Acta Sci. Math. 46 (1983), 287—293

Note on operators of class $C_0(1)$

SHLOMO ROSENOER

1. Introduction. Let H be a separable Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. The ultraweak topology on $B(H)$ is the weak topology relative to the family of functionals φ of the form

$$\varphi(T) = \sum_{n=1}^{\infty} (Tx_n, y_n) \tag{1}$$

where $\{x_n\}, \{y_n\} \subset H$ and $\sum_{n=1}^{\infty} (\|x_n\|^2 + \|y_n\|^2) < \infty$.

The following theorem occurs in HADWIN and NORDGREN [1].

Theorem 1. *Let $\mathcal{L}$ be an ultraweakly closed subspace of $B(H)$ and φ an ultraweakly continuous linear functional on $\mathcal{L}$ with $\|\varphi\| \leqq 1$. Then for every $\varepsilon > 0$ there is an extension of φ to $B(H)$ which is a functional of the form* (1) *with*

$$\left(\sum_{n=1}^{\infty} \|x_n\|^2\right)^{1/2} \left(\sum_{n=1}^{\infty} \|y_n\|^2\right)^{1/2} < 1 + \varepsilon.$$

Let $\mathcal{A}$ be a unital ultraweakly closed subalgebra of $B(H)$. We say that $\mathcal{A}$ *has property* $D_\sigma(1)$ if every ultraweakly continuous linear functional φ on $\mathcal{A}$ can be represented in the form $\varphi(T) = (Tx, y)$ with some x, y in H. If in addition $r \geqq 1$ and, for every $s > r$, x and y can be chosen so that $\varphi(T) = (Tx, y)$ for all T in $\mathcal{A}$ and $\|x\|\,\|y\| \leqq s\|\varphi\|$, then we say that $\mathcal{A}$ *has property* $D_\sigma(r)$. An operator T is said to *have property* D_σ *or* $D_\sigma(r)$ if $\mathcal{A}(T)$ has the respective property, where $\mathcal{A}(T)$ denotes the unital ultraweakly closed algebra generated by T.

The main purpose of this note is to show that the operators of the class $C_0(1)$ have property $D_\sigma(1)$. From this we deduce that the commutant of the Volterra operator V defined by $(Vf)(x) = \int_0^x f(y)\,dy$ in $L^2(0, 1)$ is the minimal unital ultraweakly closed algebra with linearly ordered invariant subspace lattice containing V.

Received December 1, 1980.

In conclusion we prove that a contraction with sufficiently large spectra cannot be reductive.

2. Main result. For any subset $\mathcal{S}$ of $B(H)$, $\mathcal{S}'$ denotes its commutant, lat $\mathcal{S}$ the collection of closed subspaces in H invariant under every operator in $\mathcal{S}$, and alg lat $\mathcal{S}$ the algebra of all operators in $B(H)$ leaving each element of lat $\mathcal{S}$ invariant. $\mathcal{S}$ is called *reflexive* if $U(\mathcal{S})=\text{alg lat}\,\mathcal{S}$, where $U(\mathcal{S})$ is the weakly closed unital algebra generated by $\mathcal{S}$.

Let N be a positive integer. The class $C_0(N)$ of operators is defined as the set of completely non-unitary contractions $T\in B(H)$ for which $T^n\to 0$, $T^{*n}\to 0$ (strongly) and $\dim(I-T^*T)(H)=\dim(I-TT^*)(H)=N$. The operators of class $C_0(1)$ admit the following description [2]. Let U be the canonical unilateral shift, that is the operator of multiplication by the independent variable λ in H^2, and let $m(\lambda)$ be an inner function. Denote by $H(m)$ the subspace $H^2\ominus mH^2$ and define the operator $S(m)$ in $H(m)$ by

$$S(m) = P_{H(m)}U.$$

Then every operator of class $C_0(1)$ is unitarily equivalent to $S(m)$ for an appropriate inner function m. Alternatively, one can view the operators of class $C_0(1)$ as restrictions of the backward shift U^* to its invariant subspaces.

