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On ranges of adjoint operators in Hilbert space

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Introduction

Let A be a given densely defined operator in the (complex) Hilbert space H. Let further y and z be elements in H. The relation

$$(1) \qquad (Ax, z) = (x, y) \quad (x \in \mathscr{D}(A)),$$

where $\mathscr{D}(A)$ stands as usual for the domain of A, is fundamental for the definition of A^*, the adjoint of A. Namely, z is in $\mathscr{D}(A^*)$ if

$$\sup \{|(Ax, z)|: x \in \mathscr{D}(A), \|x\| \leqq 1\} < \infty$$

holds, that is by the Riesz Representation Theorem if and only if there is an y in H satisfying (1). The reverse problem is the characterization of $\mathscr{R}(A^*)$, the range of $A^* : y$ is in $\mathscr{R}(A^*)$ if there is an element, z in $\mathscr{D}(A^*)$ for which (1) holds. We shall show that this is the case if and only if

$$\sup \{|(x, y)|: x \in \mathscr{D}(A), \|Ax\| \leqq 1\} < \infty$$

holds (Theorem 1).

As an application we obtain results concerning the factorization of a given densely defined operator C in H in the form $C \subset A^*B$ by which we mean that B is an operator in H defined at least on $\mathscr{D}(C)$, and for any x in $\mathscr{D}(C)$, $Bx \in \mathscr{D}(A^*)$ and $Cx = A^*(Bx)$. In general, as a Zorn's argument shows, $\mathscr{R}(A^*) \supset \mathscr{R}(C)$ is sufficient for such a factorization, but we produce a minimal B in the sense that

$$(2) \qquad \|Bx\| \leqq \|u\| \quad \text{for} \quad x \in \mathscr{D}(C),\ u \in \mathscr{D}(A^*);\ Cx = A^*u.$$

The question of the boundedness of B is also analyzed in the hope that we shall be able to answer the question raised by R. G. DOUGLAS [1] concerning the factorization of unbounded operators, especially with a bounded cofactor.

Our constant reference is [2].

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Results

Theorem 1. *Let y and A be a unit vector and a densely defined operator, respectively, in a Hilbert space H. The following two assertions are equivalent:*

(i) *There exists a unique vector z in H such that*

$$y = A^*z \quad and \quad \|z\| \leqq \|u\| \quad for \quad u\in\mathcal{D}(A^*),\ y = A^*u. \tag{3}$$

(ii) $M_y := \sup\{|(x, y)|:\ x\in\mathcal{D}(A),\ \|Ax\| \leqq 1\} < \infty$.

If (i) *and* (ii) *are valid, then* $M_y = \|z\|$.

Proof. (ii) simply follows from (i) since for any x in $\mathcal{D}(A)$,

$$|(x, y)| = |(x, A^*z)| = |(Ax, z)| \leqq \|z\| \cdot \|Ax\|;$$

we see also that $M_y \leqq \|z\|$.

(ii) implies (i): Assuming (ii) we have $|(x, y)| \leqq M_y\|Ax\|$ for any x in $\mathcal{D}(A)$. So the map $Ax \mapsto (x, y)$ is a bounded linear functional on $\mathcal{R}(A)$. It has a unique bounded linear extension to $\overline{\mathcal{R}(A)}$, the norm closure of $\mathcal{R}(A)$. By the Riesz Representation Theorem there exists a unique vector, z, in $\overline{\mathcal{R}(A)}$ for which (1) holds. Then z is in $\mathcal{D}(A^*)$ and $y = A^*z$.

If $u \neq z$ is from $\mathcal{D}(A^*)$ and $y = A^*u$, then $(Ax, z) = (x, A^*z) = (x, A^*u) = (Ax, u)$ for every x in $\mathcal{D}(A)$. Since z is, while u is not in $\overline{\mathcal{R}(A)}$, it follows that

$$\|z\| = \sup\{|(Ax, z)|:\ x\in\mathcal{D}(A),\ \|Ax\| \leqq 1\} = $$
$$= \sup\{|(Ax, u)|:\ x\in\mathcal{D}(A),\ \|Ax\| \leqq 1\} < \|u\|$$

Thus (3) holds and the z with this property is unique. The proof is complete.

