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# On the polar decomposition of an operator

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## 1. Introduction

An operator means a bounded linear operator on a Hilbert space. An operator $T$ can be decomposed into $T=UP$ where $U$ is a partial isometry and $P=|T|=(T^*T)^{1/2}$ with $N(U)=N(P)$, where $N(X)$ denotes the kernel of an operator $X$. The kernel condition $N(U)=N(P)$ uniquely determines $U$ and $P$ of this polar decomposition $T=UP$ [2]. In this paper, $T=UP$ denotes the right-handed polar decomposition which satisfies the kernel condition $N(U)=N(P)$. In order to prove our results, this kernel condition $N(U)=N(P)$ is essential. When $T=UP$ where $U$ is partial isometry and $P=|T|$, but the kernel condition $N(U)=N(P)$ is not necessarily satisfied, we say that $T=UP$ is merely "a decomposition" (not the polar decomposition) of $T$. When $T$ commutes with $S$ and $S^*$, we say that $T$ doubly commutes with $S$.

Our two main results are as follows. When $T_1=U_1P_1$ and $T_2=U_2P_2$ are polar decompositions of $T_1$ and $T_2$ with $N(U_1)=N(P_1)$ and $N(U_2)=N(P_2)$, respectively, then $T_1$ doubly commutes with $T_2$ if and only if $U_1^*, U_1$ and $P_1$ commute with $U_2^*, U_2$ and $P_2$. As an application of this result we show that for every normal operator $T$, there exists a unitary $U$ such that $T=UP=PU$ and $U$ and $P$ commute with $V^*, V$ and $|A|$ of the polar decomposition $A=V|A|$ of any operator $A$ which commutes with $T$ and $T^*$. This second result yields a familiar and well-known result, see RIESZ and SZ.-NAGY [4].

An operator $T$ is called quasinormal if $T$ commutes with $T^*T$ and hyponormal if $T^*T\geqq TT^*$. The inclusion relation of these classes of nonnormal operators is as follows:

$$\text{Normal} \subset \text{Quasinormal} \subset \text{Hyponormal}$$

and the inclusions above are all proper [2].

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## 2. A necessary and sufficient condition for $T_1T_2=T_2T_1$ and $T_1T_2^*=T_2^*T_1$

Theorem 1. *If* $T=UP$ *is the polar decomposition of an operator* $T$, *then* $U$ *and* $P$ *commute with* $A$ *and* $A^*$, *where* $A$ *denotes any operator which commutes with* $T$ *and* $T^*$.

Proof. Let $A$ be an operator such that $AT=TA$ and $AT^*=T^*A$. Then $(T^*T)A=A(T^*T)$, that is, $P^2A=AP^2$ where $P=|T|$, whence $PA=AP$, or equivalently $PA^*=A^*P$. The conditions $AT-TA=0$ and $PA=AP$ yield $AUP-UPA=(AU-UA)P=0$, so that $AU-UA$ annihilates $\overline{R(P)}$. If $x\in N(P)=N(U)$, then $Px=0$ and $Ux=0$, so that $PAx=APx=0$, that is, $Ax\in N(P)=N(U)$, hence $UAx=0$, therefore $AU-UA$ annihilates $N(P)$ too, and it follows that $AU-UA=0$ on $H=\overline{R(P)}\oplus N(P)$. Similarly, the conditions $AT^*-T^*A=0$ and $PA=AP$ imply $APU^*-PU^*A=P(AU^*-U^*A)=0$. By taking adjoint of this equation, $(UA^*-A^*U)P=0$, so that $UA^*-A^*U$ annihilates $\overline{R(P)}$. If $x\in N(P)=N(U)$, then $Px=0$ and $Ux=0$, so that $PA^*x=A^*Px=0$ (since $PA^*=A^*P$ holds), therefore $A^*x\in N(P)=N(U)$, $UA^*x=0$, whence $UA^*-A^*U$ annihilates $N(P)$, too, and it follows that $UA^*-A^*U=0$ and so the proof is complete.