Theorem 2. *Every operator of the class $C_0(1)$ has property $D_\sigma(1)$.*

Proof. By virtue of the preceding remark, it is enough to show that if U is a (cyclic) unilateral shift in H and $L\in\text{lat}\,U^*$, then $T=U^*\,|\,L$ has property $D_\sigma(1)$. Suppose φ is an ultraweakly continuous functional on $\mathcal{A}(T)$ with $\|\varphi\|\leqq 1$, and $\varepsilon>0$. By Theorem 1, we may assume that for every $S\in\mathcal{A}(T)$,

$$\varphi(S) = \sum_{n=1}^{\infty}(Sx_n, y_n).$$

where $\{x_n\}, \{y_n\}\subset L$ and $\left(\sum_{n=1}^{\infty}\|x_n\|^2\right)^{1/2}\left(\sum_{n=1}^{\infty}\|y_n\|^2\right)^{1/2}<1+\varepsilon$. Let $\mathbf{H}$ denote the infinite Hilbert sum $H\oplus H\oplus\ldots\oplus H\oplus\ldots$. Then the vectors $\mathbf{x}=x_1\oplus x_2\oplus\ldots\oplus\ldots\oplus x_n\oplus\ldots$ and $\mathbf{y}=y_1\oplus y_2\oplus\ldots\oplus y_n\oplus\ldots$ are in $\mathbf{H}$ and the operator $\mathbf{U}=U\oplus U\oplus\ldots\oplus U\oplus\ldots$ is in $B(\mathbf{H})$. Let $M=\bigvee_{n=1}^{\infty}\mathbf{U}^n\mathbf{y}$. Since $\mathbf{U}\,|\,M$ is a cyclic completely non-unitary isometry, it is unitarily equivalent to U. Hence there is an isometry W from H into $\mathbf{H}$ such that $W(H)=M$ and

$$WU = \mathbf{U}W. \tag{2}$$

Let $T_n=P_nW$, where P_n is the projection in $\mathbf{H}$ onto the nth coordinate subspace. Clearly $T_n \in B(H)$ and for every $x \in H$,

$$Wx = T_1 x \oplus T_2 x \oplus \ldots \oplus T_n x \oplus \ldots. \tag{3}$$

From (2) and (3) it follows that $T_n U = UT_n$ for every n. Let $y_0 = W^* \mathbf{y}$. Then $T_n y_0 = y_n$ and $\|y_0\|^2 = \|\mathbf{y}\|^2 = \sum_{n=1}^{\infty} \|y_n\|^2$. Since

$$(W^* \mathbf{x}, z) = (\mathbf{x}, Wz) = \sum_{n=1}^{\infty} (x_n, T_n z) = \sum_{n=1}^{\infty} (T_n^* x_n, z),$$

for every $z \in H$, we can assert that the series $\sum_{n=1}^{\infty} T_n^* x_n$ converges weakly to some $x_0 \in H$ and

$$\|x_0\| = \|W^* \mathbf{x}\| \leqq \|\mathbf{x}\| = \left(\sum_{n=1}^{\infty} \|x_n\|^2 \right)^{1/2}.$$

Moreover, since L is a hyperinvariant subspace of U^*, x_0 is actually in L. Now for every $S \in \mathscr{A}(T)$,

$$\varphi(S) = \sum_{n=1}^{\infty} (Sx_n, y_n) = \sum_n (Sx_n, T_n y_0) = \sum_n (T_n^* S x_n, y_0) =$$

$$= \sum_n (ST_n^* x_n, y_0) = \big(S(\sum_n T_n^* x_n), y_0\big) = (Sx_0, y_0) = (Sx_0, Py_0),$$

where P denotes the projection in H onto L. Finally,

$$\|x_0\| \, \|Py_0\| \leqq \|x_0\| \, \|y_0\| \leqq \left(\sum_{n=1}^{\infty} \|x_n\|^2 \right)^{1/2} \left(\sum_{n=1}^{\infty} \|y_n\|^2 \right)^{1/2} < 1 + \varepsilon,$$

which completes the proof.

The proofs of the assertions of Corollary 3 below are either obvious or can be found in [1] (if we note that by [2, Corollary VI. 4.3.7], for every operator T of class $C_0(1)$ we have $\{T\}' = \mathscr{A}(T)$). For the definitions of an attainable family and direct integral see also [1].

