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Weighted translation semi-groups with operator weights

MARY EMBRY-WARDROP*

1. Introduction. If φ is a continuous nonzero complex-valued function on $\mathcal{R}^+$ and $(S_t f)(x)=[\varphi(x)/\varphi(x-t)]f(x-t)$ for $x\geqq t$ and 0 otherwise, then S is a semi-group of linear transformations on $\mathcal{L}^2(\mathcal{R}^+,\mathcal{C})$. S is a strongly continuous semi-group of bounded operators if φ satisfies certain boundedness conditions. These semi-groups, called *weighted translation semi-groups (w.t.s.) with symbol* φ, were introduced in [4] and the subnormal w.t.s. characterized in [5].

In [4] it was shown that S is quasinormal if and only if $\varphi(x)=Ma^x$ for some constants M and a. In this case $S_t=a^t L_t$, where L is the forward translation semi-group on $\mathcal{L}^2(\mathcal{R}^+,\mathcal{C})$. In [6] we proved that any strongly continuous quasinormal semi-group S on a separable Hilbert space $\mathcal{H}$ is unitarily equivalent to the direct sum of a normal semigroup and a pure quasinormal semi-group Q on $\mathcal{L}^2(\mathcal{R}^+,\mathcal{K})$ for some Hilbert space $\mathcal{K}$. Furthermore, $Q_t=\overline{h(t)}L_t$ where L is the forward translation semi-group on $\mathcal{L}^2(\mathcal{R}^+,\mathcal{K})$, h is a strongly continuous self-adjoint semi-group on $\mathcal{K}$, and $(\overline{h(t)}f)(x)=h(t)f(x)$ a.e. for each f in $\mathcal{L}^2(\mathcal{R}^+,\mathcal{K})$. Thus, the pure quasinormal semi-groups behave like quasinormal w.t.s.

In this paper, we shall introduce w.t.s. on $\mathcal{L}^2(\mathcal{R}^+,\mathcal{K})$ for which the symbol φ is $\mathcal{K}$-operator-valued and study a few of their properties.

In Section 2, we specify which operator-valued functions φ will be allowed. This class of semi-groups gives a rich supply of easily constructed examples. In particular, every pure quasinormal semi-group is (unitarily equivalent to) a weighted translation semi-group. Section 3 is devoted to characterizing subnormal w.t.s. on $\mathcal{L}^2(\mathcal{R}^+,\mathcal{K})$. In Theorem 3 we show that S with symbol φ is subnormal if and only if φ^2 is the compression of a strongly continuous self-adjoint semi-group; equivalently, there exists an operator measure on an interval $[0,a]$ such that $\varphi(x)^2=$

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$= \int_0^a r^x d\varrho(r)$. This last condition is precisely the characterization of subnormal w.t.s. in [5] in the numerical case $\mathscr{K} = \mathscr{C}$.

Throughout the paper, we shall assume all Hilbert spaces to be separable. $\mathscr{L}^2(\mathscr{R}^+, \mathscr{K})$ is the Hilbert space of (equivalence classes of) square integrable weakly measurable functions from the nonnegative reals $\mathscr{R}^+$ to the separable Hilbert space $\mathscr{K}$. $\mathscr{B}(\mathscr{K})$ or $\mathscr{B}(\mathscr{L}^2(\mathscr{R}^+, \mathscr{K}))$ stands for the Banach algebra of continuous linear operators on $\mathscr{K}$ or $\mathscr{L}^2(\mathscr{R}^+, \mathscr{K})$, respectively. A function $S: \mathscr{R}^+ \to \mathscr{B}(\mathscr{K})$ is a *semi-group* if $S_0 = I$, the identity operator, and $S_t S_r = S_{t+r}$ for all r and t in $\mathscr{R}^+$. A function $\varphi: \mathscr{R}^+ \to \mathscr{B}(\mathscr{K})$ is *strongly continuous* if $\lim_{t\to r} \|\varphi(t)f - \varphi(r)f\| = 0$ for each f in $\mathscr{K}$ and r in $\mathscr{R}^+$. In this case, we write $\text{s-}\lim_{t\to r} \varphi(t) = \varphi(r)$. The *forward translation semi-group* L $((L_t f)(x) = f(x-t)$ if $x \geqq t$ and 0 otherwise) on $\mathscr{L}^2(\mathscr{R}^+, \mathscr{K})$ plays a special role in ideas developed in this paper.