Our main result is the following extension of Theorem 1 which gives a necessary and sufficient condition under which an operator doubly commutes with another.

Theorem 2. *Let* $T_1=U_1P_1$ *and* $T_2=U_2P_2$ *be the polar decompositions of* $T_1$ *and* $T_2$, *respectively. Then the following conditions are equivalent:*

(A) $T_1$ *doubly commutes with* $T_2$.

(B) $U_1^*, U_1$ *and* $P_1$ *commute with* $U_2^*, U_2$ *and* $P_2$.

(C) *The following five equations are satisfied:* (1) $P_1P_2=P_2P_1$, (2) $U_1P_2=P_2U_1$, (3) $P_1U_2=U_2P_1$, (4) $U_1U_2=U_2U_1$ *and* (5) $U_1^*U_2=U_2U_1^*$.

Proof. (B)↔(C). (B)→(C) is trivial and (B) follows from (C) by taking adjoints of (2), (3), (4) and (5).

(A)→(C). Assume (A), then by Theorem 1, we have

$$(*)\qquad T_1P_2=P_2T_1 \quad\text{and}\quad T_1^*P_2=P_2T_1^*,$$

$$(**)\qquad T_1U_2=U_2T_1 \quad\text{and}\quad T_1^*U_2=U_2T_1^*.$$

By $(*)$ and by Theorem 1, we have (1), (2), and also by $(**)$ and by Theorem 1, we have (3), (4), and $U_1U_2^*=U_2^*U_1$, or equivalently (5).

(B)→(A). (A) easily follows from (B). Hence the proof is complete.

Theorem 2 yields the following well-known fact. In Theorem 2, $U_1^*U_1$ and $U_1U_1^*$ commute with $U_2$, $P_2$ and $T_2$, that is, both the initial space and the final space of $U_1$ reduce $U_2, P_2$ and $T_2$. Similarly, both the initial space and the final space of $U_2$ reduce $U_1, P_1$ and $T_1$. In Section 4, Theorem 2 will be extended to Theorem 5 in the intertwining case.

Corollary 1. *Let $T_1 = U_1 P_1$ and $T_2 = U_2 P_2$ be the polar decompositions of $T_1$ and $T_2$, respectively. If $T_1$ doubly commutes with $T_2$, then $T_1 T_2 = (U_1 U_2)(P_1 P_2)$ is the polar decomposition of $T_1 T_2$.*

Proof. By (4) and (5) in (C) of Theorem 2, we have

$$U_1 U_2 (U_1 U_2)^* U_1 U_2 = U_1 U_2 U_2^* U_1^* U_1 U_2 = U_1 U_1^* U_1 U_2 U_2^* U_2 = U_1 U_2$$

since $U_1$ and $U_2$ are both partial isometries, whence $U_1 U_2$ is a partial isometry. By (1) in (C) of Theorem 2, we have

$$|T_1 T_2|^2 = (T_1 T_2)^* (T_1 T_2) = (T_1^* T_1)(T_2^* T_2) = P_1^2 P_2^2 = (P_1 P_2)^2.$$

$N(U_2 U_1) = N(U_1 U_2) = N(P_1 P_2)$ is obtained by (2) and (4) in (C) of Theorem 2 as follows: $x \in N(U_2 U_1) \leftrightarrow U_2 U_1 x = 0 \leftrightarrow U_1 x \in N(U_2) = N(P_2) \leftrightarrow P_2 U_1 x = 0 \leftrightarrow U_1 P_2 x = 0 \leftrightarrow P_2 x \in N(U_1) = N(P_1) \leftrightarrow P_1 P_2 x = 0 \leftrightarrow x \in N(P_1 P_2)$, so the proof is complete.

Theorem 2 easily implies the following result which is a more precise statement than Theorem 1 on the polar decomposition.

