} \hat{\mu}(n) e^{in\tau}$.

Now let $\{a_n, n=0, \pm 1, \pm 2, \dots\}$ be a bounded sequence of numbers. Suppose for each $k=0, \pm 1, \dots$ the limit

$$\mu_k = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{N} a_n \overline{a_{n-k}} \tag{2.1}$$

exists. The following result is due to Wiener and Wintner [5].

Theorem 2. (i) *There exists a positive Borel measure* μ *on* T *such that* $\hat{\mu}(k) = \mu_k$, $k=0, \pm 1, \pm 2, \dots$ *and*

(ii) $\lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{N} a_n e^{in\lambda} = 0$ *for every* λ *with* $0 \leqq \lambda \leqq 2\pi$ *which is a continuity point of* μ.

Since we shall only consider real sequences $\{a_n\}$, we shall restrict ourselves to one-sided sequences $\{a_n, n=1, 2, \dots\}$ and (ii) becomes

$$\lim_{N\to\infty} \frac{1}{N} \sum_{1}^{N} a_n e^{in\lambda} = 0 \tag{2.2}$$

for λ a continuity point of μ.

3. Nonrandom sequences. Let $S = \{k_1, k_2, \dots\}$ be a sequence of positive integers. For each $n=1, 2, \dots$ let $S_n = \{k_1, \dots, k_n\}$, and for each $r=1, 2, \dots$ let $S_n^{(r)} = \{k_1+r, \dots, k_n+r\}$. Assume that for each $r=1, \dots$

$$\nu_r = \lim_{n\to\infty} \frac{1}{n} |S_n \cap S_n^{(r)}| \tag{3.1}$$

exists, where $|A|$ is the cardinality of A. Let $\chi_S(\cdot)$ be the indicator function of S. Then we have

Theorem 3. *Suppose* S *has positive density, i.e.,*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\chi_S(j)=d>0. \tag{3.2}$$

Let $\nu_0=1$. *Then* μ_k *exists for all* $k=0,\pm1,\ldots$ *for the sequence* $\{\chi_S(j)\}$ *and* $\mu_k=d\nu_k$, $k=0,\pm1,\ldots$. *Let* μ *be the measure with Fourier coefficients* $\hat\mu(k)=\mu_k$. *Then we have*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}e^{ik_j\lambda}=0 \tag{3.3}$$

for every λ *which is a continuity point of* μ.

The proof follows easily from Theorem 2. We see that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\chi_S(j)\chi_S(j+r)=\lim_{n\to\infty}\frac{\alpha(n)}{n}\frac{1}{\alpha(n)}|S_{\alpha(n)}\cap S^{(r)}_{\alpha(n)}|=d\nu_r,$$

$r=1,2,\ldots$, where $\alpha(n)$ is the number of ones among $\chi_S(j)$, $j=1,\ldots,n$. Let $\mu_r=\mu_{-r}=d\nu_r$, $r=0,1,2,\ldots$, and let μ be the measure guaranteed by Theorem 2. Then we have

$$\lim\frac{1}{\alpha(n)}\sum_{j=1}^{\alpha(n)}e^{ik_j\lambda}=\lim\frac{n}{\alpha(n)}\frac{1}{n}\sum_{j=1}^{n}\chi_S(j)e^{ij\lambda}=\frac{1}{d}\mu(\{\lambda\})=0$$

for $0<\lambda<2\pi$ and λ a continuity point of μ. But $\alpha(n)\to\infty$ and $\dfrac{\alpha(n)}{n}\to d$.

This result allows us to give many simple examples of ergodic sequences of integers. If $\nu_1=1$, and hence $\nu_k=1$ for all k, then μ is the measure which puts mass d at $e^{2\pi i}$ and every Borel set of T which does not include the point $e^{2\pi i}$ has μ-measure zero, and we have an ergodic sequence.

We can apply the theorem in two ways. One way is to look at simple measures on T, calculate their Fourier coefficients and then construct ergodic sequences which give rise to these coefficients. The other is to look at certain sequences and verify the appropriate conditions.

Here is a simple example of the first technique. Consider the measure μ which puts mass 1/2 each on $e^{\pi i}$ and $e^{2\pi i}$. Then $\hat\mu(n)=0$ for n odd and $\hat\mu(n)=1$ for n even. If S is the sequence of even integers then the numbers $\chi_S(j)$ satisfy the conditions of Theorem 2. However the sequence $k_n=2n$ is not ergodic since $\lim\limits_{n}\frac{1}{n}\sum_{j=1}^{n}e^{\pi ik_j}=1$. Now let $\{r_k, k=1,\ldots\}$ be an increasing sequence of positive integers such that $\lim\limits_{n}(r_{k+1}-r_k)=\infty$, $\lim\limits_{n}r_{k+1}/r_k=1$. Modify the sequence S in the following way. When k is even leave the elements of S between r_k and r_{k+1} as they are. When k is odd, add one to each k_n between r_k and r_{k+1}. The resulting sequence $S'=\{k'_1, k'_2, \ldots\}$ will be ergodic from Theorem 3, and the fact that now $\lim_n \frac{1}{n} \sum_{j=1}^{n} e^{\pi i k'_j}=0$. Clearly we can play the same game for any measure μ which puts mass $1/k$ on each of $e^{2\pi i j/k}$, $j=0, \ldots, k-1$.