A semi-group S of operators is *normal* if $S_t^* S_t = S_t S_t^*$ for all t, *quasinormal* if $S_t(S_t^* S_t) = (S_t^* S_t) S_t$ for all t and *subnormal* if S is the restriction of a normal semi-group to an invariant subspace. An *operator measure* ϱ on $[a, b]$ is a function defined on the Borel sets of $[a, b]$ with values in $\mathscr{B}(\mathscr{K})$ such that $\varrho(\emptyset) = 0$, $\varrho(E)$ is a positive Hermitian operator for each Borel set E, $\varrho(E) \ll \varrho(F)$ whenever $E \subseteqq F$ and $\varrho(E) = \text{s-}\lim_{n\to\infty} \sum_{i=1}^{n} \varrho(E_i)$ whenever E is the union of a collection of pairwise disjoint sets E_i. If the values of ϱ are projections and $\varrho[a, b] = I$, then ϱ is a *spectral measure* on $[a, b]$. Two integral representations which reoccur frequently in this paper are as follows:

1) [8, Theorem 22.3.1, p. 588]. If H is a strongly continuous self-adjoint semi-group of operators, there exists a spectral measure ϱ on an interval $[0, a]$ such that

$$H_t = \int_0^a r^t \, d\varrho(r),$$

2) [5, Theorem 2.1]. S is a strongly continuous subnormal semi-group if and only if there exists an operator measure ϱ on an interval $[0, a]$ such that $\varrho([0, a]) = I$ and

$$S_t^* S_t = \int_0^a r^t \, d\varrho(r).$$

We shall also say that a semi-group S on $\mathscr{H}$ is the *compression* of a semi-group T on $\mathscr{K}$ if $\mathscr{H} \subseteqq \mathscr{K}$ and $S_t = P T_t P$ for each t where P is the orthogonal projection of $\mathscr{K}$ onto $\mathscr{H}$.

2. Weighted translation semi-groups. Let $\varphi: \mathscr{R}^+ \to \mathscr{B}(\mathscr{K})$ have properties:

i) for each x in $\mathscr{R}^+$, $\varphi(x)$ is a one-to-one positive Hermitian operator,

ii) $\{\varphi(x): x \in \mathscr{R}^+\}$ is abelian,

iii) φ is strongly continuous,

iv) there exist numbers M and a such that for all x and t in $\mathcal{R}^+$,

$$\varphi(x+t)^2 \ll M^2 a^{2t} \varphi(x)^2.$$

Such a φ will be called a *symbol.* We are requiring $\varphi(x)$ to be positive Hermitian for simplicity. We use the other requirements to prove that the mapping $t \to \varphi(x-t)^{-1}\varphi(x)(L_t f)(x)$ defines a semi-group which is strongly continuous.

Conditions i) and iv) imply that if $t \leqq x$, there exists a unique element C of $\mathcal{B}(\mathcal{K})$ such that $\varphi(x)=\varphi(t)C$. In this case, we write $C=\varphi(t)^{-1}\varphi(x)$. Even if $\varphi(x)$ is not one-to-one, this factorization of $\varphi(x)$ can be obtained [2]; however, $\varphi(x)^{-1}\varphi(x)$ would be the projection onto the closure of the range of $\varphi(x)$ and the semi-group that we are interested in constructing would not have $S_0=I$.

Lemma 1. *Let φ be a symbol on $\mathcal{K}$. Then*

i) $\varphi(x)^{-1}\varphi(x)=I$ for all x,

ii) $\varphi(r)$ *commutes with* $\varphi(t)^{-1}\varphi(x)$ *for all* r *whenever* $t \leqq x$,

iii) $\varphi(t)^{-1}\varphi(x)$ *commutes with* $\varphi(a)^{-1}\varphi(b)$ *whenever* $t \leqq x$ *and* $a \leqq b$,

iv) $[\varphi(r)^{-1}\varphi(t)][\varphi(t)^{-1}\varphi(s)]=\varphi(r)^{-1}\varphi(s)$ *whenever* $r \leqq t \leqq s$,

v) $\varphi(t)^{-1}\varphi(x)$ *is one-to-one and positive Hermitian whenever* $t \leqq x$ *and satisfies* $\|\varphi(t)^{-1}\varphi(x)\| \leqq Ma^{x-t}$,

vi) $\operatorname{s-lim}_{t\to 0} \varphi(x)^{-1}\varphi(x+t)=I$,

vii) $[\varphi(x)^{-1}\varphi(x+t)]^2=(\varphi(x)^2)^{-1}\varphi(x+t)^2$ *for all* x *and* t.