Corollary 2 (The polar decomposition). *Every operator $T$ can be expressed in the form $U|T|$ where $U$ is a partial isometry with $N(U) = N(|T|)$. This kernel condition uniquely determines $U$; $U$ and $|T|$ commute with $V^*$, $V$ and $|A|$ of the polar decomposition $A = V|A|$ of any operator $A$ commuting with $T$ and $T^*$.*

Proof. The first half of the result follows by [2] and the second follows by Theorem 2 since we put $T = T_2$ and $A = T_1$ in Theorem 2.

Theorem 2 also yields the following result which is a characterization of normal operators.

Corollary 3. *Let $T = UP$ be the polar decomposition of an operator $T$. Then $T$ is normal if and only if $U$ commutes with $P$ and $U$ is unitary on $N(T)^\perp$.*

Proof. Put $T = T_1 = T_2$ in Theorem 2, then the condition ($A$) in Theorem 2 is equivalent to the normality of $T$ and the condition (C) is equivalent to that $U$ commutes with $P$ and $U^* U = U U^*$. So $U$ is unitary on the initial space of $U$ which equals $N(T)^\perp$.

Theorem 3. *Let $T$ be normal. Then there exists a unitary operator $U$ such that $T = UP = PU$ and both $U$ and $P$ commute with $V^*$, $V$ and $|A|$ of the polar decomposition $A = V|A|$ of any operator commuting with $T$ and $T^*$.*

Proof. Let $T=U_1P=PU_1$ be the polar decomposition of a normal operator $T$ and let $A=V|A|$ be the polar decomposition of $A$. By Corollary 3, $U_1^*U_1=U_1U_1^*$, that is, the initial space $M$ of $U_1$ coincides with the final space $N$, so that $M$ reduces $U_1$; consequently $U_1P_M=P_MU_1$ where $P_M=U_1^*U_1$ denotes the projection of $H$ onto $M$. Put $U=U_1P_M+1-P_M$ by the standard technique [4], then $U_1^*U_1= =U_1U_1^*$ and $U_1P_M=P_MU_1$ yield the following:

$$U^*U=(P_MU_1^*+1-P_M)(U_1P_M+1-P_M)=1,$$

$$UU^*=(U_1P_M+1-P_M)(P_MU_1^*+1-P_M)=1.$$

Hence $U$ is unitary and we show that $U$ is the desired unitary as follows. As $P_MP=P$, that is, $PP_M=P$, so we have

$$UP=(U_1P_M+1-P_M)P=U_1P_MP+P-P_MP=U_1P=T$$

and similarly we have $T=PU=PU_1$, therefore $T=UP=PU$. By Corollary 2' $U_1$ and $P$ commute with $V^*$, $V$ and $|A|$, so $P_M=U_1^*U_1$ commutes with $V^*$, $V$ and $|A|$, that is, $P_M|A|=|A|P_M$, $P_MV=VP_M$ and $P_MV^*=V^*P_M$. By Corollary 2, $P$ commutes with $V^*$, $V$ and $|A|$. Hence we have only to show that $U$ commutes with $V^*$, $V$ and $|A|$.

Clearly,

$$VU=V(U_1P_M+1-P_M)=VU_1P_M+V(1-P_M)=$$

$$=U_1VP_M+V(1-P_M)=(U_1P_M+1-P_M)V=UV.$$

Similarly we have $V^*U=UV^*$ and $|A|U=U|A|$, so the proof is complete.

We remark that $U$ and $P$ commute with $A=V|A|$ in Theorem 3, so that Theorem 3 yields the following well-known result.

Theorem A. [4] *Every normal operator $T$ can be written in the form $UP$ where $P$ is positive and $U$ may be taken to be unitary and such that $U$ and $P$ commute with each other and with all operators commuting with $T$ and $T^*$.*

Corollary 4. *Let $T_1=U_1P_1$ be the polar decomposition of an operator $T_1$, and let $T_2=U_2P_2$ be the decomposition described in Theorem 3 of a normal operator $T_2$. Then the following conditions are equivalent.*

(A) *$T_1$ commutes with $T_2$.*

(B) *$U_1^*$, $U_1$ and $P_1$ commute with $U_2^*$, $U_2$ and $P_2$.*

(C) *$U_1$ and $P_1$ commute with $U_2$ and $P_2$.*

Proof. As $T_2$ is normal, (A) implies $T_1T_2^*=T_2^*T_1$ by the Fuglede—Putnam Theorem [2], so by Theorem 3, $U_2$ and $P_2$ commute with $U_1^*$, $U_1$ and $P_1$, whence (B) is shown. (C) trivially follows from (B) and also (A) easily follows from (C), so the proof is complete.