Theorem 2. *Let A and C be densely defined operators in a Hilbert space H. The following three assertions are equivalent:*

(i) *There exists an operator B in H such that*

$$C \subset A^*B \quad and\ B\ fulfils\ (2). \tag{4}$$

(ii) $\mathcal{R}(C) \subset \mathcal{R}(A^*)$.

(iii) $M_y(C) := \sup\{|(x, Cy)|:\ x\in\mathcal{D}(A),\ \|Ax\| \leqq 1\} < \infty \quad (y\in\mathcal{D}(C))$.

Proof. (i) clearly implies (ii). Further (ii) implies (iii) since for any y in $\mathcal{D}(C)$ there exists (by assumption) a u in $\mathcal{D}(A^*)$ such that $Cy = A^*u$ whence for any x in $\mathcal{D}(A)$,

$$|(x, Cy)| = |(x, A^*u)| = |(Ax, u)| \leqq \|u\| \cdot \|Ax\|,$$

and thus (iii) follows.

Lastly assume (iii) and prove (i). For a fixed x in $\mathcal{D}(C)$ there exists, by Theorem 1, a unique vector z in H such that (3) holds with $y = Cx$. Writing

$z=Bx$ we get just (2) as desired. We have to show only that Bx is a linear function of x if x varies on $\mathscr{D}(C)$.

Recall that, as the proof of Theorem 1 indicates, Bx is in $\overline{\mathscr{R}(A)}$ for any x in $\mathscr{D}(C)$. Thus if x, x' are arbitrary vectors from $\mathscr{D}(C)$, for any y belonging to $\mathscr{D}(A)$ we have

$$0 = \big(C(x+x'), y\big)-(Cx, y)-(Cx' y) = \big(A^*B(x+x'), y\big)-(A^*Bx, y)-(A^*Bx', y) =$$
$$= \big(B(x+x'), Ay\big)-(Bx, Ay)-(Bx', Ay) = \big(B(x+x')-Bx+Bx', Ay\big),$$

which shows that $B(x+x')=Bx+Bx'$. The proof of $B(\lambda x)=\lambda Bx$ for a scalar λ is similar. The proof is complete.

The following is analogous to [1, Theorem 2, (3)] due to Douglas.

Corollary 1. *If C of Theorem 2 is closed then*

$$\sup \{\|Bx\|:\ x\in\mathscr{D}(C),\ \|x\|+\|Cx\| \leqq 1\} < \infty.$$

In particular, B is bounded if C is.

Proof. By assumption, C has a closed graph. Hence we have to show that the linear operator given by $\{x, Cx\}\mapsto Bx$ $\big(x\in\mathscr{D}(C)\big)$ also has a closed graph. In other words, assuming that $x_n\to x$, $Cx_n\to Cx$ and $Bx_n\to u$, we must conclude $u=Bx$. Since $Cx_n\to Cx$ means that $A^*Bx_n\to Cx$, by the closedness of A^* we get $A^*u=Cx=A^*Bx$. But since Bx_n is in $\overline{\mathscr{R}(A)}$, u is in $\overline{\mathscr{R}(A)}$, too. As $(Ay, u)= =(y, A^*u)=(y, A^*Bx)=(Ay, Bx)$ for every $y\in\mathscr{D}(A)$, it follows that

$$\|u\| = \sup \{|(Ay, u)|:\ y\in\mathscr{D}(A),\ \|Ay\| \leqq 1\} =$$
$$= \sup \{|(Ay, Bx)|:\ y\in\mathscr{D}(A),\ \|Ay\| \leqq 1\} \leqq \|Bx\|,$$

whence by the uniqueness of Bx we have $u=Bx$ indeed.