Proof. i) follows immediately from definition of $\varphi(x)^{-1}\varphi(x)$. ii) by definition $\varphi(x)=\varphi(t)[\varphi(t)^{-1}\varphi(x)]$. Since $\{\varphi(s)\}$ is abelian and $\varphi(t)$ is one-to-one, $\varphi(r)$ commutes with $\varphi(t)^{-1}\varphi(x)$. Therefore, $\varphi(x)=\varphi(t)^{1/2}[\varphi(t)^{-1}\varphi(x)]\varphi(t)^{1/2}$ and v) now follows from the fact that $\varphi(x)$ and $\varphi(t)^{1/2}$ are one-to-one positive Hermitian operators. The inequality in v) follows from condition iv) of the definition of φ. iii) follows from ii) and the facts that each $\varphi(x)$ is one-to-one and $\{\varphi(x)\}$ is abelian. iv) $\varphi(r)^{-1}\varphi(s)$ is the unique operator satisfying $\varphi(s)=\varphi(r)[\varphi(r)^{-1}\varphi(s)]$. But $[\varphi(r)^{-1}\varphi(t)][\varphi(t)^{-1}\varphi(s)]$ also satisfies this equation. vi) Note that for each k in $\mathcal{K}$,

$$\|[\varphi(x)^{-1}\varphi(x+t)-I]\,\varphi(x)k\| = \|\varphi(x+t)k-\varphi(x)k\|.$$

Since φ is strongly continuous, then $\operatorname{s-lim}_{t\to 0} \varphi(x)^{-1}\varphi(x+t)=I$ on the range of $\varphi(x)$ which is dense in $\mathcal{K}$. Since $\|\varphi(x)^{-1}\varphi(x+t)-I\| \leqq Ma^t+1 \leqq M_0$ for t in $[0, 1]$, we see that $\operatorname{s-lim}_{t\to 0} \varphi(x)^{-1}\varphi(x+t)=I$ on all of $\mathcal{K}$. vii) Since $\varphi(x+t)^2 \ll M^2a^{2t}\varphi(x)^2$ and $\{\varphi(x)\}$ is abelian, then $\varphi(x+t)^4 \ll M^4a^{4t}\varphi(x)^4$ and $(\varphi(x)^2)^{-1}\varphi(x+t)^2$ can be defined in a fashion similar to $\varphi(x)^{-1}\varphi(x+t)$: that is, $(\varphi(x)^2)^{-1}\varphi(x+t)^2$ is the unique operator C satisfying $\varphi(x+t)^2=\varphi(x)^2C$. Since $[\varphi(x)^{-1}\varphi(x+t)]^2$ also satisfies this equation (using the definition of $\varphi(x)^{-1}\varphi(x+t)$ and the fact that it commutes with $\varphi(x+t)$), the proof of vii) is complete.

Now let φ be a symbol on $\mathscr{K}$. For each t in $\mathscr{R}^+$ define the operator S_t on $\mathscr{L}^2(\mathscr{R}^+, \mathscr{K})$ by

$$(1)\qquad (S_t f)(x) = \begin{cases} \varphi(x-t)^{-1}\varphi(x)f(x-t) & \text{if } x \geqq t \\ 0 & \text{if } x < t. \end{cases}$$

An argument directly paralleling one in [7, p. 211] can be given to show that

$$(2)\qquad \|S_t\| = \operatorname*{ess\,sup}_{x\in\mathscr{R}^+} \|\varphi(x)^{-1}\varphi(x+t)\|.$$

Theorem 2. *If φ is a symbol on $\mathscr{K}$, then S is a strongly continuous semi-group of operators on $\mathscr{L}^2(\mathscr{R}^+, \mathscr{K})$.*

Proof. Note that $(S_0 f)(x)=\varphi(x)^{-1}\varphi(x)f(x)=f(x)$ by Lemma 1 i) so that $S_0=I$ on $\mathscr{L}^2(\mathscr{R}^+, \mathscr{K})$. A straightforward computation, making use of Lemma 1 iii) and iv), shows that $S_{t+r}=S_t S_r$ for all $t, r\geqq 0$. It remains to be shown that S is strongly continuous. By equation (2) and Lemma 1 v) we have

$$(3)\qquad \|S_t\| \leqq Ma^t.$$

We argue as in [4, p. 211]. Assume first that $a=1$. Let f be a continuous function of compact support in $\mathscr{L}^2(\mathscr{R}^+, \mathscr{K})$. Then

$$\|S_t f-f\|^2 = \int \|\varphi(x)^{-1}\varphi(x+t)f(x)-f(x+t)\|^2\,dx.$$

Let $b=\operatorname{ess\,sup}|f|$, $\operatorname{supp} f\subseteqq[0, k]$ and $g(x)=b$ if $x\in[0, k+1]$ and $g(x)=0$ otherwise. Then $g\in\mathscr{L}^2(\mathscr{R}^+, \mathscr{K})$ and for $t\leqq 1$,

$$\|\varphi(x)^{-1}\varphi(x+t)f(x)-f(x+t)\| \leqq (M+1)g(x).$$

By Lemma 1 vi) and the continuity of f,

$$\lim_{t\to 0}\|\varphi(x)^{-1}\varphi(x+t)f(x)-f(x+t)\| = 0.$$

Thus, by the Lebesgue dominated convergence theorem, $\lim_{t\to 0}\|S_t f-f\|^2=0$. S is strongly continuous on a dense subset of $\mathscr{L}^2(\mathscr{R}^+, \mathscr{K})$ and consequently on all of $\mathscr{L}^2(\mathscr{R}^+, \mathscr{K})$ since S is uniformly bounded by M.