## 3. Nonnormal operators

Theorem 4. *Suppose that $N(T) \subset N(T^*)$ and let $T = UP$ be the polar decomposition of $T$. Then there exists an isometry $U_1$ such that $T = U_1 P$ and both $U_1$ and $P$ commute with $V^*$, $V$ and $|A|$ of the polar decomposition $A = V|A|$ of any operator $A$ commuting with $T$ and $T^*$. In case $N(T) = N(T^*)$, $U_1$ can be chosen to be unitary.*

Proof. The condition $N(T) \subset N(T^*)$ implies $N(T)^\perp \supset N(T^*)^\perp = \overline{R(T)}$, so that $U$ is a partial isometry from the initial space $M = N(T)^\perp$ into $M$, whence $M$ reduces $U$; consequently $UP_M = P_M U$ where $P_M$ denotes the projection of $H$ onto $M$ and $P_M = U^*U$. Put $U_1 = UP_M + 1 - P_M$. In the same way as in the proof of Theorem 3, $U_1^* U_1 = 1$, $U_1 P = UP = T$, and the commutativity stated in Theorem 4 can be shown. If $N(T) = N(T^*)$, then $U$ is unitary on $M$, so that $U_1$ defined above turns out to be unitary since $U_1 U_1^* = 1$ can also be shown.

Remark 1. If $T$ is invertible or hyponormal, then $N(T) \subset N(T^*)$ holds, so that Theorem 4 holds for these operators.

Corollary 5. *Let $T$ be quasinormal. Then there exists an isometry $U$ such that $T = UP = PU$ and $U$ and $P$ commute with $V^*, V$ and $|A|$ of the polar decomposition $A = V|A|$ of any operator $A$ commuting with $T$ and $T^*$.*

Proof. If $T$ is quasinormal, then $T$ is hyponormal, so that $T$ satisfies $N(T) \subset N(T^*)$. $T$ commutes with $T^*T$ by the quasinormality of $T$, so that $P = (T^*T)^{1/2}$ commutes with $T$ and $T^*$. Put $A = P$ in Theorem 4, so the isometry $U$ chosen in Theorem 4 commutes with $P$ and the rest follows from Theorem 4.

We remark that Theorem 3 can be alternatively derived from Theorem 4 and Corollary 5.

## 4. Intertwining case

Theorem 2 yields the following result which is closely related to the Fuglede—Putnam theorem.

Theorem 5. *Let $T_k = U_k P_k$ be the polar decompositions of $T_k$ for $k = 1, 2$ and 3. Then the following conditions are equivalent.*

(A) $$T_1 T_2 = T_2 T_3 \quad \text{and} \quad T_1^* T_2 = T_2 T_3^*.$$

(B) (1) $P_3 P_2 = P_2 P_3$, (2) $P_1 U_2 = U_2 P_3$, (3) $U_3 P_2 = P_2 U_3$, (4) $U_1 U_2 = U_2 U_3$ *and* (5) $U_1^* U_2 = U_2 U_3^*$.