Corollary 3. *Let (X, μ) be a measure space and $\{\mathscr{A}(T_x)\}_{x \in X}$ an attainable family of algebras, where T_x is an operator of class $C_0(1)$ for every $x \in X$. Denote by $\mathscr{A}$ the direct integral of the algebras $\mathscr{A}(T_x)$: $\mathscr{A} = \int_X^{\oplus} \mathscr{A}(T_x)\, d\mu(x)$. Let $\mathscr{A}_1$ be a unital ultraweakly closed subalgebra of $\mathscr{A}$. Then,*

(a) *the weak and ultraweak topologies coincide on $\mathscr{A}$,*

(b) *$\mathscr{A}_1$ has property $D_\sigma(1)$,*

(c) *if $\mathscr{A}_2$ is a unital ultraweakly closed subalgebra of $\mathscr{A}$ and* $\operatorname{lat} \mathscr{A}_1 \subseteq \operatorname{lat} \mathscr{A}_2$, *then $\mathscr{A}_1 \supseteq \mathscr{A}_2$,*

(d) *if* $\mathscr{A}_1$ *is reflexive, then every unital ultraweakly closed subalgebra of* $\mathscr{A}_1$ *is reflexive,*

(e) *if* T_x *reflexive for almost every* $x \in X$, *then* $\mathscr{A}_1$ *is reflexive,*

(f) $\mathscr{A}_1 = \mathscr{A}_1'' \cap \text{alg lat } \mathscr{A}_1$,

(g) $\mathscr{A}_1^{(2)} = \{S \oplus S \mid S \in \mathscr{A}_1\}$ *is reflexive.*

We now give two examples to illustrate Theorem 2.

Corollary 4. *For every positive integer* n, *let* H_n *be a finite dimensional Hilbert space and* J_n *a Jordan cell in* $B(H_n)$ *(with respect to some orthonormal basis in* H_n*). If* $T_n J_n = J_n T_n$ *and* $T = T_1 \oplus T_2 \oplus \ldots \oplus T_n \oplus \ldots$ *is a bounded opera- then* $\mathscr{A}(T) = \{T\}' \cap \text{alg lat } T$.

Proof. It suffices to note that, by a theorem of BRICKMAN and FILLMORE [3], for any operator S in a finite dimensional space, $\mathscr{A}(S) = \{S\}' \cap \text{alg lat } S$, and then we apply corollary 3 (f).

Next we consider the Volterra operator V defined by $(Vf)(x) = \int_0^x f(y)\,dy$ in $L^2(0, 1)$. It is well known that V is quasinilpotent and unicellular. FOIAŞ and WILLIAMS [7] gave an example of a unicellular operator in $\{V\}'$ whose spectrum contains more than one point. Here we prove that every unicellular operator commuting with V is an ultraweak generator of the commutant of V.

Corollary 5. *Suppose* $\mathscr{B}$ *is an ultraweakly closed unital algebra strictly contained in* $\{V\}'$. *Then* lat $\mathscr{B}$ *is not a chain. In particular, if* T *commutes with* V *and* T *is unicellular, then* $\mathscr{A}(T) = \{V\}'$.

Proof. SARASON [4] pointed out that V commutes with the operator $S(m)$ where $m(\lambda) = \exp\{(\lambda+1)/(\lambda-1)\}$. Since $\{S(m)\}' = \mathscr{A}(S(m))$, $V \in \mathscr{A}(S(m))$. On the other hand, by [5], the commutant of V is the weak closure of the polynomials of V, so that $\{V\}' = \mathscr{A}(V) \subseteq \mathscr{A}(S(m))$. Now suppose lat $\mathscr{B}$ is a chain. Since lat V is a maximal chain, lat $V =$ lat $\mathscr{B}$. By Corollary 3 (c), $\mathscr{B}$ coincides with $\{V\}'$. The obtained contradiction proves the assertion.

Now we note that Theorem 2 yields the following factorization theorem. By H_0^1 we denote the subspace of H^1 consisting of the functions vanishing at 0.