Remark 1. If in Theorem 2 the operator A is bounded and C is closed, further if we take $\mathscr{D}(B)=\mathscr{D}(C)$, then B is closed. Indeed, if $x_n\to x$ and $Bx_n\to u$, where $x_n\in\mathscr{D}(C)$ $(n=1, 2, \dots)$, then $Cx_n=A^*Bx_n\to A^*u$ so that $A^*u=Cx=A^*Bx$, and an argument similar to that appearing in the proof of Corollary 1 shows $u=Bx$.

Theorem 3. *The following four assertions are equivalent:*

(i) *The operator B in Theorem 2 (i) is bounded.*

(ii) $\mathscr{R}(A^*)\supset\mathscr{R}(C)$ *and*

$$\sup \{\inf [\|z\|:\ z\in\mathscr{D}(A^*),\ Cy = A^*z]:\ y\in\mathscr{D}(C),\ \|y\| \leqq 1\} < \infty.$$

(iii) $\sup \{|(x, Cy)|:\ x\in\mathscr{D}(A),\ \|Ax\| \leqq 1,\ y\in\mathscr{D}(C),\ \|y\| \leqq 1\} < \infty.$

(iv) $\mathscr{D}(C^*) \supset \mathscr{D}(A)$ *and*

$$\sup \{\|C^*x\|:\ x\in\mathscr{D}(A), \|Ax\| \leqq 1\} < \infty.$$

Proof. Assume first (i). We know from Theorem 2 that for any y in $\mathscr{D}(C)$,

$$\inf[\|z\|:\ z\in\mathscr{D}(A^*),\ Cy=A^*z]=\|By\|\leqq\|B\|\,\|y\|$$

holds. This proves (ii). But (ii) implies (iii) since we know also from Theorem 2 that

$$\sup\{|(x, Cy)|:\ x\in\mathscr{D}(A),\ \|Ax\|\leqq 1\}=\inf[\|z\|:\ z\in\mathscr{D}(A)^*,\ Cy=A^*z]$$

for any y in $\mathscr{D}(C)$. For the same reason (iii) implies (i). But (iv) also follows from (iii) since by (iii) $\mathscr{D}(A)\subset\mathscr{D}(C^*)$ and since for any x in $\mathscr{D}(A)$,

$$\|C^*x\|=\sup\{|(C^*x, y)|:\ y\in\mathscr{D}(C),\ \|y\|\leqq 1\}=$$
$$=\sup\{|(x, Cy)|:\ y\in\mathscr{D}(C),\ \|y\|\leqq 1\}.$$

Finally (iv) implies (iii) since for any x in $\mathscr{D}(A)$, x is in $\mathscr{D}(C^*)$ and

$$|(x, Cy)|=|(C^*x, y)|\leqq\|C^*x\|\cdot\|y\|$$

holds for any y in $\mathscr{D}(C)$.

Remark 2. Assuming that A^* is densely defined or, what is the same, that A^{**} exists, assertions (i)—(iv) in Theorem 3 are equivalent to

(iv)′ $\sup\{\|C^*x\|:x\in\mathscr{D}(A^{**}),\ \|A^{**}x\|\leqq 1\}<\infty$.

Indeed, since $A^{**}\supset A$ in this case, (iv)′ implies (iv). On the other hand, (i) implies now that $C^*\supset(A^*B)^*\supset B^*A^{**}$ and that

$$\|C^*x\|=\|B^*A^{**}x\|\leqq\|B^*\|\cdot\|A^{**}x\|$$

holds for any x in $\mathscr{D}(A^{**})$, which proves (iv)′.

References

[1] R. G. Douglas, On majorization, factorization, and range inclusion of operators on Hilbert spaces, *Proc. Amer. Math. Soc.*, **17** (1966), 413—415.

[2] F. Riesz and B. Sz.-Nagy, *Functional Analysis*, Ungar (New York; 1960).

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