Proof. We put $\hat{A}$ and $\hat{T}$ on $H\oplus H$ as follows:

$$\hat{A} = \begin{pmatrix} T_1 & 0 \\ 0 & T_3 \end{pmatrix} \quad \text{and} \quad \hat{T} = \begin{pmatrix} 0 & T_2 \\ 0 & 0 \end{pmatrix}.$$

Let $\hat{A}=\hat{U}_1\hat{P}_1$ and $\hat{T}=\hat{U}_2\hat{P}_2$ be the polar decompositions of $\hat{A}$ and $\hat{T}$, respectively, where $\hat{U}_1, \hat{P}_1, \hat{U}_2$ and $\hat{P}_2$ are as follows on $H\oplus H$ :

$$\hat{U}_1 = \begin{pmatrix} U_1 & 0 \\ 0 & U_3 \end{pmatrix}, \quad \hat{P}_1 = \begin{pmatrix} P_1 & 0 \\ 0 & P_3 \end{pmatrix}, \quad \hat{U}_2 = \begin{pmatrix} 0 & U_2 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \hat{P}_2 = \begin{pmatrix} 0 & 0 \\ 0 & P_2 \end{pmatrix}.$$

The condition (A) assures that $\hat{A}\hat{T}=\hat{T}\hat{A}$ and $\hat{A}^*\hat{T}=\hat{T}\hat{A}^*$, so by Theorem 2, these relations are equivalent to that $\hat{U}_1^*$, $\hat{U}_1$ and $\hat{P}_1$ commute with $\hat{U}_2$ and $\hat{P}_2$. Then, by simple calculations, $\hat{P}_1\hat{P}_2=\hat{P}_2\hat{P}_1 \leftrightarrow$ (1), $\hat{U}_2\hat{P}_1=\hat{P}_1\hat{U}_2 \leftrightarrow$ (2), $\hat{U}_1\hat{P}_2=\hat{P}_2\hat{U}_1 \leftrightarrow$ (3), $\hat{U}_1\hat{U}_2=\hat{U}_2\hat{U}_1 \leftrightarrow$ (4), and $\hat{U}_1^*\hat{U}_2=\hat{U}_2\hat{U}_1^* \leftrightarrow$ (5), whence the proof is complete.

Combining the techniques in Corollary 4 and Theorem 5, we have

Corollary 6. *Let* $T_1=U_1P_1$, $T_3=U_3P_3$ *be the decompositions described in Theorem* 3 *of some normal operators* $T_1, T_3$, *and let* $T_2=U_2P_2$ *be the polar decomposition of an operator* $T_2$. *Then the following conditions are equivalent:*

(A) $T_1T_2=T_2T_3$.

(B) (1), (2), (3), (4), *and* (5) *in Theorem* 5 *hold.*

(C) (1), (2), (3), *and* (4) *in Theorem* 5 *hold.*

Let $\{p_\alpha\}$ be a family of polynomials of $T$ and $T^*$. A property $\Sigma$ of $T$ is said to be algebraic definite (resp. semidefinite) with $\{p_\alpha\}$ provided that $T$ has $\Sigma$ if and only if $p_\alpha(T, T^*)=0$ (resp. $p_\alpha(T, T^*)\geqq 0$) for all $\alpha$ [1].

Next we show an application of Theorem 5.

Corollary 7. *Let* $T_k=U_kP_k$ *be the polar decompositions of* $T_k$ *for* $k=1, 2$ *and* 3 *and let* $T_1T_2=T_2T_3$ *and* $T_1^*T_2=T_2T_3^*$. *Then*

(1) $\overline{R(T_2)}$ *reduces* $U_1, P_1$ *and* $T_1$; $N(T_2)$ *reduces* $U_3, P_3$ *and* $T_3$.

(2) $U_1 \mid \overline{R(T_2)}$ (*resp.* $P_1 \mid \overline{R(T_2)}$, $T_1 \mid \overline{R(T_2)}$) *is unitary equivalent to* $U_3 \mid N(T_2)^\perp$ (resp. $P_3 \mid N(T_2)^\perp$, $T_3 \mid N(T_2)^\perp$).

(3) *When* $T_2$ *has dense range, then if* $U_3$ (*resp.* $P_3$ *and* $T_3$) *has an algebraic definite property* $\Sigma$ *with polynomials* $\{p_\alpha\}$, *then so has* $U_1$ (*resp.* $P_1$ *and* $T_1$).