Now assume that a is arbitrary in (3) and let $T_t=a^{-t}S_t$ and $\varrho(t)=a^{-t}\varphi(t)$. Then ϱ is a symbol on $\mathscr{K}$ and defines T by (1). Hence, the preceding result implies, that T is strongly continuous on $\mathscr{L}^2(\mathscr{R}^+, \mathscr{K})$; the same must be true for S.

Hereafter, if φ is a symbol on $\mathscr{K}$ and S is the semi-group defined by (1) we shall say that (S, φ) *is a weighted translation semi-group (w.t.s.) on* $\mathscr{L}^2(\mathscr{R}^+, \mathscr{K})$. Note that $(S_t^* f)(x)=\varphi(x)^{-1}\varphi(x+t)f(x+t)$ and, consequently, by Lemma 1 for f in $\mathscr{L}^2(\mathscr{R}^+, \mathscr{K})$,

$$(4)\qquad (S_t^* S_t f)(x) = \varphi(x)^{-2}\varphi(x+t)^2 f(x) \quad \text{a.e.}$$

Thus, if P_t is the positive square root of $S_t^*S_t$, then $(P_tf)(x)=\varphi(x)^{-1}\varphi(x+t)f(x)$ by Lemma 1 vii) and v). A straightforward argument shows that $S_t=L_tP_t$ and $\ker L_t=\ker S_t$ where L is the forward translation semi-group on $\mathscr{L}^2(\mathscr{R}^+,\mathscr{K})$. That is, P_t is the positive factor and L_t the isometric factor in the polar decomposition of S_t.

The following examples give two ways in which to construct symbols and the associated w.t.s.

Example 1. Let $\varphi\colon\mathscr{R}^+\to\mathscr{B}(\mathscr{K})$ and assume that φ is one-to-one positive Hermitian-valued, nonincreasing and strongly continuous. If $\{\varphi(x)\}$ is abelian, then it follows that φ satisfies the properties of a symbol. Consequently, (S,φ) is a strongly continuous semi-group.

Example 2. Let φ be a strongly continuous self-adjoint semi-group of operators on $\mathscr{K}$. It follows easily that φ satisfies properties i)—iii) of a symbol. Moreover, there exists a spectral measure ϱ such that $\varphi(x)=\int_0^a r^x d\varrho(r)$ [8, p. 588]. The inequality $\varphi(x+t)^2\ll a^{2t}\varphi(x)^2$ readily follows. In this case, (S,φ) has a simpler form than the general w.t.s.:

$$(S_tf)(x)=\varphi(x-t)^{-1}\varphi(x)f(x-t)=\varphi(t)f(x-t)\quad\text{if}\quad x\geqq t.$$

We shall see in the following section that these are the only quasinormal w.t.s. Indeed, every pure quasinormal semi-group is unitarily equivalent to (S,φ) where φ is a strongly continuous self-adjoint semi-group (Corollary 6).

In the next section, it will be convenient to consider symbols φ for which $\varphi(0)=I$. There is no loss of generality in making this assumption for if φ is a symbol, define $\varphi_1(x)=\varphi(0)^{-1}\varphi(x)$. Then $\varphi_1(0)=I$ by Lemma 5. Furthermore, by Lemma 5 $\varphi_1(x)$ is a one-to-one positive Hermitian operator, $\{\varphi_1(x)\}$ is abelian and φ_1 is strongly continuous. To see that $\varphi_1(x+t)^2\ll M^2a^{2t}\varphi_1(x)^2$ we argue as follows. By definition of φ, $\|\varphi(x+t)k\|\leqq Ma^t\|\varphi(x)k\|$ for all k in $\mathscr{K}$. Therefore, $\|[\varphi(0)^{-1}\varphi(x+t)]\varphi(0)k\|\leqq Ma^t\|[\varphi(0)^{-1}\varphi(x)]\varphi(0)k\|$. Consequently, $\|\varphi_1(x)k\|\leqq$ $\leqq Ma^t\|\varphi_1(x)k\|$ for all k in the range of $\varphi(0)$, a dense subset of $\mathscr{K}$. Thus, the inequality holds for all k so that φ_1 satisfies condition iv) of the definition of a symbol.