Theorem 6. *Let* m *be an inner function,* $f, g \in H(m) = H^2 \ominus mH^2$ and $\inf\{\|f\bar{g} - h\|_1 \mid h \in H_0^1\} \leq 1$. *Then for every* $\varepsilon > 0$, *there exist* $f_1, g_1 \in H(m)$ *such that* $\|f_1\|_2 \|g_1\|_2 < 1 + \varepsilon$ *and* $f\bar{g} - f_1 \bar{g}_1 \in H_0^1$.

Proof. Let $T = PU \mid H(m)$, where P is the projection onto $H(m)$, and denote by φ the functional on $\mathscr{A}(T)$ defined by $\varphi(X) = (Xf, g)$. Choose $X \in \mathscr{A}(T)$ with $\|X\| = 1$. Then, by [2, Corollary VI. 4.3.7], there exists an analytic Toeplitz operator Y, $\|Y\|=1$, such that $PY\,|\,H(m)=X$. Since L^1/H_0^1 is the pre-dual of H^∞,

$$|\varphi(X)| = |(Xf, g)| = |(Yf, g)| \leqq 1,$$

so that $\|\varphi\|\leqq 1$. By Theorem 2, φ can be represented in the form $\varphi(X)=(Xf_1, g_1)$ where $f_1, g_1\in H(m)$ and $\|f_1\|_2\|g_1\|_2<1+\varepsilon$. But then, for every $Y\in\{U\}'$,

$$(Yf_1, g_1) = (PYf_1, g_1) = (PYf, g) = (Yf, g),$$

so that $f\bar{g}-f_1\bar{g}_1\in H_0^1$.

In Proposition 7 below we shall prove that for certain operator algebras the possibility of such "factorization" implies property $D_\sigma(r)$. $BC(H)$ denotes the ideal of compact operators in $B(H)$. $\omega_{x,y}$ is a functional on $B(H)$ defined by $\omega_{x,y}(T)=(Tx, y)$.

Proposition 7. *Let $\mathcal{A}$ be a unital ultraweakly closed operator algebra. Suppose that $\mathcal{A}'$ has a cyclic vector and that $\mathcal{A}\cap BC(H)$ is ultraweakly dense in $\mathcal{A}$. Then $\mathcal{A}$ has property $D_\sigma(r)$ if and only if for every $\varepsilon>0$ and every pair $x, y\in H$ such that $\|\omega_{x,y}\,|\,\mathcal{A}\|\leqq 1$ there are $\xi, \eta\in H$ such that $\|\xi\|\,\|\eta\|<r+\varepsilon$ and $\omega_{x,y}=\omega_{\xi,\eta}$ on $\mathcal{A}$.*

Proof. The "only if" part is obvious. Let us prove the "if" part. Choose an arbitrary $\varepsilon>0$ and let φ be an ultraweakly continous functional on $\mathcal{A}$ with $\|\varphi\|\leqq 1$. By Theorem 1, φ can be represented in the form $\varphi(T)=\sum_{n=1}^{\infty}(Tx_n, y_n)$ where $\left(\sum_{n=1}^{\infty}\|x_n\|^2\right)^{1/2}\left(\sum_{n=1}^{\infty}\|y_n\|^2\right)^{1/2}<1+\varepsilon$. Choose a number N such that $\sum_{n=N+1}^{\infty}(\|x_n\|^2+\|y_n\|^2)<\varepsilon$. If x_0 is a cyclic vector for $\mathcal{A}'$, there exist $\{T_i\}_{i=1}^N$ in $\mathcal{A}'$ satisfying the inequalities

$$\|T_i x_0-x_i\| < \varepsilon/2\left(\sum_{i=1}^{N}\|y_i\|\right)^{-1} \quad (i = 1, 2, \dots, N).$$

Then for every $T\in\mathcal{A}$ with $\|T\|\leqq 1$,

$$\left|\varphi(T)-\left(Tx_0, \sum_{i=1}^{N}T_i^* y_i\right)\right| \leqq \left|\sum_{i=1}^{N}(Tx_i, y_i)-\left(Tx_0, \sum_{i=1}^{N}T_i^* y_i\right)\right|+\left|\sum_{i=N+1}^{\infty}(Tx_i, y_i)\right| \leqq$$

$$\leqq \left|\sum_{i=1}^{N}(T(x_i-T_ix_0), y_i)\right|+\left|\sum_{i=N+1}^{\infty}(Tx_i, y_i)\right| \leqq$$

$$\leqq \sum_{i=1}^{N}\|x_i-T_ix_0\|\,\|y_i\|+\sum_{i=N+1}^{\infty}\|x_i\|\,\|y_i\| < \varepsilon/2+\varepsilon/2 = \varepsilon.$$