Now let x be a normal number to the base two in the unit interval and let $\{x_n, n=1, 2, \ldots\}$ be its coordinates. The measure μ corresponding to this sequence then has Fourier coefficients $\hat{\mu}(0)=1/2$ and $\hat{\mu}(k)=1/4$, $k\neq 0$. It then follows from Theorem 1 that $\mu\{e^{i\lambda}\}=0$ for $0<\lambda<2\pi$ and therefore the sequence $\{k_n, n=1, \ldots\}$ consisting of those integers for which $x_n=1$ is ergodic by Theorem 2.

4. Random sequences. Let $(\Omega, \mathscr{F}, P)$ be a probability space and let τ be a measure preserving transformation mapping Ω onto Ω. Now let $A\in\mathscr{F}$ with $0<P(A)$. It follows from the individual ergodic theorem that there exists $\Omega_0\in\mathscr{F}$ with $P(\Omega_0)=1$ such that for $\omega\in\Omega_0$ the following limit relations hold

$$(4.1) \qquad \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \chi_A(\tau^j\omega)\chi_A(\tau^{j-k}\omega) = P\{A\cap\tau^k A\}, \quad k=0,1,2,\ldots.$$

Now let $\omega\in\Omega_0$ and consider the sequence $\{\chi_A(\tau^j\omega), j=1, 2, \ldots\}$. By Theorem 2 there is a measure μ on T with $\hat{\mu}(k)=P\{A\cap\tau^k A\}$, $k=0, \pm 1, \pm 2, \ldots$. Now suppose τ is mixing. Then we have $\lim_{k\to\infty} P\{A\cap\tau^k A\}=P^2\{A\}$. Moreover from Theorem 1 we see that μ is continuous except at $e^{2\pi i}$. We summarize in

Theorem 4. *Let τ measure-preserving and mixing, and let $A\in\mathscr{F}$ with $0<P(A)$. Then for almost all ω the sequence $\{k_n(A, \omega), n=1, 2, \ldots\}$ consisting of those integers for which $\chi_A(\tau^j\omega)=1$ is an ergodic sequence.*

Theorem 3 enables us to give a simple proof of a theorem of NIEDERREITER [4]. Let r be a positive integer and suppose we are given r and α with $0<\alpha<1$. Then Niederreiter exhibited an ergodic sequence with $\nu_r=\alpha$.

We shall show that the existence of such a sequence follows from Theorem 3. For let $\{X_n(\omega), n=1, 2, \ldots\}$ be a sequence of independent Bernoulli random variables with $P\{X_n(\omega)=1\}=\alpha=1-P\{X_n(\omega)=0\}$. It follows from the law of large numbers that there exists a set of probability one such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} X_j(\omega)X_{j+r}(\omega) = \begin{cases} \alpha, & r=0 \\ \alpha^2, & r>0 \end{cases}$$

for ω in this set. If ω is in this set and $\{k_n(\omega)\}$ is the sequence of integers for which $X_j(\omega)=1$ then $\{k_n(\omega)\}$ is ergodic and $\nu_r=\alpha$ for all $r>0$.

This method can easily be generalized to yield for certain values of $r_1, r_2, \alpha_1, \alpha_2$ ergodic sequences for which $\nu_{r_1}=\alpha_1, \nu_{r_2}=\alpha_2$. Whether this can be done in full generality is not clear.

5. Concluding remarks. The method used in this paper does not apply when a sequence has density zero. For example it is easy to show that the sequence $\{n^k, n=1, \dots\}$, $k>1$ and an integer, is not ergodic. On the other hand NIEDERREITER [4] has shown that the sequence of integer parts of n^k when $k>1$ is *not* an integer is ergodic. In both cases we have $v_r=0$, $r=1, 2, \dots$.

Moreover even when a sequence has positive density and each v_k exists and is positive, the situation is not entirely clear. For example, it is possible to construct for each ε such that $0<\varepsilon<1$ a nonergodic sequence with density $d>1-\varepsilon$ and also each $v_k>1-\varepsilon$. Thus we are a long way having convenient necessary and sufficient conditions for ergodicity of a sequence.

Another unresolved question concerns the individual ergodic theorem. As mentioned in Section 1, if $S=\{k_1, k_2, \dots\}$ is an ergodic sequence then the mean ergodic theorem holds for every unitary operator U. Now suppose τ is a measure-preserving transformation on a probability space $(\Omega, \mathcal{F}, P)$. We can then ask if $\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} f(\tau^{k_j}\omega)$ exists a.e. for every $f\in L_1(\Omega, \mathcal{F}, P)$. When S consists of all positive integers, this is of course the individual ergodic theorem. However, when S is significantly different nothing is known.

References

[1] JULIUS BLUM and BENNETT EISENBERG, Generalized summing sequences and the mean ergodic theorem, *Proc. Amer. Math. Soc.*, **42** (1974), 423—429.

[2] J. R. BLUM, B. EISENBERG and L. S. HAHN, Ergodic theory and the measure of sets in the Bohr group, *Acta Sci. Math.*, **34** (1973), 17—34.

[3] YITZHAK KATZNELSON, *An Introduction to Harmonic Analysis*, John Wiley and Sons (New York, 1968).

[4] H. NIEDERREITER, On a paper of Blum, Eisenberg, and Hahn concerning ergodic theory and the distribution of sequences in the Bohr group, *Acta Sci. Math.*, **37** (1975), 103—108.

[5] NORBERT WIENER and AUREL WINTNER, Harmonic analysis and ergodic theory, *Amer. J. Math.*, **63** (1941), 415—426.

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