3. Subnormal weighted translation semi-groups. Throughout this section, we assume $\varphi(0)=I$ when φ is a symbol.

Example 3. Let ϱ be an abelian operator measure on $[0,a]$ with $\varrho[0,a]=I$. Define $\varphi(x)^2=\int_0^a r^x d\varrho(r)$ where $\varphi(x)\gg 0$ for each x. It will follow from Lemma 4 that φ is a symbol. Indeed, we see in the following theorem that these are exactly the symbols which define the subnormal w.t.s.

Theorem 3. *Let* (S, φ) *be a w.s.t. on* $\mathcal{L}^2(\mathcal{R}^+, \mathcal{K})$. *The following statements are equivalent:*

i) (S, φ) *is subnormal,*

ii) φ^2 *is the compression of a strongly continuous self-adjoint semi-group,*

iii) *there exists an operator measure* ϱ *on* $[0, a]$ *with* $\varrho[0, a]=I$ *such that*

$$\varphi(x)^2 = \int_0^a r^x \, d\varrho(r).$$

Before proving Theorem 3, we shall prove a lemma which includes the equivalence of ii) and iii).

Lemma 4. *Let* $\mathcal{K}$ *be a Hilbert space and* $h\colon \mathcal{R}^+ \to \mathcal{B}(\mathcal{K})$. *The following are equivalent:*

1. h *is a compression of a strongly continuous self-adjoint semi-group,*
2. *there exists an operator measure on a finite interval* $[0, a]$ *such that* $\varrho[0, a]=I$ *and*

$$h(x) = \int_0^a r^x d\varrho(r),$$

3. h *satisfies the following four conditions:*

   i) $h(0)=I$,

   ii) h *is strongly continuous,*

   iii) *there exists a number* a *such that* $h(x+t) \ll a^t h(x)$ *for all* x *and* t *in* $\mathcal{R}^+$,

   iv) $\sum_{i,j=0}^{n} \langle h(x_i+x_j)k_i, k_j\rangle \geqq 0$ *for all finite collections* $\{x_0, \dots, x_n\}$ *in* $\mathcal{R}^+$ *and* $\{k_0, \dots, k_n\}$ *in* $\mathcal{K}$.

Proof. We shall show that $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$.

$1 \Rightarrow 2$. Assume that $h(t)=PH(t)P$ where P is the projection of a larger Hilbert space onto $\mathcal{K}$ and H is a strongly continuous self-adjoint semi-group on the larger space. There exists a spectral measure [8, p. 588] on an interval $[0, a]$ such that $H(t)=\int_0^a r^t d\mu(r)$. Consequently, $h(t)=\int_0^a r^t dP\mu(r)P$ and $P\mu P$ is an operator measure on $\mathcal{K}$ with $(P\mu P)[0, a]=I$ on $\mathcal{K}$.

$2 \Rightarrow 3$. Assume 2 holds. 3 i) and iii) are immediate. 3 ii) follows from an application of the monotone convergence theorem. To see that 3 iv) holds, observe that if E is any measurable subset of $[0, a]$, then

$$\sum_{i,j=0}^{n} r^{x_i+x_j} \langle \varrho(E)k_i, k_j\rangle = \left\langle \varrho(E) \sum_{i=0}^{n} r^{x_i}k_i, \sum_{j=0}^{n} r^{x_j}k_j \right\rangle \geqq 0$$

and consequently,

$$\sum_{i,j=0}^{n} \int_{0}^{a} r^{x_i+x_j} \langle d\varrho(r) k_i, k_j \rangle \geqq 0.$$

3⇒1. The techniques used in this part of the proof are standard and will only be outlined. They are patterned after proofs in [1] and [3]. Assume that h satisfies the properties given in 3. Let M be the set of all functions $f: \mathscr{R} \to \mathscr{K}$ such that $f(x)=0$ except possibly for a finite number of real x. If f and g are in M, define

$$(f, g) = \sum_{a,b} \langle h(a+b) f(a), g(b) \rangle.$$

(See [3, p. 1254] for details.) Since $(f, f) \geqq 0$ by hypothesis, it is easily checked that $(\, , \,)$ is a semi-inner product on M. Let $M_0 = \{f: (f, f)=0\}$ and $H_0 = M/M_0$. Let $(\, , \,)$ also be the inner product on H_0 induced by $(\, , \,)$ on M and let $\mathscr{H}$ be the completion of H_0.

For each t in $\mathscr{R}^+$ define $H(t): M \to M$ by $(H(t)f)(x) = f(x-t)$. Then H is a semi-group and for f and g in M

$$(H(t)f, g) = \sum_{a,b} \langle h(a+b) f(a-t), g(b) \rangle =$$

$$= \sum_{a,b} \langle h(a+b+t) f(a), g(b) \rangle = (f, H(t) g).$$

It follows from the Cauchy—Schwarz inequality that M_0 is invariant under $H(t)$; consequently $H(t)$ induces a self-adjoint semi-group of linear transformations on H_0. If we can show that $H(t)$ is a bounded transformation, then $H(t)$ can be extended continuously to $\mathscr{H}$.