(4) *When* $T_2$ *is injective, then if* $U_1$ (*resp.* $P_1$ *and* $T_1$) *has an algebraic definite property* $\Sigma$ *with polynomials* $\{p_\alpha\}$, *then so has* $U_3$ (*resp.* $P_3$ *and* $T_3$).

Proof. (1) By (5), (4) and (2) in Theorem 5

$$(U_2U_2^*)U_1 = U_2U_3U_2^* = U_1(U_2U_2^*), \quad (U_2U_2^*)P_1 = U_2P_3U_2^* = P_1(U_2U_2^*),$$

whence $\overline{R(T_2)}$ reduces $U_1, P_1$ and also $T_1$. By (4), (5) and (2) in Theorem 5,

$$(U_2^* U_2)U_3 = U_2^* U_1 U_2 = U_3(U_2^* U_2), \quad (U_2^* U_2)P_3 = U_2^* P_1 U_2 = P_3(U_2^* U_2),$$

whence $N(T_2)$ reduces $U_3$, $P_3$ and also $T_3$.

(2) By (2) and (1) in Theorem 5,

$$\text{(i)} \qquad P_1 U_2 P_2 x = U_2 P_3 P_2 x = U_2 P_2 P_3 x \quad \text{for all } x.$$

Let $P_1' = P_1 \mid \overline{R(T_2)}$ and $P_3' = P_3 \mid N(T_2)^\perp$. Let $V$ be defined by $Vy = U_2 y$ for all $y \in N(T_2)^\perp$. This $V$ maps from $N(T_2)^\perp = N(P_2)^\perp = \overline{R(P_2)}$ onto $\overline{R(T_2)}$, so $V$ is a surjective isometry, i.e., $V$ is unitary. As $P_2 x$ and $P_2 P_3 x$ belong to $N(T_2)^\perp$ and $U_2 P_2 x$ belongs to $\overline{R(T_2)}$, (i) implies $P_1' V y = V P_3' y$ for all $y \in N(T_2)^\perp$, so $P_1'$ is unitary equivalent to $P_3'$. Similarly (4) and (3) in Theorem 5 yield

$$\text{(ii)} \qquad U_1 U_2 P_2 x = U_2 U_3 P_2 x = U_2 P_2 U_3 x \quad \text{for all } x.$$

Let $U_1' = U_1 \mid \overline{R(T_2)}$ and $U_3' = U_3 \mid N(T_2)^\perp$. As $P_2 x$ and $P_2 U_3 x$ belong to $N(T_2)^\perp$ and $U_2 P_2 x$ belongs to $\overline{R(T_2)}$, (ii) implies $U_1' V y = V U_3' y$ for all $y \in N(T_2)^\perp$. The third unitary equivalence relation follows by the first and second relations obtained above.

(3) When $T_2$ has dense range, then by (2), $U_1 \mid \overline{R(T_2)} = U_1$ is unitary equivalent to $U_3' = U_3 \mid N(T_2)^\perp$. If $U_3$ has an algebraic definite property, then $U_3'$ also has it, and consequently so has $U_1$. The rest can be shown similarly.

(4) When $T_2$ is injective, then by (2), $U_3 \mid N(T_2)^\perp = U_3$ is unitary equivalent to $U_1' = U_1 \mid \overline{R(T_2)}$ and the proof goes in a similar way as above.

We remark that the algebraic definite property can be replaced by semidefinite property in (3) and (4) of Corollary 7. Also we remark that in [3] only the equivalence relation between $T_1 \mid \overline{R(T_2)}$ and $T_3 \mid N(T_2)^\perp$ is shown, and in [1] the algebraic definite property relation between $T_1$ and $T_3$ is shown when $T_2$ has dense range, and in [5] also when $T_2$ has dense range and injective.

*Added in proof.* Theorem 2 is also found in M. Takesaki, Theory of operator algebras I, Springer, 1979, however, the proof we gave here is more elementary in that it merely uses kernel conditions and avoids operator algebraic considerations. We would express our thanks to Professor J. Tomiyama for his valuable comments after reading our preprint.

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