Hence it follows that the set of functionals $\omega_{x,y}$ is norm-dense in the family of all ultraweakly continuous functionals on $\mathcal{A}$. Let ω_{x_n,y_n} be a sequence converging to φ in the norm topology. We may assume that $\|\omega_{x_n, y_n} \mid \mathscr{A}\| < 1+\varepsilon$. Choose ξ_n, η_n in H such that $\omega_{x_n, y_n} = \omega_{\xi_n, \eta_n}$ on $\mathscr{A}$ and $\|\xi_n\| = \|\eta_n\| < r^{1/2}(1+\varepsilon)^{1/2}$. Passing to subsequences, we may suppose that $\xi_n \xrightarrow{w} \xi$, $\eta_n \xrightarrow{w} \eta$ for some ξ, η in H. If $K \in \mathscr{A} \cap BC(H)$, $\varphi(K) = \lim_n (K\xi_n, \eta_n)$. On the other hand,

$$|(K\xi_n, \eta_n) - (K\xi, \eta)| \leqq |(K(\xi_n - \xi), \eta_n)| + |(K\xi, \eta_n - \eta)| \leqq$$
$$\leqq r^{1/2}(1+\varepsilon)^{1/2} \|K(\xi_n - \xi)\| + |(K\xi, \eta_n - \eta)| \to 0,$$

so that $\varphi(K) = (K\xi, \eta)$. Now if $T \in \mathscr{A}$ and if $\{K_\alpha\}$ is a net of compact operators in $\mathscr{A}$ which converges ultraweakly to T, then $\varphi(T) = \lim_\alpha \varphi(K_\alpha) = \lim_\alpha (K_\alpha \xi, \eta) = (T\xi, \eta)$. Finally, since the norm on any Hilbert space is lower semicontinuous, we have $\|\xi\| \, \|\eta\| \leqq r(1+\varepsilon)$.

3. Reductive contractions with rich spectra. The properties D_σ or $D_\sigma(r)$ might be very useful in applications to various problems on invariant subspaces. Let us introduce the following definition. If G is an open non-empty subset of the complex plane, we say that $\sigma \subseteqq \mathbf{C}$ is *rich in* G if for every h in $H^\infty(G)$,

$$\sup_{z \in G} |h(z)| = \sup_{z \in \sigma \cap G} |h(z)|$$

where $H^\infty(G)$ denotes, as usual, the algebra of bounded functions analytic in G.

Recently, Brown, Chevreau and Pearcy [8] proved that if T is contraction in $B(H)$ whose spectrum $\sigma(T)$ is rich in the open unit disk D, then lat T is not trivial. Recall that an operator T is called *reductive* if lat T = lat T^*. Here we show that if $\sigma(T)$ is sufficiently large, then T cannot be reductive.

Theorem 8. *Let $T \in B(H)$ be a contraction. Suppose that $\sigma(T)$ has the following property: if $\sigma(T) = \sigma_1 \cup \sigma_2$ where σ_1 and σ_2 are closed subsets of* cl D, *then either there exists a non-empty open set G such that σ_1 is rich in G, or σ_2 is rich in D. Then T is not reductive.*

Proof. Clearly we may assume that T is completely non-unitary. Suppose T is reductive. Let H_1 denote the subspace of H spanned by all eigenvectors of T. Then $H_1 \in \text{lat}\, T$, so that $H_2 = H_1^\perp \in \text{lat}\, T$. Then $T = T_1 \oplus T_2$, where $T_i = T \mid H_i$. Clearly $\sigma(T) = \sigma(T_1) \cup \sigma(T_2)$. We claim that T_1 is normal. Indeed, since T is reductive, so is T_1, and every eigenvector of T_1 is also an eigenvector of T_1^*. Thus, if $x_1, x_2, \ldots, x_n$ is any finite set of eigenvectors of T_1 (and T_1^*), then $T_1 T_1^* \left(\sum_{k=1}^n \alpha_k x_k\right) = T_1^* T_1 \left(\sum_{k=1}^n \alpha_k x_k\right)$ for all scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$. Since H_1 is spanned by eigenvectors, we conclude that $T_1 T_1^* = T_1^* T_1$, that is T_1 is normal. Now T_1 is a normal reductive operator whose set of eigenvectors is total in H_1. By Sarason [6], there is no non-empty open set G such that $\sigma(T_1)$ is rich in G.