To prove that $H(t)$ is bounded, we need to show that there exists K such that $(H(t)f, H(t)f) \leqq K(f, f)$ for all f in M. Equivalently,

$$\sum_{a,b} \langle h(a+b+2t) f(a), f(b) \rangle \leqq K \sum_{a,b} \langle h(a+b) f(a), f(b) \rangle.$$

The argument given by Bram [1, p. 76] can be duplicated in this situation to show that this inequality holds with $K = a^{2t}$ (we use condition iii) here).

Thus, H is a semi-group of self-adjoint operators on $\mathscr{H}$. We next show that H is strongly continuous. Let $f \in M$ and compute

$$(H(t)f - f, H(t)f - f) = \operatorname{Re} \sum_{a,b} \langle [h(a+b+2t) - 2h(a+b+t) + h(a+b)] f(a), f(b) \rangle.$$

Since h is strongly continuous on $\mathscr{K}$, the right-hand side converges to 0 as $t \to 0$. We conclude that H is strongly continuous on $\mathscr{H}$.

We complete the proof by identifying $\mathscr{K}$ with a subspace of $\mathscr{H}$ and h with the compression of H to that subspace. For each k in $\mathscr{K}$ define $(Uk)(x) = k$ if $x=0$ and $(Uk)(x)=0$ otherwise. Then $Uk \in M$, U is linear, and $(Uk, Uk) =$

$=\langle h(0)k, k\rangle=\|k\|^2$ by condition i). Therefore, we may consider Uk to be an element of M/M_0 and consequently of $\mathscr{H}$. $\|Uk\|_{\mathscr{H}}=\|k\|_{\mathscr{K}}$ so that U is an isometry from $\mathscr{K}$ onto a subspace of $\mathscr{H}$. UU^* is the projection P of $\mathscr{H}$ onto that subspace. We complete the proof by showing that $U^*PH(t)PU=h(t)$, so that h is unitarily equivalent to this compression of the strongly continuous selfadjoint semi-group H. For k and j in $\mathscr{K}$,

$$\langle U^*PH(t)PUk, j\rangle = \langle U^*H(t)Uk, j\rangle = (H(t)Uk, Uj) =$$
$$= \sum_{a,b} \langle h(a+b)(Uk)(a-t), (Uj)(b)\rangle = \langle h(t)k, j\rangle$$

and $h(t)=U^*PH(t)PU$, as desired.

Remark. If h satisfies Lemma 4.2, then $h(x)=PH(x)P$ where P is a projection and H a self-adjoint semi-group. Therefore, if $h(x)k=0$, then $H(x/2)Pk=0$ and $h(x/2)k=0$. Consequently, we can construct a sequence $x_n\to 0$ for which $h(x_n)k=0$. Since h is strongly continuous and $h(0)=I$, then $k=0$ and we see that $h(x)$ is one-to-one. Indeed, we see that h satisfies all of the properties of a symbol except possibly $\{h(x)\}$ being abelian.

Proof of Theorem 3. ii)$\Leftrightarrow$iii) by Lemma 4.

Assume that i) holds and (S, φ) is subnormal. By [5, Theorem 2.1] there exists an operator measure ϱ in $\mathscr{B}(\mathscr{L}^2(\mathscr{R}^+, \mathscr{K}))$ such that $\varrho[0, a]=I$ and

$$S_t^* S_t = \int_0^a r^t \, d\varrho(r).$$

By equation (4) then for each f in $\mathscr{L}^2(\mathscr{R}^+, \mathscr{K})$,

$$\varphi(x)^{-2}\varphi(x+t)^2 f(x) = \int_0^a r^t \big(d\varrho(r) f\big)(x)$$

except on a set of measure zero. We conclude then that for a given finite collection $f_0, \ldots, f_n$ of elements of $\mathscr{L}^2(\mathscr{R}^+, \mathscr{K})$ and all positive rational numbers t this equation holds except on a set E of measure zero. In particular, if $k_0, \ldots, k_n$ are elements of $\mathscr{K}$ and for $i=0, \ldots, n, f_i(x)=k_i$ for x in $[0, 1]$ and zero otherwise, then

$$\varphi(x)^{-2}\varphi(x+t)^2 k_i = \int_0^a r^t \big(d\varrho(r) f_i\big)(x)$$

for t rational and x in $[0, 1]\cap E$. Consequently, if $t_0, \ldots, t_n$ are rational and $x\in[0, 1]\cap E$, then

$$\sum_{i,j=0}^{n} \langle \varphi(x)^{-2}\varphi(x+t_i+t_j)^2 k_i, k_j\rangle = \sum_{i,j=0}^{n} \int_0^a r^{t_i+t_j} \langle d\big(\varrho(r) f_i\big)(x), f_j(x)\rangle.$$

We argue as in the proof of Lemma 4 to see that the right-hand side of the last equation is nonnegative. Therefore,

$$\sum_{i,j=0}^{n} \langle \varphi(x)^{-2}\varphi(x+t_i+t_j)^2 k_i, k_j\rangle \geqq 0$$

for all x in E. Using arguments similar to those in Lemma 1, we can show that $\text{s-}\lim_{x\to 0} \varphi(x)^{-2}\varphi(x+t)^2=\varphi(t)^2$ for all real t. Consequently,

$$\sum_{i,j=0}^{n} \langle \varphi(t_i+t_j)^2 k_i, k_j\rangle \geqq 0$$

for all finite collections $k_0, \ldots, k_n$ in $\mathscr{K}$ and $t_0, \ldots, t_n$ in the rationals (and hence, in the reals since φ^2 is strongly continuous). We now apply Lemma 4 to φ^2 and see that iii) holds.

Conversely, assume that iii) holds: ϱ_0 is an operator measure on $[0, a]$ such that $\varrho_0[0, a]=I$ and

$$\varphi^2(x) = \int_0^a r^x \, d\varrho_0(r). \tag{5}$$

Note that for each Borel set E, $\left(\int_E r^x d\varrho_0(r)\right)^2 \ll \varphi(x)^4$, so by the factorization theorem [2] $\varphi(x)^{-2}\int_E r^x d\varrho_0(r)$ is the unique operator C on $\mathscr{K}$ satisfying $\varphi(x)^2 C = \int_E r^x d\varrho_0(r)$, $\ker C = \ker \int_E r^x d\varrho_0(r)$, $C(\mathscr{K}) \subseteqq \varphi(x)^2(\mathscr{K})^-$ and $\|C\| \leqq 1$. Thus, we can define

$$\varrho(E, x) = \varphi(x)^{-2} \int_E r^x d\varrho_0(r) \tag{6}$$

for all x in $\mathscr{R}^+$. Since $\{\varphi(x)\}$ is abelian, so is ϱ_0 and it follows that $\varrho(E, x)$ is positive Hermitian and further, that if $F \subseteqq E$,

$$\varphi(x)[\varrho(E, x)-\varrho(F, x)]\varphi(x) = \int_{E-F} r^x \, d\varrho_0(r).$$

Since $\varphi(x)$ is one-to-one, $\varrho(E, x)$ is a monotone $\mathscr{B}(\mathscr{K})$-valued function on the Borel sets of $[0, a]$. Further, if

$$E = \bigcup_n E_n, \quad E_j \cap E_i = \emptyset \quad \text{for } (i \neq j), \text{ then}$$

$$\varphi(x)^2\varrho(E, x) = \int_E r^x \, d\varrho_0(r) = \text{s-}\lim_{n\to\infty} \sum_{i=1}^{n} \int_{E_i} r^x \, d\varrho_0(r) = \text{s-}\lim_{n\to\infty} \sum_{i=1}^{n} \varphi(x)^2 \varrho(E_i, x).$$

Thus, $\sum_{i=1}^{n} \varrho(E_i, x)$ converges strongly to $\varrho(E, x)$ on the dense set $\varphi(x)^2$ and consequently, on $\mathscr{K}$ since $\left\|\sum_{i=1}^{n} \varrho(E_i, x)\right\| \leqq 1$. Finally, observe that $\varrho(\emptyset, x)=0$ and $\varrho([0, a], x)=I$ for all x.

Define the set function ϱ with values in $\mathscr{B}(\mathscr{L}^2(\mathscr{R}^+, \mathscr{H}))$ by $(\varrho(E)f)(x) = \varrho(E, x)f(x)$. From our previous remarks concerning $\varrho(E, x)$, it follows that ϱ is monotone, $\varrho(\emptyset)=0$ and $\varrho([0, a])=I$. Using an argument similar to that used in Theorem 2 $\varrho(E)=\text{s-}\lim_{n\to\infty} \sum_{i=1}^{n} \varrho(E_i)$ when $E=\bigcup_n E_i$, $E_i \cap E_j=\emptyset$ (for $i\neq j$). Therefore, ϱ defines an operator measure. Finally, from (6) and (5), we have

$$\varphi(x)^2 \int_0^a s^t \, d\varrho(s, x) = \int_0^a s^{x+t} \, d\varrho_0(s) = \varphi(x+t)^2.$$

Therefore,

$$\varphi(x)^{-2} \varphi(x+t)^2 = \int_0^a s^t \, d\varrho(s, x).$$

We combine the last equation with equation (4) to conclude that

$$S_t^* S_t = \int_0^a s^t \, d\varrho(s).$$

Once again, we invoke [5, Theorem 2.1] and conclude that S is subnormal.