By assumption $\sigma(T_2)$ is rich in D. Now T_2 is a reductive completely non-unitary contraction with rich spectrum in D. Then $\sigma(T_2)$ coincides with the left essential spectrum of T_2, for otherwise, as pointed out in [8], T_2 or T_2^* (and therefore both of them) has an eigenvector, which contradicts the definition of H_1. Now by [8], T_2 has property D_σ and there exist a non-zero multiplicative ultraweakly continuous functional φ on $\mathscr{A}(T_2)$. Let

$$\varphi(S) = (Sx, y), \quad S \in \mathscr{A}(T_2).$$

Let $\mathscr{I}$ denote the null-space of φ. Then $\mathscr{I}$ is an ideal in $\mathscr{A}(T_2)$ such that the subspace $M = \mathrm{cl}\, \mathscr{I}x = \mathrm{cl}\, \{Sx, S \in \mathscr{I}\}$ is in $\mathrm{lat}\, T_2$. On the other hand, x is not in M, for $y \in M^\perp$, but $\varphi(I) = (x, y) \neq 0$. If we denote by N the subspace spanned by x and M, then $N \in \mathrm{lat}\, T_2$ and $\dim(N \ominus M) = 1$. Since T_2 is reductive, $N \ominus M \in \mathrm{lat}\, T_2$, which again contradicts the definition of H_1. This contradiction leads to the desired conclusion.

Corollary 9. *If T is a contraction and $\sigma(T)$ is an annulus $\{z \mid r \leqq |z| \leqq 1\}$, $0 \leqq r < 1$, then T is not reductive.*

Proof. Suppose $\sigma(T) = \sigma_1 \cup \sigma_2$ with $\sigma_1, \sigma_2 \subseteqq \mathrm{cl}\, D$ and σ_2 not rich in D. Then there are $\lambda \in \mathbf{C}$, $|\lambda| = 1$ and $\varepsilon, 0 < \varepsilon < 1 - r$ such that $\{z \mid |z - \lambda| < \varepsilon\} \cap \sigma_2 = \emptyset$. But then $G = \{z \mid |z - \lambda| < \varepsilon\} \cap D \subseteqq \sigma_1$, so that σ_1 is rich in G, which completes the proof.

Of course, an example of a rich subset of D not satisfying the conditions of Theorem 8 can be easily constructed.

References

[1] D. Hadwin and E. Nordgren, Subalgebras of reflexive algebras, preprint.

[2] B. Sz.-Nagy and C. Foiaş, *Harmonic analysis of operators on Hilbert space*, Akadémiai Kiadó, (Budapest), North-Holland (Amsterdam), American Elsevier (New York) (1970).

[3] L. Brickman and P. Fillmore, The invariant subspace lattice of a linear transformation, *Canad. J. Math.*, **19** (1967), 810—822.

[4] D. Sarason, A remark on the Volterra operator, *J. Math. Anal. Appl.*, **12** (1965), 244—246.

[5] D. Sarason, Generalized interpolation in H^∞, *Trans. Amer. Math. Soc.*, **127** (1967), 179—203.

[6] D. Sarason, Weak-star density of the polynomials, *J. Reine Angew. Math.*, **252** (1972), 1—15.

[7] C. Foiaş and J. Williams, Some remarks on the Volterra operators, *Proc. Amer. Math. Soc.*, **31** (1972), 177—184.

[8] S. Brown, B. Chevreau and C. Pearcy, Contractions with rich spectra have invariant subspaces, *J. Operator Theory*, **1** (1979), 123—136.

BOL'SHAYA SERPUHOVSKAYA 31, KORPUS 6, APT. 229A,
113093 MOSCOW, U.S.S.R.