In [8, Theorem 22.3.1], it is shown that if T is a strongly continuous semi-group of self-adjoint operators, then T has a holomorphic extension whose maximal domain of analyticity is either the whole plane or the right half-plane. It follows immediately from Theorem 3 that the symbol φ of a subnormal w.t.s. has a holomorphic extension. Therefore, if two such symbols φ_1 and φ_2 agree on an infinite set with cluster point in their common domain of analyticity, they must agree everywhere.

Prior to characterizing quasinormal w.t.s., we restate a general characterization of quasinormal semi-groups [6, Theorem 6] in the w.t.s. terminology.

Theorem 5. *Let Q be a strongly continuous semi-group on a separable Hilbert space $\mathscr{H}$. Q is quasinormal if and only if Q is unitarily equivalent to the direct sum of a strongly continuous normal semi-group N and a w.t.s. (S, φ) where φ is a strongly continuous self-adjoint semi-group.*

A quasinormal semi-group is *pure* if there exists no nontrivial invariant subspace on which it is normal.

Corollary 6. *Every strongly continuous pure quasinormal semi-group is unitarily equivalent to a w.t.s. (S, φ) where φ is a strongly continuous self-adjoint semi-group.*

Corollary 7. *Let (S, φ) be a w.t.s. The following are equivalent:*

i) *(S, φ) is quasinormal,*

ii) *φ is a strongly continuous self-adjoint semi-group,*

iii) *there exists a spectral measure* ϱ *on* $[0, a]$ *such that*

$$\varphi(x) = \int_0^a r^x \, d\varrho(r).$$

Proof. Observe first that if (S, φ) is any w.t.s., then (S, φ) has no normal part. If $S_t^* S_t f = S_t S_t^* f$ for all t, then $\varphi(x)^{-2}\varphi(x+t)^2 f(x)=0$ for $0 \leqq x \leqq t$ and for all t. But $\varphi(x)^{-2}\varphi(x+t)^2$ is one-to-one so that $f=0$. The equivalence of i) and ii) now follows immediately from Theorem 5. The equivalence of ii) and iii) can either be derived from Theorem 5 or from [8, p. 588].

Example 4. Let s be a strongly continuous subnormal semi-group on a separable Hilbert space $\mathcal{H}$. Let $\varphi(x)=(s_x^* s_x)^{1/2}$. By [5, Theorem 2.1], there exists an operator measure ϱ on $[0, a]$ with $\varrho[0, a]=I$ and $\varphi(x)^2 = \int_0^a r^x d\varrho(r)$. We noted in the remark after Lemma 4 that φ satisfies all properties of a symbol except $\{\varphi(x)\}$ being abelian. If we assume $\{\varphi(x)\}$ abelian, then φ is a symbol and it follows from Theorem 3 that (S, φ) is a subnormal w.t.s.

During the development of the material in this paper, several questions arose which remain unanswered.

1. If φ is the symbol of a subnormal w.t.s. (S, φ), does there exist a strongly continuous semi-group s such that $\varphi(x)=(s_x^* s_x)^{1/2}$? In the last example, we saw that if φ is of this type, it does generate a subnormal w.t.s. However, if we start with a subnormal (S, φ), then by Theorem 3 $\varphi(x)^2 = \int_0^a r^t d\varrho(r)$. Thus, by [5, Theorem 2.1] φ acts like the positive part of some subnormal semi-group. The trick is to construct a function $u: \mathcal{R}^+ \to \mathcal{B}(\mathcal{H})$ such that each $u(x)$ is an isometry and $u\varphi$ is a strongly continuous semi-group.

2. More generally, we can ask whether each of the functions $h(x) = \int_0^a r^x d\varrho(r)$ in Lemma 4 is the square of the positive part of some strongly continuous subnormal semi-group. (Here, we do not require $\{h(x)\}$ to be abelian as we do for symbols.)

3. When are two w.t.s. (S, φ) and (T, ψ) unitarily equivalent or similar? In [5] it was shown in the numerical case, $\mathcal{H} = \mathcal{C}$, that similarity occurs if and only if there exist constants m and M such that $0 < m \leqq |\varphi(x)/\psi(x)| \leqq M < \infty$ for all x in $\mathcal{R}^+$ and in [4], it was shown that (T, ψ) is unitarily equivalent to (S, φ) if and only if $|\varphi(x)/\psi(x)|$ is constant on $\mathcal{R}^+$. Other questions were answered in [4] and [5] for the numerical case which may have interesting analogues in the operator case.

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DEPARTMENT OF MATHEMATICS
CENTRAL MICHIGAN UNIVERSITY
MT. PLEASANT, MICHIGAN 48859, U.S.A.